ISOMORPHISMS OF ROW AND COLUMN
FINITE MATRIX RINGS

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Abstract. This paper investigates the ring-theoretic similarities and the categorical dissimilarities between the ring $RFM(R)$ of row finite matrices and the ring $RCFM(R)$ of row and column finite matrices. For example, we prove that two rings $R$ and $S$ are Morita equivalent if and only if the rings $RCFM(R)$ and $RCFM(S)$ are isomorphic. This resembles the result of V. P. Camillo (1984) for $RFM(R)$. We also show that the Picard groups of $RFM(R)$ and $RCFM(R)$ are isomorphic, even though the rings $RFM(R)$ and $RCFM(R)$ are never Morita equivalent.

1. Introduction

Let $R$ be a ring with identity, let $RFM(R)$ be the ring of row-finite matrices over $R$, let $RCFM(R)$ be the ring of row and column-finite matrices over $R$, let $FC(R)$ be the ring of matrices with a finite number of nonzero columns, and let $FM(R)$ be the ring of all matrices with only a finite number of nonzero entries. All matrices are considered countably indexed. The theme of this paper is that while the rings $RFM(R)$ and $RCFM(R)$ are categorically quite different, they share many ring-theoretic properties. For example, Camillo ([3]) has shown that rings $R$ and $S$ are Morita equivalent if and only if $RFM(R)$ and $RFM(S)$ are isomorphic.

We prove

**Theorem A.** Rings $R$ and $S$, with identity, are Morita equivalent if and only if $RCFM(R)$ and $RCFM(S)$ are isomorphic.

Similar to Camillo’s proof, the key argument in the proof of Theorem A involves understanding the isomorphisms between $RCFM(R)$ and $RCFM(S)$. Consequently, we also prove

**Proposition B.** Every (ring) isomorphism between $RCFM(R)$ and $RCFM(S)$ restricts to an isomorphism between $FM(R)$ and $FM(S)$.

Proposition B is fundamental in the study of the Picard groups of $RCFM(R)$ and $R$ and, again, we find a ring-theoretic similarity between $RCFM(R)$ and $RFM(R)$. For example, in [2], the authors show that the Picard group of $R$ is isomorphic to the Picard group of $RFM(R)$. We prove

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Theorem C. The Picard group of \( RFM(R) \) is isomorphic to the Picard group of \( RCFM(R) \).

To prove this theorem, we first show that an automorphism of \( RFM(R) \) is the product of an inner automorphism and an automorphism that restricts to an automorphism of \( FM(R) \).

Finally, while the rings \( RFM(R) \) and \( RCFM(R) \) share ring-theoretic properties as seen in the above results, our last theorem shows that they are very different in a categorical sense. We prove

Theorem D. There do not exist rings \( R \) and \( S \) (with identity) such that \( RFM(S) \) is Morita equivalent to \( RCFM(R) \).

So, for example, although the Picard groups of \( RFM(R) \) and \( RCFM(R) \) are isomorphic, the rings themselves are not Morita equivalent.

The main tool for all these applications appears in section 3, in which we prove that \( FM(R) \) is the largest two-sided ideal in \( RCFM(R) \) satisfying some technical property. See Lemma 1. Section 4 is devoted towards proving our above-mentioned results.

2. Notation and preliminaries

Let \( R \) be a ring with identity. We will write the action of homomorphisms of left modules on the right.

For every \( i, j \in \mathbb{N} \), \( e_{ij} \in FM(R) \) is the basic matrix having 1 in the \( ij \)-place and zero in each other. We denote \( e_i = e_{ii} \). For any finite subset \( X \subseteq \mathbb{N} \) we denote \( e_X = \sum_{x \in X} e_x \). For any matrix \( \alpha \) and \( i, j \in \mathbb{N} \), \( \alpha(i, j) \) denotes the \((i, j)\)-entry of \( \alpha \).

