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SETS OF SAMPLING AND INTERPOLATION IN BERGMAN SPACES

ALEXANDER P. SCHUSTER

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Abstract. Properties of the unions of sampling and interpolation sets for Bergman spaces are discussed in conjunction with the examples given by Seip (1993). Their relationship to the classical interpolation sequences is explored. In addition, the role played by canonical divisors in the study of these sets is examined and an example of a sampling set is constructed in the disk.

1. Introduction

For $0 < p < \infty$, the Bergman space A^p is the set of functions analytic in the unit disk $\mathbb{D}$ with

$$\|f\|_p^p = \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA denotes Lebesgue area measure. A sequence Γ of distinct points in $\mathbb{D}$ is said to be a *set of sampling for* A^p if there exist positive constants K_1 and K_2 such that

$$K_1\|f\|_p^p \le \sum_{z\in\Gamma}(1-|z|^2)^2|f(z)|^p \le K_2\|f\|_p^p \tag{1}$$

for all $f \in A^p$. The weight function in the sum is chosen to make these sets invariant under analytic automorphisms of $\mathbb{D}$.

Given a sequence Γ, let T_p be the linear operator which maps $f \in A^p$ to the sequence $\{f(z)(1-|z|^2)^{\frac{2}{p}}\}_{z\in\Gamma}$. Γ is said to be a *set of interpolation for* A^p if $T_p(A^p) \supseteq \ell^p$.

Seip [10] completely characterizes sets of sampling and interpolation for A^2 in terms of a density condition using results which may be extended to $0 < p < \infty$. In [9], Seip constructs a family of sets which may be sampling or interpolating, depending on the parameters defining the family.

We say that Γ is an A^p *zero set* if there is a non-trivial function $f \in A^p$ which vanishes precisely on Γ. By a theorem of Horowitz [6], it actually suffices for f to be zero at least on Γ. It is clear from (1) that a set of sampling cannot be an A^p zero set and it is also not difficult to show that a set of interpolation for A^p must

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be an A^p zero set. Horowitz [6] proved that A^p zero sets vary with p and that the union of A^p zero sets is not necessarily an A^p zero set. One goal of this paper is to strengthen those results using the ideas of [9] and [10]. Also, an interesting family of sets in $\mathbb{D}$ is presented. In addition, there will be some discussion of basic properties of sampling and interpolation sets and how they relate to other concepts encountered in the study of Bergman spaces, such as canonical divisors.

2. Seip's description of sampling and interpolation sets

In the definition (1) of sampling sets, the upper inequality requires the sequence to be "thin", while the lower inequality requires it to be fairly dense and evenly distributed over the disk. Thus one might well ask whether there are any sampling sets at all. In fact, for the Hardy spaces H^p, the answer is negative. To see this, recall first that H^p consists of the functions analytic in $\mathbb{D}$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

If Γ were a set of sampling for H^p, an inequality of the form

$$K_1 \|f\|_{H^p}^p \leq \sum_{z\in\Gamma} (1-|z|^2)|f(z)|^p \leq K_2 \|f\|_{H^p}^p \tag{2}$$

would hold for all $f \in H^p$. But the upper inequality in (2) applied to $f \equiv 1$ forces Γ to be a Blaschke sequence, a fact which contradicts the lower inequality.

However, the Bergman spaces A^p do have sampling sets and Seip has described them all with a density condition that we will define shortly. Sets of interpolation for A^p are characterized in a similar fashion. The interpolation sets for H^p, defined slightly differently from their Bergman counterparts, have been described completely using a separation condition. See [12].

In order to state Seip's theorems, we need a few definitions. The *pseudo-hyperbolic metric* ρ is defined on $\mathbb{D}$ by $\rho(z,\zeta) = |\phi_\zeta(z)|$, where

$$\phi_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}, \qquad z, \zeta \in \mathbb{D}.$$

A sequence $\Gamma = \{z_k\}$ is *uniformly discrete* if there is a $\delta > 0$ such that $\rho(z_i, z_j) \geq \delta$ for all $i \neq j$. For Γ uniformly discrete and $1/2 < r < 1$, let

$$D(\Gamma, r) = (\log(\frac{1}{1-r}))^{-1} \sum_{1/2<|z_k|<r} \log(\frac{1}{|z_k|}). \tag{3}$$

The lower and upper uniform densities are defined, respectively, to be

$$D^-(\Gamma) = \liminf_{r\to 1} \inf_{\zeta\in\mathbb{D}} D(\phi_\zeta(\Gamma), r)$$

and

$$D^+(\Gamma) = \limsup_{r\to 1} \sup_{\zeta\in\mathbb{D}} D(\phi_\zeta(\Gamma), r).$$

The following results were proved in [10] for $p = 2$ and stated for $0 < p < \infty$ in [5]. For a proof of the case $0 < p < \infty$ see [8].

