PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1651–1658
S 0002-9939(97)03849-5

ISOMORPHISMS OF ROW AND COLUMN FINITE MATRIX RINGS

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(Communicated by Ken Goodearl)

Abstract. This paper investigates the ring-theoretic similarities and the categorical dissimilarities between the ring $RFM(R)$ of row finite matrices and the ring $RCFM(R)$ of row and column finite matrices. For example, we prove that two rings R and S are Morita equivalent if and only if the rings $RCFM(R)$ and $RCFM(S)$ are isomorphic. This resembles the result of V. P. Camillo (1984) for $RFM(R)$. We also show that the Picard groups of $RFM(R)$ and $RCFM(R)$ are isomorphic, even though the rings $RFM(R)$ and $RCFM(R)$ are never Morita equivalent.

1. Introduction

Let R be a ring with identity, let $RFM(R)$ be the ring of row-finite matrices over R, let $RCFM(R)$ be the ring of row and column-finite matrices over R, let $FC(R)$ be the ring of matrices with a finite number of nonzero columns, and let $FM(R)$ be the ring of all matrices with only a finite number of nonzero entries. All matrices are considered countably indexed. The theme of this paper is that while the rings $RFM(R)$ and $RCFM(R)$ are categorically quite different, they share many ring-theoretic properties. For example, Camillo ([3]) has shown that rings R and S are Morita equivalent if and only if $RFM(R)$ and $RFM(S)$ are isomorphic. We prove

Theorem A. *Rings R and S, with identity, are Morita equivalent if and only if $RCFM(R)$ and $RCFM(S)$ are isomorphic.*

Similar to Camillo's proof, the key argument in the proof of Theorem A involves understanding the isomorphisms between $RCFM(R)$ and $RCFM(S)$. Consequently, we also prove

Proposition B. *Every (ring) isomorphism between $RCFM(R)$ and $RCFM(S)$ restricts to an isomorphism between $FM(R)$ and $FM(S)$.*

Proposition B is fundamental in the study of the Picard groups of $RCFM(R)$ and R and, again, we find a ring-theoretic similarity between $RCFM(R)$ and $RFM(R)$. For example, in [2], the authors show that the Picard group of R is isomorphic to the Picard group of $RFM(R)$. We prove

Received by the editors January 8, 1996.
1991 *Mathematics Subject Classification.* Primary 16D30, 16S50, 16W20.
This paper was written while the first author was visiting the Universidad de Murcia with a grant from DGICYT (SAB 95-0215).
The second and third authors have been supported by DGICYT (PB-0300-C02-02).

Theorem C. *The Picard group of $RFM(R)$ is isomorphic to the Picard group of $RCFM(R)$.*

To prove this theorem, we first show that an automorphism of $RFM(R)$ is the product of an inner automorphism and an automorphism that restricts to an automorphism of $FM(R)$.

Finally, while the rings $RFM(R)$ and $RCFM(R)$ share ring-theoretic properties as seen in the above results, our last theorem shows that they are very different in a categorical sense. We prove

Theorem D. *There do not exist rings R and S (with identity) such that $RFM(S)$ is Morita equivalent to $RCFM(R)$.*

So, for example, although the Picard groups of $RFM(R)$ and $RCFM(R)$ are isomorphic, the rings themselves are not Morita equivalent.

The main tool for all these applications appears in section 3, in which we prove that $FM(R)$ is the largest 2-sided ideal in $RCFM(R)$ satisfying some technical property. See Lemma 1. Section 4 is devoted towards proving our above-mentioned results.

2. Notation and preliminaries

Let R be a ring with identity. We will write the action of homomorphisms of left modules on the right.

For every $i, j \in \mathbf{N}$, $e_{ij} \in FM(R)$ is the basic matrix having $1 \in R$ in the ij-place and zero in each other. We denote $e_i = e_{ii}$. For any finite subset $X \subset \mathbf{N}$ we denote $e_X = \sum_{x \in X} e_x$. For any matrix α and $i, j \in \mathbf{N}$, $\alpha(i, j)$ denotes the (i, j)-entry of α.