The following facts will be used without explicit mention. \( FC(R) \) is a two-sided ideal of \( RFM(R) \) and it is generated by \( \{ e_i \mid i \in \mathbb{N} \} \) as a left ideal. \( FM(R) \) is a two-sided ideal of \( RCFM(R) \) and it is generated by \( \{ e_i \mid i \in \mathbb{N} \} \) both as left ideal and as right ideal. Actually, \( RCFM(R) \) is the idealizer of \( FM(R) \) in \( RFM(R) \). \( FM(R) \) is the right ideal of \( RFM(R) \) generated by \( \{ e_i \mid i \in \mathbb{N} \} \) (see[5]). Moreover,

\[
FM(R) = \bigcup_{X \subseteq \mathbb{N} \atop X \text{ finite}} e_X RFM(R)
\]

\[
= \bigcup_{X \subseteq \mathbb{N} \atop X \text{ finite}} RCFM(R) e_X = \bigcup_{X \subseteq \mathbb{N} \atop X \text{ finite}} e_X RCFM(R)
\]

and

\[
FC(R) = \bigcup_{X \subseteq \mathbb{N} \atop X \text{ finite}} RFM(R) e_X.
\]

We observe that if \( f \in RFM(R) \) and \( fe_i = 0 \) for all \( i \in \mathbb{N} \), then \( f = 0 \) and the symmetric property also holds.

Finally, \( RFM(R) \) is isomorphic to \( \text{End}_R(R^\mathbb{N}) \) and to \( \text{End}_{FM(R)}(FM(R)) \) (by right multiplications).

3. The Fundamental Lemma

To prove the results posed in the introduction, we first show that \( FM(R) \) is the largest two-sided ideal of \( RCFM(R) \) in a certain sense.
Lemma 1. Let $R$ be a ring with identity and suppose that there exists a family \( \{f_{ij}\}_{ij \in \mathbb{N}} \) of nonzero elements of $RCFM(R)$ such that:

1. $f_{ij}f_{kl} = \delta_{jk} f_{il}$, for every $i, j, k, l \in \mathbb{N}$.
2. $J = \sum_{i} RCFM(R)f_i$ is a two-sided ideal in $RCFM(R)$ (where $f_i = f_{ii}$).

Then $J \subseteq FM(R)$.

Proof. Set $I = FM(R)$. If $f_{ij} \in I$ for some $i, j$ then $f_{kl} = f_{kl}f_{ij}f_{jl} \in I$, for every $k, l \in \mathbb{N}$ (because $I$ is a two-sided ideal of $A$) and hence $J \subseteq I$. Therefore one may assume that $f_{ij} \notin I$ for all $i, j \in \mathbb{N}$.

For each $n \in \mathbb{N}$ set $X_n = \{ i \in \mathbb{N} \mid f_1 f_{1n} \neq 0 \}$. We claim that $X_n \neq \emptyset$. To see this, first note that $f_1 f_{1n} = f_{1n} \neq 0$ and so there is an $i \in \mathbb{N}$ such that $0 \neq e_i f_1 \notin I$. Thus $e_i f_1 = e_i e X$, for some finite subset $X \subseteq \mathbb{N}$ and hence $0 \neq e_i f_{1n} = e_i f_1 e X f_{1n} = e_i e X f_{1n}$, for every $k \in \mathbb{N}$ and hence $f_{1n} = f_1 e X f_{1n} \in I$ which contradicts our assumption.

We recursively construct two sequences, $(i_j)_{j \in \mathbb{N}}$ and $(k_j)_{j \in \mathbb{N}}$, of natural numbers such that the first sequence consists of elements from $X_n$, while second one is strictly increasing. This will ultimately generate a contradiction to our assumption that $f_{ij} \notin I$ for all $i, j \in \mathbb{N}$.