Theorem S1 (Seip). *A sequence Γ of distinct points in the disk is a set of sampling for A^p if and only if it is a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > \frac{1}{p}$.*

Theorem S2 (Seip). *A sequence Γ of distinct points in the disk is a set of interpolation for A^p if and only if Γ is uniformly discrete and $D^+(\Gamma) < \frac{1}{p}$.*

3. Unions of sets of sampling and interpolation

In [9], Seip constructs a family of examples of sets of sampling and interpolation. Let $a > 1, b > 0$ and let

$$\Lambda(a,b) = \{a^m(bn+i)\}_{m,n\in\mathbb{Z}},$$

where $\mathbb{Z}$ is the set of integers. $\Lambda(a,b)$ is a sequence of points in $\mathbb{H}^+$, the upper half-plane. An analytic isomorphism from $\mathbb{D}$ to $\mathbb{H}^+$ is given by

$$\psi(z) = i(\frac{1+z}{1-z})$$

and we define

$$\Gamma(a,b) = \psi^{-1}(\Lambda(a,b))$$

so $\Gamma(a,b)$ is a sequence of distinct points in $\mathbb{D}$. In fact, $\Gamma(a,b)$ is uniformly discrete and Seip shows that

$$D^-(\Gamma(a,b)) = D^+(\Gamma(a,b)) = \frac{2\pi}{b\log a}.$$

The following is a direct consequence of Theorems S1 and S2.

Corollary S3. *Let $0 < p < \infty$ and $\Gamma = \Gamma(a,b)$ as above. Then Γ is a set of interpolation for A^p if and only if $\dfrac{2\pi}{b\log a} < \dfrac{1}{p}$ and Γ is a set of sampling for A^p if and only if $\dfrac{2\pi}{b\log a} > \dfrac{1}{p}$.*

Horowitz, in [6], shows that if $p < q$, there is an A^p zero set which is not an A^q zero set. As a corollary to Seip's work, we are able to strengthen this result.

Theorem 1. *If $0 < p < q < \infty$, then there is an A^p interpolation set which is a set of sampling for A^q.*

Proof. Choose $a > 1$ and $b > 0$ such that

$$\frac{1}{q} < \frac{2\pi}{b\log a} < \frac{1}{p}.$$

This shows how strongly A^p zero sets depend on p.

In [4], Hedenmalm divides $\Gamma(a,b)$ into two disjoint subsequences A and B, where

$$A = \psi^{-1}\left(\{a^m(bn+i)\}_{m\in\mathbb{Z},n\equiv 0(\mathrm{mod}\,2)}\right) \qquad \text{and}$$
$$B = \psi^{-1}\left(\{a^m(bn+i)\}_{m\in\mathbb{Z},n\equiv 1(\mathrm{mod}\,2)}\right)$$

and he states that

$$D^-(A) = D^+(A) = D^-(B) = D^+(B) = \frac{\pi}{b\log a}.$$

The main results of this section employ a generalization of this idea. Our goal is to divide $\Gamma(a,b)$ into disjoint subsequences in a more general way, while keeping track of whether each subsequence is a set of sampling or interpolation for A^p, for various values of p. Before we can prove most of the theorems, we require a few technical lemmas.

Lemma 1. *Let A and B be disjoint sequences in $\mathbb{D}$. Then*

$$\begin{aligned} D^-(A) + D^-(B) \le D^-(A \cup B) &\le D^-(A) + D^+(B) \\ &\le D^+(A \cup B) \le D^+(A) + D^+(B). \end{aligned}$$

Proof. It is clear from (3) that

$$D(\phi_\zeta(A), r) + D(\phi_\zeta(B), r) = D(\phi_\zeta(A \cup B), r) \tag{4}$$

for all $\zeta \in \mathbb{D}$ and all $r < 1$. Therefore, $D^-(A) + D^-(B) \le D^-(A \cup B)$ and $D^+(A) + D^+(B) \ge D^+(A \cup B)$.