The following facts will be used without explicit mention. $FC(R)$ is a two-sided ideal of $RFM(R)$ and it is generated by $\{e_i \mid i \in \mathbf{N}\}$ as a left ideal. $FM(R)$ is a two-sided ideal of $RCFM(R)$ and it is generated by $\{e_i \mid i \in \mathbf{N}\}$ both as left ideal and as right ideal. Actually, $RCFM(R)$ is the idealizer of $FM(R)$ in $RFM(R)$. $FM(R)$ is the right ideal of $RFM(R)$ generated by $\{e_i \mid i \in \mathbf{N}\}$ (see[5]). Moreover,

$$\begin{aligned} FM(R) &= \bigcup_{\substack{X \subset \mathbf{N} \\ X \text{finite}}} e_X RFM(R) \\ &= \bigcup_{\substack{X \subset \mathbf{N} \\ X \text{ finite}}} RCFM(R) e_X = \bigcup_{\substack{X \subset \mathbf{N} \\ X \text{ finite}}} e_X RCFM(R) \end{aligned}$$

and

$$FC(R) = \bigcup_{\substack{X \subset \mathbf{N} \\ X \text{finite}}} RFM(R) e_X.$$

We observe that if $f \in RFM(R)$ and $fe_i = 0$ for all $i \in \mathbf{N}$, then $f = 0$ and the symmetric property also holds.

Finally, $RFM(R)$ is isomorphic to $\mathrm{End}({}_R R^{(\mathbf{N})})$ and to $\mathrm{End}({}_{FM(R)} FM(R))$ (by right multiplications).

3. The Fundamental Lemma

To prove the results posed in the introduction, we first show that $FM(R)$ is the largest two-sided ideal of $RCFM(R)$ in a certain sense.

Lemma 1. *Let R be a ring with identity and suppose that there exists a family $\{f_{ij}\}_{ij\in\mathbf{N}}$ of nonzero elements of $RCFM(R)$ such that:*

1. *$f_{ij}f_{kl} = \delta_{jk}f_{il}$, for every $i, j, k, l \in \mathbf{N}$.*
2. *$J = \sum_{\mathbf{N}} RCFM(R)f_i$ is a two-sided ideal in $RCFM(R)$ (where $f_i = f_{ii}$).*

Then $J \subseteq FM(R)$.

Proof. Set $I = FM(R)$. If $f_{ij} \in I$ for some i, j then $f_{kl} = f_{ki}f_{ij}f_{jl} \in I$, for every $k, l \in \mathbf{N}$ (because I is a two-sided ideal of A) and hence $J \subseteq I$. Therefore one may assume that $f_{ij} \notin I$ for all $i, j \in \mathbf{N}$.

For each $n \in \mathbf{N}$ set $X_n = \{i \in \mathbf{N} \mid f_1e_if_{1n} \neq 0\}$. We claim that $X_n \neq \emptyset$. To see this, first note that $f_1f_{1n} = f_{1n} \neq 0$ and so there is an $i \in \mathbf{N}$ such that $0 \neq e_if_1 \in I$. Thus $e_if_1 = e_if_1e_X$, for some finite subset $X \subset \mathbf{N}$ and hence $0 \neq e_if_{1n} = e_if_1f_{1n} = e_if_1e_Xf_{1n} = \sum_{x\in X} e_if_1e_xf_{1n}$ so that there is an $x \in X$ such that $f_1e_xf_{1n} \neq 0$.

Next we prove that X_n is an infinite set. Suppose X_n is finite. For every $k \in \mathbf{N}$, let $P_k = \{r \in \mathbf{N} \mid e_kf_1e_r \neq 0\}$. Then $e_kf_{1n} = e_kf_1e_{P_k}f_{1n} = e_kf_1e_{P_k\cap X_n}f_{1n} = e_kf_1e_{X_n}f_{1n}$, for every $k \in \mathbf{N}$ and hence $f_{1n} = f_1e_{X_n}f_{1n} \in I$ which contradicts our assumption.

We recursively construct two sequences, $(i_j)_{j\in\mathbf{N}}$ and $(k_j)_{j\in\mathbf{N}}$, of natural numbers such that the first sequence consists of elements from X_n, while second one is strictly increasing. This will ultimately generate a contradiction to our assumption that $f_{ij} \notin I$ for all $i, j \in \mathbf{N}$.