Let $i_1$ be the first element of $X_1$, $Z_1 = \{ r \in \mathbb{N} \mid e_r f_1 e_{i_1} \neq 0 \}$ or $e_i f_1 f_r \neq 0 \}$ and $k_1 = \max Z_1$. For every $n > 1$, let

\[
Y_n = \{ r \in \mathbb{N} \mid e_m f_1 e_r \neq 0 \text{ or } e_r f_{1n} e_m \neq 0 \text{ for some } m \leq k_{n-1} \}.
\]

It is clear that $Y_n$ is a finite set. Now we define $i_n$ to be the first element of $X_n - Y_n$ and $Z_n = \{ r \in \mathbb{N} \mid e_r f_{1n} e_i \neq 0 \}$. Note that $Z_n$ is a finite set and is not empty because $i_n \in X_n$. Further since $i_n \notin Y_n$, $r > k_{n-1}$ for every $r \in Z_n$. In particular $k_n = \max Z_n > k_{n-1}$.

Let $\alpha$ be the $\mathbb{N} \times \mathbb{N}$ matrix over $R$ given by

\[
\alpha(i, j) = \begin{cases} (f_1 e_{i_n} f_{1n})(i, j) & \text{if } k_{n-1} < i, j \leq k_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}
\]

Obviously $\alpha \in RCFM(R)$. Let $K_n = \{ i \in \mathbb{N} \mid k_{n-1} < i \leq k_n \}$. Note that $e_{K_n} f_1 e_{i_n} f_{1n} = f_1 e_{i_n} f_{1n} e_{K_n} = f_1 e_{i_n} f_{1n}$, because if $e_1 f_{1n} e_{i_n} f_{1n} \neq 0$ then $x \in Z_n \subseteq K_n$ and similarly $f_1 e_{i_n} f_{1n} e_{x} \neq 0$ implies that $x \in Z_n \subseteq K_n$. Hence we have that $e_{K_n} \alpha e_{K_n} = f_1 e_{i_n} f_{1n}$ for every $n \in \mathbb{N}$.

Now we show two properties of $\alpha$. First, we assert that $\alpha = f_1 \alpha$. Indeed, if $j \in \mathbb{N}$, then $j \in K_n$ for some $n$. Therefore $a e_j = \alpha e_{K_n} e_j = f_1 e_{i_n} f_{1n} e_j = f_1 \cdot f_1 e_{i_n} f_{1n} e_j = f_1 \alpha e_{K_n} e_j = f_1 \alpha e_j$. We conclude that $\alpha = f_1 \alpha$.

Second, we assert that $\alpha f_n \neq 0$ for all $n \in \mathbb{N}$. To see this, note that $e_{K_n} \alpha f_n = f_1 e_{i_n} f_{1n} f_n = f_1 e_{i_n} f_{1n} \neq 0$. Thus $\alpha f_n \neq 0$ for all $n \in \mathbb{N}$.

But as $\alpha \in RCFM(R)$, $f_1 \alpha \in f_1 RCFM(R) \subseteq J$ because $J$ is two-sided. Therefore $f_1 \alpha = \sum_{i \in J} f_1 \alpha f_i$, where $F$ is a finite subset of $\mathbb{N}$, so that $f_1 \alpha f_n = 0$ for almost all $n \in \mathbb{N}$, which contradicts the second property of $\alpha$.

It is obvious that condition 1 of Proposition 1 cannot be deleted. The following example shows that condition 2 of Proposition 1 is also not superfluous.
Example 2. There exist a ring $R$, a family $\{f_{ij}\}_{ij \in \mathbb{N}} \subseteq RCFM(R)$ such that $f_{ij}f_{kl} = \delta_{ik}f_{jl}$ for every $i, j, k, l \in \mathbb{N}$, and a left ideal, $J = \sum RCFM(R)f_i$, that properly contains $FM(R)$.