Consider now the second inequality and let $\epsilon > 0$. There is an r_0 such that $r \ge r_0$ and $\zeta \in \mathbb{D}$ imply

$$D(\phi_\zeta(A \cup B), r) > D^-(A \cup B) - \epsilon \qquad \text{and} \qquad D(\phi_\zeta(B), r) < D^+(B) + \epsilon.$$

By (4) we have that if $r > r_0$ and $\zeta \in \mathbb{D}$, then

$$D(\phi_\zeta(A), r) > D^-(A \cup B) - D^+(B) - 2\epsilon.$$

Since ϵ was arbitrary we see that $D^-(A) \ge D^-(A \cup B) - D^+(B)$, which is the desired result. The proof of the third inequality is similar. Notice that if $D^-(A) = D^+(A)$ and $D^-(B) = D^+(B)$, then $D^-(A \cup B) = D^-(A) + D^-(B) = D^+(A \cup B)$. Also, if $E \subseteq F$, $D^-(E) = D^+(E)$ and $D^-(F) = D^+(F)$, then $D^+(F \setminus E) = D^-(F \setminus E) = D^-(F) - D^-(E)$.

Let $\mathbb{N}$ be the set of natural numbers. For $u, v, k \in \mathbb{N} \cup \{0\}$ with $u \le v \le k-1$, define

$${}^v_u\Lambda_k(a, b) = \{a^m(bn + i)\}_{m \in \mathbb{Z}, n \equiv u (\operatorname{mod} k), n \equiv u+1 (\operatorname{mod} k), \ldots, n \equiv v (\operatorname{mod} k)}$$

and

$${}^v_u\Gamma_k(a, b) = \psi^{-1}({}^v_u\Lambda_k(a, b)).$$

We then obtain

Lemma 2.

$$D^-({}^v_u\Gamma_k(a, b)) = D^+({}^v_u\Gamma_k(a, b)) = \frac{2\pi}{b \log a}\left(\frac{v - u + 1}{k}\right).$$

Proof. By symmetry, we notice that

$$D^-({}^v_u\Gamma_k(a, b)) = D^-({}^{v-u}_0\Gamma_k(a, b)) \tag{5}$$

for $u \le v \le k-1$. Then, by Lemma 1 and (5),

$$\begin{aligned} D^-({}^v_u\Gamma_k(a, b)) &= D^-\left(\bigcup_{j=u}^{v} {}^j_j\Gamma_k(a, b)\right) \\ &= \sum_{j=u}^{v} D^-({}^j_j\Gamma_k(a, b)) = (v - u + 1) D^-({}^0_0\Gamma_k(a, b)) \\ &= (v - u + 1) D^-(\Gamma(a, bk)) = \frac{2\pi}{b \log a}\left(\frac{v - u + 1}{k}\right). \end{aligned}$$

The argument involving the upper uniform density is similar.

Lemma 3. *Let $0 < p_1, \dots, p_n < \infty$ and let $1/p = \sum_{i=1}^n 1/p_i$. If $r > p$, then there exist $a > 1, b > 0$ and $k, l_1, \dots, l_n \in \mathbb{N}$ with $\sum_{i=1}^n l_i = k$ such that*

$$\frac{2\pi}{b \log a} > \frac{1}{r} \qquad \text{and} \qquad \frac{2\pi}{b \log a} \frac{l_i}{k} < \frac{1}{p_i} \qquad i = 1, \dots, n.$$

If $0 < r < p$, then one can find a, b, k, l_i as above with all the inequalities reversed.

Proof. The proof is straightforward and will be omitted. For details see [8].

Horowitz [6] showed that the union of two A^p zero sets is an $A^{\frac{p}{2}}$ zero set but for $r > p/2$, it need not be an A^r zero set. We improve this result in the following way.

Theorem 2. *Let $0 < p_1, \dots, p_n < \infty$ and let F_i be an A^{p_i} zero set. Then $F = \cup_{i=1}^n F_i$ is an A^p zero set, where $1/p = \sum_{i=1}^n 1/p_i$. The result is sharp in the strong sense that if $r > p$, then the union may be a set of sampling for A^r.*

Proof. For $i = 1, \dots, n$, let f_i be an A^{p_i} function which vanishes on F_i without being identically zero. Then $f_1 \dots f_n$ is zero on F and an application of the generalized Hölder's inequality shows that $f_1 \dots f_n \in A^p$. This proves the first part of the theorem.

Now suppose $r > p$. Choose a, b, k, l_i as in the first statement of Lemma 3. For $i = 1, \dots, n$, let $m_i = \sum_{j=0}^i l_j$, where l_0 is defined to be zero. Let

$$F_i = {}_{m_{i-1}}^{m_i - 1}\Gamma_k(a, b).$$

Then the F_i's form a disjoint partition of $\Gamma(a, b)$. Now,

$$D^+(F_i) = \frac{2\pi}{b \log a} \frac{l_i}{k} < \frac{1}{p_i}, \qquad i = 1, \dots, n, \qquad \text{and}$$

$$D^-(\cup_{i=1}^n F_i) = D^-(\Gamma(a, b)) = \frac{2\pi}{b \log a} > \frac{1}{r}.$$

By Theorems S1 and S2, F_i is an A^{p_i} zero set for $i = 1, \dots, n$ and $\cup_{i=1}^n F_i = \Gamma(a, b)$ is a set of sampling for A^r.