Let i_1 be the first element of X_1, $Z_1 = \{r \in \mathbf{N} \mid e_rf_1e_{i_1} \neq 0 \text{ or } e_{i_1}f_1e_r \neq 0\}$ and $k_1 = \max Z_1$. For every $n > 1$, let

$$Y_n = \{r \in \mathbf{N} \mid e_mf_1e_r \neq 0 \text{ or } e_rf_{1n}e_m \neq 0 \text{ for some } m \leq k_{n-1}\}.$$

It is clear that Y_n is a finite set. Now we define i_n to be the first element of $X_n - Y_n$ and $Z_n = \{r \in \mathbf{N} \mid e_rf_1e_{i_n} \neq 0 \text{ or } e_{i_n}f_{1n}e_r \neq 0\}$. Note that Z_n is a finite set and is not empty because $i_n \in X_n$. Further since $i_n \notin Y_n$, $r > k_{n-1}$ for every $r \in Z_n$. In particular $k_n = \max Z_n > k_{n-1}$.

Let α be the $\mathbf{N} \times \mathbf{N}$ matrix over R given by

$$\alpha(i,j) = \begin{cases} (f_1e_{i_n}f_{1n})(i,j) & \text{if } k_{n-1} < i, j \leq k_n \text{ for some } n \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $\alpha \in RCFM(R)$. Let $K_n = \{i \in \mathbf{N} \mid k_{n-1} < i \leq k_n\}$. Note that $e_{K_n}f_1e_{i_n}f_{1n} = f_1e_{i_n}f_{1n}e_{K_n} = f_1e_{i_n}f_{1n}$, because if $e_xf_1e_{i_n}f_{1n} \neq 0$ then $x \in Z_n \subseteq K_n$ and similarly $f_1e_{i_n}f_{1n}e_x \neq 0$ implies that $x \in Z_n \subseteq K_n$. Hence we have that $e_{K_n}\alpha = \alpha e_{K_n} = f_1e_{i_n}f_{1n}$ for every $n \in \mathbf{N}$.

Now we show two properties of α. First, we assert that $\alpha = f_1\alpha$. Indeed, if $j \in \mathbf{N}$, then $j \in K_n$ for some n. Therefore $\alpha e_j = \alpha e_{K_n}e_j = f_1e_{i_n}f_{1n}e_j = f_1 \cdot f_1e_{i_n}f_{1n}e_j = f_1\alpha e_{K_n}e_j = f_1\alpha e_j$. We conclude that $\alpha = f_1\alpha$.

Second, we assert that $\alpha f_n \neq 0$ for all $n \in \mathbf{N}$. To see this, note that $e_{K_n}\alpha f_n = f_1e_{i_n}f_{1n}f_n = f_1e_{i_n}f_{1n} \neq 0$. Thus $\alpha f_n \neq 0$ for all $n \in \mathbf{N}$.

But as $\alpha \in RCFM(R)$, $f_1\alpha \in f_1RCFM(R) \subseteq J$ because J is two-sided. Therefore $f_1\alpha = \sum_{j\in F} f_1\alpha f_j$, where F is a finite subset of $\mathbf{N}$, so that $f_1\alpha f_n = 0$ for almost all $n \in \mathbf{N}$, which contradicts the second property of α. $\square$

It is obvious that condition 1 of Proposition 1 cannot be deleted. The following example shows that condition 2 of Proposition 1 is also not superfluous.

Example 2. There exist a ring R, a family $\{f_{ij}\}_{ij\in\mathbf{N}} \subseteq RCFM(R)$ such that $f_{ij}f_{kl} = \delta_{jk}f_{il}$ for every $i, j, k, l \in \mathbf{N}$, and a left ideal, $J = \sum RCFM(R)f_i$, that properly contains $FM(R)$.

Proof. Let $\beta : \mathbf{N} \to \mathbf{N}^2$ be a bijection. This bijection induces an isomorphism of R-bimodules $\beta : {}_RR_R^{(\mathbf{N})} \to {}_R(R^{(\mathbf{N})})_R^{(\mathbf{N})}$ which induces two ring isomorphisms β_l: $\mathrm{End}({}_RR^{(\mathbf{N})}) \to \mathrm{End}({}_R(R^{(\mathbf{N})})^{(\mathbf{N})})$ and β_r : $\mathrm{End}(R_R^{(\mathbf{N})}) \to \mathrm{End}((R^{(\mathbf{N})})_R^{(\mathbf{N})})$. We recall that $\mathrm{End}({}_R(R^{(\mathbf{N})})^{(\mathbf{N})})$ is isomorphic to the ring $RCM(RFM(R))$ of row-convergent matrices over $RFM(R)$ and $\mathrm{End}((R^{(\mathbf{N})})_R^{(\mathbf{N})})$ is isomorphic to the ring $CCM(CFM(R))$ of column convergent matrices over $CFM(R)$ ([4, Theorem 106.1]). Now having in mind the nature of these isomorphisms, one can check they induce an isomorphism σ between $RCFM(R)$ and $RCCM(RCFM(R)) = RCM(RFM(R)) \cap CCM(CFM(R))$. Further, one can check that $\sigma(FM(R)) = FM(FM(R))$.