Proof. Let $\beta : \mathbb{N} \to \mathbb{N}^2$ be a bijection. This bijection induces an isomorphism of $R$-bimodules $\beta : R(R^{(N)}) \to (R^{(N)})[R]$ which induces two ring isomorphisms

$$\beta_1 : \text{End}(R^{(N)}) \to \text{End}(R(R^{(N)})), \quad \beta_r : \text{End}(R^{(N)}) \to \text{End}((R^{(N)})^{(N)}).$$

We recall that $\text{End}(R^{(N)})$ is isomorphic to the ring $RCM(RFM(R))$ of row-convergent matrices over $RFM(R)$ and $\text{End}((R^{(N)})^{(N)})$ is isomorphic to the ring $CCM(CFM(R))$ of column convergent matrices over $CFM(R)$ ([4, Theorem 106.1]). Now having in mind the nature of these isomorphisms, one can check they induce an isomorphism $\sigma$ between $RCFM(R)$ and $RCFM(RFM(R)) = RCM(RFM(R)) \cap CCM(CFM(R))$. Further, one can check that $\sigma(FM(R)) = FM(RFM(R))$.

Let $K = RCCM(RCFM(R))$ and $f'_{ij} \in K$ such that $f'_{ij}$ has 1 in the $ij$-place and zero elsewhere. Let $J = \sum Kf'_{ij}$. Clearly, $FM(RFM(R))$ is contained properly in $J$ and taking $f_{ij} = \sigma^{-1}(f'_{ij})$ we have the desired family. We show explicitly that $J$ is not a two-sided ideal. Take $x \in K$ such that $x$ has $e_{1j} \in FM(R)$ in the $1j$-entry and zero elsewhere. Since $J \subseteq RCFM(RFM(R))$, $x \notin J$. But, $f_{11}e_{1j} = e_{1j}$ for every $j \in \mathbb{N}$, and hence $x = f_{11}x$.

4. Comparing $RFM(R)$ with $RCFM(R)$

In this section, we prove the results mentioned in the introduction. We begin with Theorem A and Proposition B. The key to the main result of [3] is, in essence, that an isomorphism $\phi : RFM(R) \to RFM(S)$ satisfies $\phi(FC(R)) = FC(S)$, where $FC$ is the ring of matrices with only finitely many non-zero columns. In our setting, with $RCFM(R)$ taking the place of $RFM(R)$, this result translates into Proposition B from the Introduction.

Proposition 3. Let $R$ and $S$ be any two rings with identity.

(a) Every ring isomorphism $\delta : RCFM(R) \to RCFM(S)$ satisfies $\delta(FM(R)) = FM(S)$.

(b) Every ring isomorphism $\delta : RCFM(R) \to RCFM(S)$ extends, in a unique way, to an isomorphism $\delta' : RFM(R) \to RFM(S)$.

(c) Every ring isomorphism $\sigma : RFM(R) \to RFM(S)$ such that $\sigma(FM(R)) = FM(S)$ satisfies $\sigma(RCFM(R)) = RCFM(S)$.

(d) There is a group monomorphism $\phi : \text{Aut}(RCFM(R)) \to \text{Aut}(RFM(R))$ via $\phi(\delta) = \delta'$ using (b) above. Moreover, the image of $\phi$ is the subgroup of automorphisms of $RFM(R)$ that restrict to automorphisms of $FM(R)$.

Proof. (a) follows immediately from Lemma 1. Using the fact that $RFM(R)$ is isomorphic to $\text{End}(FM(R)FM(R))$, (b) is straightforward. We get (c) from the fact that $RCFM(R)$ is the idealizer of $FM(R)$ in $RFM(R)$. Finally, (d) follows from (a), (b), and (c).
Theorem 4. Let $R$ and $S$ be rings with identity. $R$ and $S$ are Morita equivalent rings if and only if $RCFM(R)$ and $RCFM(S)$ are isomorphic rings.