Let $K = RCCM(RCFM(R))$ and $f'_{ij} \in K$ such that f'_{ij} has $1 \in RCFM(R)$ in the ij-place and zero elsewhere. Let $J = \sum K f'_{ii}$. Clearly, $FM(FM(R))$ is contained properly in J and taking $f_{ij} = \sigma^{-1}(f_{ij})$ we have the desired family. We show explicitly that J is not a two-sided ideal. Take $x \in K$ such that x has $e_{1j} \in FM(R)$ in the $1j$-entry and zero elsewhere. Since $J \subseteq RFM(RFM(R))$, $x \notin J$. But, $f_{11}e_{1j} = e_{1j}$ for every $j \in \mathbf{N}$, and hence $x = f_{11}x$. □

4. Comparing $RFM(R)$ with $RCFM(R)$

In this section, we prove the results mentioned in the introduction. We begin with Theorem A and Proposition B. The key to the main result of [3] is, in essence, that an isomorphism $\phi : RFM(R) \to RFM(S)$ satisfies $\phi(FC(R)) = FC(S)$, where FC is the ring of matrices with only finitely many non-zero columns. In our setting, with $RCFM(R)$ taking the place of $RFM(R)$, this result translates into Proposition B from the Introduction.

Proposition 3. *Let R and S be any two rings with identity.*

(a) *Every ring isomorphism $\delta : RCFM(R) \to RCFM(S)$ satisfies*
$$\delta(FM(R)) = FM(S).$$
(b) *Every ring isomorphism $\delta : RCFM(R) \to RCFM(S)$ extends, in a unique way, to an isomorphism $\delta' : RFM(R) \to RFM(S)$.*

(c) *Every ring isomorphism $\sigma : RFM(R) \to RFM(S)$ such that*
$$\sigma(FM(R)) = FM(S)$$
satisfies $\sigma(RCFM(R)) = RCFM(S)$.

(d) *There is a group monomorphism*
$$\phi : \mathrm{Aut}(RCFM(R)) \to \mathrm{Aut}(RFM(R))$$
via $\phi(\delta) = \delta'$ using (b) *above. Moreover, the image of ϕ is the subgroup of automorphisms of $RFM(R)$ that restrict to automorphisms of $FM(R)$.*

Proof. (a) follows immediately from Lemma 1. Using the fact that $RFM(R)$ is isomorphic to $End({}_{FM(R)}FM(R))$, (b) is straightforward. We get (c) from the fact that $RCFM(R)$ is the idealizer of $FM(R)$ in $RFM(R)$. Finally, (d) follows from (a), (b), and (c). □

Now we can prove an analogue to Camillo's result for $RCFM(R)$.

Theorem 4. *Let R and S be rings with identity. R and S are Morita equivalent rings if and only if $RCFM(R)$ and $RCFM(S)$ are isomorphic rings.*

Proof. Assume first that R and S are Morita equivalent rings. Let ${}_RP$ be a progenerator such that $\mathrm{End}({}_RP) \cong S$ as rings. By [1, Lemma 1.2] we have that there exists a ring isomorphism $\alpha^* : RFM(R) \to RFM(\mathrm{End}({}_RP))$ such that $\alpha^*(FM(R)) = FM(\mathrm{End}({}_RP))$. Let $\beta : RFM(\mathrm{End}({}_RP)) \to RFM(S)$ be induced (coordinate-wise) by the isomorphism $\mathrm{End}({}_RP) \cong S$, and let $\delta = \beta \circ \alpha^*$. Then it is clear that $\delta(FM(R)) = FM(S)$. Since $RCFM(R)$ (resp. $RCFM(S)$) is the idealizer of $FM(R)$ (resp. $FM(S)$), $\delta(RCFM(R)) = RCFM(S)$.