Proof. Assume first that $R$ and $S$ are Morita equivalent rings. Let $R_P$ be a progenitor such that $\text{End}(R_P) \cong S$ as rings. By [1, Lemma 1.2] we have that there exists a ring isomorphism $\alpha^* : RCFM(R) \to RCFM(\text{End}(R_P))$ such that $\alpha^*(FM(R)) = FM(\text{End}(R_P))$. Let $\beta : RCFM(\text{End}(R_P)) \to RCFM(S)$ be induced (coordinate-wise) by the isomorphism $\text{End}(R_P) \cong S$, and let $\delta = \beta \circ \alpha^*$. Then it is clear that $\delta(FM(R)) = FM(S)$. Since $RCFM(R)$ (resp. $RCFM(S)$) is the idealizer of $FM(R)$ (resp. $FM(S)$), $\delta(RCFM(R)) = RCFM(S)$.

The converse follows from Proposition 3 together with [1, Theorem 2.5 (3 implies 1)].

It is interesting to note that there are some rings between $FM(R)$ and $RCFM(R)$ which have automorphisms that do not restrict to automorphisms of $FM(R)$, as the next example shows.

Example 5. There exist rings with identity, $R$ and $S$, such that $FM(R) \subset S \subset RCFM(R)$ and an automorphism of $S$, say $\delta$, such that $\delta(FM(R)) \neq FM(R)$.

Proof. Let $K$ be any ring with identity and let $B = RCFM(RCFM(K))$. As we saw in Example 2 there exists an injective ring homomorphism $\beta : B \to RCFM(K)$. Let $T$ be the image of $\beta$, let $R = K \times RCFM(K)$, and let $S = T \times B$. After identifying $RCFM(R)$ with $RCFM(K) \times B$, it is clear that $FM(R) \subset S \subset RCFM(R)$ and that $\delta : S \to S$ via $\delta(x, y) = (\beta(y), \beta^{-1}(x))$ is an automorphism of $S$. But a straightforward calculation shows that $\delta(FM(R)) \neq FM(R)$.

The ring $RCFM(R)$ is another ring for which there are automorphisms of $RCFM(R)$ that do not restrict to automorphisms of $FM(R)$; see [1]. Nonetheless, we show that these pathological automorphisms are “controlled” by the inner automorphisms of $RCFM(R)$. In particular, we show that every automorphism of $RCFM(R)$ is a product of an inner automorphism and an automorphism that restricts to an automorphism of $FM(R)$.

Proposition 6. For every $\sigma \in \text{Aut}(RCFM(R))$, there exists $\tau \in \text{Inn}(RCFM(R))$ such that $\sigma \tau(FM(R)) = FM(R)$.

Proof. Let $\sigma \in \text{Aut}(RCFM(R))$. Then $P = R^{(N)}\sigma(e_1)$ is a progenator as left $R$-module such that $\text{End}(R_P)$ is isomorphic to $R$ [3]. Specifically, the isomorphism $\tau : R \to \text{End}(R_P)$ is given by $(p)\tau(r) = p\sigma(e_1)D(r)$ where $D(r)$ denotes the scalar matrix defined by $r$. We consider $P$ as an $R$-bimodule using this isomorphism; explicitly, $r \cdot p \cdot s = r \tau(s) (r, s \in R, p \in P)$. Define $\tau^* : RCFM(R) \to RCFM(\text{End}(P))$ via a coordinate-wise application of $\tau$.

The map $f : P(N) \to R(N)$ given by $((p_i)_{i \in N})f = \sum_{i \in N} p_i \sigma(e_{1i})$ is an isomorphism whose inverse is given by $((r_i)_{i \in N})f^{-1} = (r_i \sigma(e_{1i}))_{i \in N}$.