The converse follows from Proposition 3 together with [1, Theorem 2.5 (3 implies 1)]. $\square$

It is interesting to note that there are some rings between $FM(R)$ and $RCFM(R)$ which have automorphisms that do not restrict to automorphisms of $FM(R)$, as the next example shows.

Example 5. There exist rings with identity, R and S, such that $FM(R) \subset S \subset RCFM(R)$ and an automorphism of S, say δ, such that $\delta(FM(R)) \neq FM(R)$.

Proof. Let K be any ring with identity and let $B = RCFM(RCFM(K))$. As we saw in Example 2 there exists an injective ring homomorphism $\beta : B \to RCFM(K)$. Let T be the image of β, let $R = K \times RCFM(K)$, and let $S = T \times B$. After identifying $RCFM(R)$ with $RCFM(K) \times B$, it is clear that $FM(R) \subset S \subset RCFM(R)$ and that $\delta : S \to S$ via $\delta(x, y) = (\beta(y), \beta^{-1}(x))$ is an automorphism of S. But an straightforward calculation shows that $\delta(FM(R)) \neq FM(R)$. $\square$

The ring $RFM(R)$ is another ring for which there are automorphisms of $RFM(R)$ that do not restrict to automorphisms of $FM(R)$; see [1]. Nonetheless, we show that these pathological automorphisms are "controlled" by the inner automorphisms of $RFM(R)$. In particular, we show that every automorphism of $RFM(R)$ is a product of an inner automorphism and an automorphism that restricts to an automorphism of $FM(R)$.

Proposition 6. *For every $\sigma \in \mathrm{Aut}(RFM(R))$, there exists $\tau \in \mathrm{Inn}(RFM(R))$ such that $\sigma\tau(FM(R)) = FM(R)$.*

Proof. Let $\sigma \in \mathrm{Aut}(RFM(R))$. Then $P = R^{(\mathbf{N})}\sigma(e_1)$ is a progenerator as left R-module such that $\mathrm{End}({}_RP)$ is isomorphic to R [3]. Specifically, the isomorphism $\tau : R \to \mathrm{End}({}_RP)$ is given by $(p)\tau(r) = p\sigma(e_1 D(r))$ where $D(r)$ denotes the scalar matrix defined by r. We consider P as an R-bimodule using this isomorphism; explicitly, $r{\cdot}p{\cdot}s = rp\tau(s)$ $(r, s \in R, p \in P)$. Define $\tau^* : RFM(R) \to RFM(\mathrm{End}(P))$ via a coordinate-wise application of τ.

The map $f : P^{(\mathbf{N})} \to R^{(\mathbf{N})}$ given by $((p_i)_{i\in\mathbf{N}})f = \sum_{i\in\mathbf{N}} p_i\sigma(e_{1i})$ is an isomorphism whose inverse is given by $((r_i)_{i\in\mathbf{N}})f^{-1} = (r_i\sigma(e_{i1}))_{i\in\mathbf{N}}$.

We identify $RFM(R)$ with $\mathrm{End}(R^{(\mathbf{N})})$ and $RFM(\mathrm{End}(P))$ with $\mathrm{End}(P^{(\mathbf{N})})$ canonically. For every $x \in RFM(R)$, the following diagram is commutative:

$$\begin{array}{ccc} R^{(\mathbf{N})} & \xrightarrow{\sigma(x)} & R^{(\mathbf{N})} \\ f\uparrow & & \uparrow f \\ P^{(\mathbf{N})} & \xrightarrow{\tau^*(x)} & P^{(\mathbf{N})} \end{array}$$

To see this, observe

$$\begin{aligned}(p)\tau^*(x)f &= ((\textstyle\sum_{i\in\mathbf{N}} p_i\tau(x_{ij}))_{j\in\mathbf{N}})f\\ &= ((\textstyle\sum_{i\in\mathbf{N}} p_i\sigma(e_1D(x_{ij})))_{j\in\mathbf{N}})f\\ &= \textstyle\sum_{j\in\mathbf{N}}\sum_{i\in\mathbf{N}} p_i\sigma(e_1D(x_{ij})e_{1j})\\ &= \textstyle\sum_{i\in\mathbf{N}} p_i\sigma(e_1\sum_{j\in\mathbf{N}} D(x_{ij})e_{1j})\\ &= \textstyle\sum_{i\in\mathbf{N}} p_i\sigma(e_{1i}x)\\ &= (p)f\sigma(x).\end{aligned}$$