We identify $RCFM(R)$ with $\text{End}(R^{(N)})$ and $RCFM(\text{End}(P))$ with $\text{End}(P^{(N)})$ canonically. For every $x \in RCFM(R)$, the following diagram is commutative:

$$R^{(N)} \xrightarrow{\sigma(x)} R^{(N)}$$

$$\uparrow f \quad \uparrow f$$

$$P^{(N)} \xrightarrow{\tau(x)} P^{(N)}$$
To see this, observe
\[
(p)\tau^*(x)f = ((\sum_{i\in\mathbb{N}} p_i \tau(x_{ij}))_{j\in\mathbb{N}})f
= ((\sum_{i\in\mathbb{N}} p_i \sigma(e_1 D(x_{ij})))_{j\in\mathbb{N}})f
= \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} p_i \sigma(e_1 D(x_{ij})e_{ij})
= \sum_{i\in\mathbb{N}} p_i \sigma(e_1 \sum_{j\in\mathbb{N}} D(x_{ij})e_{ij})
= \sum_{i\in\mathbb{N}} p_i \sigma(e_{1i} x)
= (p)f \sigma(x).
\]

Now let \(\alpha : P^{(N)} \to R^{(N)}\) be the isomorphism mentioned in [1], which induces an isomorphism \(\alpha^* : RCFM(\text{End}(P)) = \text{End}(P^{(N)}) \to \text{End}(R^{(N)}) = RFM(R)\) such that \(\alpha^*(FM(\text{End}(RP))) = FM(R)\). More concretely, \(\alpha^*\) is characterized by the property that, for every \(y \in \text{End}(P^{(N)})\), the following diagram is commutative:
\[
P^{(N)} \xrightarrow{y} P^{(N)}
\]
\[
\alpha \downarrow \downarrow \alpha
\]
\[
R^{(N)} \xrightarrow{\alpha^*(y)} R^{(N)}
\]

Therefore, the following diagram is commutative, for every \(x \in RFM(R)\):
\[
R^{(N)} \xrightarrow{\alpha(x)} R^{(N)}
\]
\[
f \uparrow \uparrow f
\]
\[
P^{(N)} \xrightarrow{\tau^*(x)} P^{(N)}
\]
\[
\alpha \downarrow \downarrow \alpha
\]
\[
R^{(N)} \xrightarrow{\alpha^* \tau^*(x)} R^{(N)}
\]

It follows that \(\sigma^{-1} \alpha^* \tau^*\) is the inner automorphism of \(RFM(R)\) induced by \(\alpha^{-1} f\) and \(\alpha^* \tau^*(FM(R)) = FM(R)\).

Recall that the Picard group of a ring \(T\) is the multiplicative group consisting of the bimodule isomorphism classes of invertible \(T\)-bimodules. We now prove Theorem C from the Introduction.

**Theorem 7.** For every ring \(R\),
\[
\text{Pic}(R) \simeq \text{Pic}(RFM(R)) \simeq \text{Pic}(FM(R)) \simeq \text{Pic}(RCFM(R)).
\]

**Proof.** It has been shown in [2] that \(\text{Pic}(R) \simeq \text{Pic}(RFM(R)) \simeq \text{Pic}(FM(R))\). On the other hand, both \(RFM(R)\) and \(RCFM(R)\) have the SBN property, so that \(\text{Pic}(RCFM(R)) \simeq \text{Out}(RCFM(R))\) and \(\text{Pic}(RFM(R)) \simeq \text{Out}(RFM(R))\); see [2]. Thus, it suffices to show that \(\text{Out}(RCFM(R)) \simeq \text{Out}(RFM(R))\).

From Proposition 3, there is a group monomorphism \(\phi : \text{Aut}(RCFM(R)) \to \text{Aut}(RFM(R))\) such that the image of \(\phi\) is the subgroup of automorphisms of \(RFM(R)\) that restrict to automorphisms of \(FM(R)\). In particular, \(\phi(\delta) = \delta'\) using (b) of Proposition 3. We claim that
\[
\phi(\text{Inn}(RCFM(R))) = \text{Inn}(RFM(R)) \cap \text{Im}(\phi).
\]
It is clear that \(\phi(\text{Inn}(RCFM(R))) \subseteq \text{Inn}(RFM(R))\). For the opposite inclusion, note that if \(\sigma \in \text{Aut}(RCFM(R))\) such that \(\phi(\sigma) \in \text{Inn}(RFM(R))\), then there exists a unit \(u \in RFM(R)\) for which \(uFM(R) = FM(R)u\). In particular, for each \(i, u \cdot e_{ii} \in FM(R)\) so that \(e_{jj} \cdot u \cdot e_{ii} = 0\) for almost all values of \(j\). Hence,
u ∈ RCFM(R) and so σ ∈ Im(RCFM(R)). This completes our claim. Therefore, φ induces an isomorphism between Out(RCFM(R)) and
\[ \frac{\text{Im}(\phi) \cdot \text{Inn}(R_{\rm FM}(R))}{\text{Im}(R_{\rm FM}(R))}. \]