Now let $\alpha : P^{(\mathbf{N})} \to R^{(\mathbf{N})}$ be the isomorphism mentioned in [1], which induces an isomorphism $\alpha^* : RFM(\mathrm{End}(P)) = \mathrm{End}(P^{(\mathbf{N})}) \to \mathrm{End}(R^{(\mathbf{N})}) = RFM(R)$ such that $\alpha^*(FM(\mathrm{End}(_RP))) = FM(R)$. More concretely, α^* is characterized by the property that, for every $y \in \mathrm{End}(P^{(\mathbf{N})})$, the following diagram is commutative:

$$\begin{array}{ccc} P^{(\mathbf{N})} & \xrightarrow{y} & P^{(\mathbf{N})} \\ \alpha\downarrow & & \downarrow\alpha \\ R^{(\mathbf{N})} & \xrightarrow{\alpha^*(y)} & R^{(\mathbf{N})} \end{array}$$

Therefore, the following diagram is commutative, for every $x \in RFM(R)$:

$$\begin{array}{ccc} R^{(\mathbf{N})} & \xrightarrow{\sigma(x)} & R^{(\mathbf{N})} \\ f\uparrow & & \uparrow f \\ P^{(\mathbf{N})} & \xrightarrow{\tau^*(x)} & P^{(\mathbf{N})} \\ \alpha\downarrow & & \downarrow\alpha \\ R^{(\mathbf{N})} & \xrightarrow{\alpha^*\tau^*(x)} & R^{(\mathbf{N})} \end{array}$$

It follows that $\sigma^{-1}\alpha^*\tau^*$ is the inner automorphism of $RFM(R)$ induced by $\alpha^{-1}f$ and $\alpha^*\tau^*(FM(R)) = FM(R)$. □

Recall that the Picard group of a ring T is the multiplicative group consisting of the bimodule isomorphism classes of invertible T-bimodules. We now prove Theorem C from the Introduction.

Theorem 7. *For every ring R,*

$$\mathrm{Pic}(R) \simeq \mathrm{Pic}(RFM(R)) \simeq \mathrm{Pic}(FM(R)) \simeq \mathrm{Pic}(RCFM(R)).$$

Proof. It has been shown in [2] that $\mathrm{Pic}(R){\simeq}\mathrm{Pic}(RFM(R)){\simeq}\mathrm{Pic}(FM(R))$. On the other hand, both $RFM(R)$ and $RCFM(R)$ have the SBN property, so that $\mathrm{Pic}(RCFM(R)) = \mathrm{Out}(RCFM(R))$ and $\mathrm{Pic}(RFM(R)){=}\mathrm{Out}(RFM(R))$; see [2]. Thus, it suffices to show that $\mathrm{Out}(RCFM(R)) \simeq \mathrm{Out}(RFM(R))$.

From Proposition 3, there is a group monomorphism $\phi : \mathrm{Aut}(RCFM(R)) \to \mathrm{Aut}(RFM(R))$ such that the image of ϕ is the subgroup of automorphisms of $RFM(R)$ that restrict to automorphisms of $FM(R)$. In particular, $\phi(\delta) = \delta'$ using (b) of Proposition 3. We claim that

$$\phi(\mathrm{Inn}(RCFM(R))) = \mathrm{Inn}(RFM(R)) \cap Im(\phi).$$

It is clear that $\phi(\mathrm{Inn}(RCFM(R))) \subseteq \mathrm{Inn}(RFM(R))$. For the opposite inclusion, note that if $\sigma \in \mathrm{Aut}(RCFM(R))$ such that $\phi(\sigma) \in \mathrm{Inn}(RFM(R))$, then there exists a unit $u \in RFM(R)$ for which $uFM(R) = FM(R)u$. In particular, for each i, $u \cdot e_{ii} \in FM(R)$ so that $e_{jj} \cdot u \cdot e_{ii} = 0$ for almost all values of j. Hence,

$u \in RCFM(R)$ and so $\sigma \in \mathrm{Inn}(RCFM(R))$. This completes our claim. Therefore, ϕ induces an isomorphism between $\mathrm{Out}(RCFM(R))$ and

$$\frac{\mathrm{Im}(\phi) \cdot \mathrm{Inn}(RFM(R))}{\mathrm{Inn}(RFM(R))}.$$

By Proposition 6, the above quotient module is isomorphic to $\mathrm{Out}(RFM(R))$. $\square$

While the previous results show that the rings $RFM(R)$ and $RCFM(S)$ share many ring-theoretical properties, they are quite different categorically. We conclude this paper with our proof of Theorem D.

Theorem 8. *For any two rings with identity, R and S the rings $RFM(R)$ and $RCFM(S)$ cannot be Morita equivalent. In particular, they are not isomorphic.*

Proof. Let $E = RFM(R)$, $B = RCFM(S)$, $I = FC(R)$, and $J = FM(S)$.

Assume that E and B are Morita equivalent rings. Then, by [6] we have that there exists a natural number $n \in \mathbf{N}$ such that E and $\mathbf{M}_n(B)$ are isomorphic rings. But B and $\mathbf{M}_n(B)$ are isomorphic. Indeed, the map $\alpha : B \to \mathbf{M}_n(B)$ given by $\alpha(X)(i,j)(a,b) = X(n(a-1)+i, n(b-1)+j)$ ($X \in B$, $1 \le i,j \le n$, $a,b \in \mathbf{N}$) is a ring isomorphism.

Let $\delta : E \to B$ be a ring isomorphism, let $\{e_{ij}\}_{ij\in\mathbf{N}}$ and $\{f_{ij}\}_{ij\in\mathbf{N}}$ be the basic matrices of E and B, respectively, and let $e'_{ij} = \delta(e_{ij})$ and $f'_{ij} = \delta^{-1}(f_{ij})$.

We show that $I = \delta^{-1}(J)$. Since $I = \sum_{\mathbf{N}} Ee_i$ is a two-sided ideal of E, we have that $\delta(I) = \sum_{\mathbf{N}} Be'_i$ is a two-sided ideal of B and the family $\{e'_{ij}\}$ verifies the conditions of Lemma 1. We conclude that $\delta(I) \subseteq J$ and so $I \subseteq \delta^{-1}(J)$. Consequently, $I = \oplus If'_i$.

Now we use analogous ideas to those found in [3]. Let $\alpha : If'_1 \to \sum_{\mathbf{N}} If'_i = I$ be any E-homomorphism. Then there exists $\overline{\alpha} : I \to I$ such that $\alpha = \overline{\alpha} \circ f'_1$. By [5], $\bar{\alpha}$ is the right multiplication by some $a \in E$. It is clear that $a \in f'_1 J$ and hence $\delta(a) \in f_1 B$. Therefore, $\delta(a) = \delta(a)\sum_{\text{finite}} f_j$ and hence $a = a\sum_{\text{finite}} f'_j$. Thus $\alpha(If'_1) \subseteq \bigoplus \sum_{\text{finite}} If'_i$. Since ${}_EIf'_1 \simeq {}_EIf'_i$, for every $i \in \mathbf{N}$, we use [3] to conclude that ${}_EIf'_1$ must be finitely generated. Let $x_1f'_1, \ldots, x_nf'_1$ be a family of generators of ${}_EIf'_1$ with $x_i \in I$. Then $If'_1 = \sum_{i=1}^n Ex_if'_1$, and hence there is a finite subset F of $\mathbf{N}$, such that $If'_1 \subseteq Ee_F$. This implies that $e_if'_1 = e_if'_1e_F$ for every $i \in \mathbf{N}$, and hence $f'_1 = f'_1e_F \in I$. Thus $f'_i = f'_{i1}f'_1f'_{1i} \in I$ for every $i \in \mathbf{N}$ and we conclude that $\delta^{-1}(J) \subseteq I$.

To finish the proof, let $x \in RFM(S) - B$, and let ρ_x denote right multiplication by x. We have the homomorphism $\delta(\rho_x)\delta^{-1} :_B I \to_B I$, and there exists $y \in RFM(R)$ such that $\delta(\rho_x)\delta^{-1} = \rho_y$. For every $a \in I$, $a(y)\delta = ((a)\delta^{-1}y)\delta = ax$. Therefore, $\delta(y) = x$ contradicting the fact that $x \notin B$. This finishes the proof. $\square$

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