By Proposition 6, the above quotient module is isomorphic to Out(RFM(R)). □

While the previous results show that the rings RFM(R) and RCFM(S) share many ring-theoretical properties, they are quite different categorically. We conclude this paper with our proof of Theorem D.

**Theorem 8.** For any two rings with identity, R and S the rings RFM(R) and RCFM(S) cannot be Morita equivalent. In particular, they are not isomorphic.

**Proof.** Let E = RFM(R), B = RCFM(S), I = FC(R), and J = FM(S).

Assume that E and B are Morita equivalent rings. Then, by [6] we have that there exists a natural number n ∈ N such that E and M_n(B) are isomorphic rings. But B and M_n(B) are isomorphic. Indeed, the map α : B → M_n(B) given by α(X)(i, j)(a, b) = X(n(a - 1) + i, n(b - 1) + j) (X ∈ B, 1 ≤ i, j ≤ n, a, b ∈ N) is a ring isomorphism.

Let δ : E → B be a ring isomorphism, let \{e_{ij}\}_{ij∈N} and \{f_{ij}\}_{ij∈N} be the basic matrices of E and B, respectively, and let e'_{ij} = δ(e_{ij}) and f'_{ij} = δ^{-1}(f_{ij}).

We show that I = δ^{-1}(J). Since I = \bigoplus_N Ee_i is a two-sided ideal of E, we have that δ(I) = \bigoplus_N Be'_i is a two-sided ideal of B and the family \{e'_i\} verifies the conditions of Lemma 1. We conclude that δ(I) ⊆ J and so I ⊆ δ^{-1}(J). Consequently, I = \bigoplus_I f'_1.

Now we use analogous ideas to those found in [3]. Let α : If'_1 → \bigoplus_N If'_1 = I be any E-homomorphism. Then there exists π : I → I such that α = π ◦ f'_1. By [5], α is the right multiplication by some a ∈ E. It is clear that a ∈ f'_1J and hence \delta(a) = f_1B. Therefore, \delta(a) = \delta(a) \bigoplus_{finite} f_1 and hence a = a \bigoplus_{finite} f_1. Thus α(If'_1) ⊆ \bigoplus_{finite} If'_1. Since \bigoplus_E If'_1 ⊆ EIf'_1, for every i ∈ N, we use [3] to conclude that EIf'_1 must be finitely generated. Let x_1f'_1, ..., x_nf'_1 be a family of generators of EIf'_1 with x_i ∈ I. Then If'_1 = \sum_{i=1}^n E x_i f'_1, and hence there is a finite subset F of N, such that If'_1 ⊆ E F. This implies that e_i f'_1 = e_i f'_1 e_F for every i ∈ N, and hence f'_1 = f'_1 e_F ⊆ I. Thus f'_1 = f'_1 f'_1 f'_1 ∈ I for every i ∈ N and we conclude that δ^{-1}(J) ⊆ I.

To finish the proof, let x ∈ RFM(S) - B, and let ρ_x denote right multiplication by x. We have the homomorphism δ(ρ_x)δ^{-1} : B → B, and there exists y ∈ RCFM(R) such that δ(ρ_y)δ^{-1} = ρ_y. For every a ∈ I, a(y)δ = ((a)δ^{-1})δ = ax. Therefore, δ(y) = x contradicting the fact that x ∉ B. This finishes the proof. □

**References**


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