A similar result holds for sets of interpolation.

Theorem 3. *Let $0 < p_1, \dots, p_n < \infty$. For $i = 1, \dots, n$, let F_i be a set of interpolation for A^{p_i}. Suppose the F_i's are disjoint and that $F = \cup_{i=1}^n F_i$ is uniformly discrete. Then F is a set of interpolation for A^p, where $1/p = \sum_{i=1}^n 1/p_i$. If $r > p$, then the union may be a set of sampling for A^r.*

Proof.

$$D^+(F) = D^+(\cup_{i=1}^n F_i) \le \sum_{i=1}^n D^+(F_i) < \sum_{i=1}^n 1/p_i = 1/p,$$

so Theorem S2 implies that F is a set of interpolation for A^p. The sharpness is proved by the same construction as in Theorem 2.

There is a result similar to Theorem 3 for sets of sampling.

Theorem 4. *Let $0 < p_1, \dots, p_n < \infty$. For $i = 1, \dots, n$, let F_i be a set of sampling for A^{p_i}. Suppose the F_i's are disjoint and that $F = \cup_{i=1}^n F_i$ is uniformly discrete. Then F is a set of sampling for A^p, where $1/p = \sum_{i=1}^n 1/p_i$. If $0 < r < p$, then the union may be a set of interpolation for A^r.*

Proof.

$$D^-(F) = D^-(\cup_{i=1}^n F_i) \geq \sum_{i=1}^n D^-(F_i) > \sum_{i=1}^n 1/p_i = 1/p,$$

so Theorem S1 implies that F is a set of sampling for A^p. To prove the sharpness, we again use the same construction as in the proof of Theorem 2, this time choosing our parameters as in the second statement in Lemma 3.

4. A family of sets in the disk

The example provided by Seip in [9] does consist of points in the disk, but it was constructed in $\mathbb{H}^+$ and is much more natural and easier to visualize there. In this section we present a family of sampling sets constructed directly in $\mathbb{D}$. Let $a > 0$. Following the strategy of Seip in [11], we let

$$A_n = \{z : r_{n-1} \leq |z| < r_n\}, \qquad n = 1, 2, 3, \ldots,$$

where r_n is defined so that $r_0 = 0$ and the hyperbolic area of A_n is $\dfrac{2^{n-2}\pi}{a}$ for all n. Divide each annulus A_n into 2^{n-1} hyperbolic rectangles, each of which has hyperbolic area $\dfrac{\pi}{2a}$. Let Γ_a consist of the centres of mass of the rectangles with respect to the measure $d\nu(z) = \dfrac{a}{8\pi}\dfrac{1}{(1-|z|^2)^2}dA(z)$.

Lemma 4. *There exist a constant C and a function g, analytic in $\mathbb{D}$ with zero set Γ_a such that $|g(z)| \leq C(1-|z|^2)^{-a}$ and $|g(z)| \geq C\rho(z, \Gamma_a)(1-|z|^2)^{-a}$ for all $z \in \mathbb{D}$.*

Proof. Define $H(z) = a\log(\dfrac{1}{1-|z|^2})$. Then H is subharmonic in $\mathbb{D}$ and

$$\Delta H(z) = \frac{4a}{(1-|z|^2)^2}.$$

By Theorem 2 of [11], there is a function g which is analytic in $\mathbb{D}$ and whose zero set $Z(g)$ is uniformly discrete, such that $|g(z)|$ is bounded above and below by constant multiples of $\rho(z, Z(g))\exp(H(z))$. Since $\exp(H(z)) = (1-|z|^2)^{-a}$, it suffices to show that $Z(g) = \Gamma_a$. Looking at the proof of the theorem, we see that this is, in fact, the case.

Theorem 5. *Γ_a is a set of sampling for A^2 if and only if $a > 1/2$. Γ_a is a set of interpolation for A^2 if and only if $a < 1/2$.*

Proof. If we replace the estimates (4) and (5) in [9] by the estimates in Lemma 4 , we see that the proofs of the main results in [9] remain valid in our situation. Note that Γ_a is uniformly discrete.

Corollary 1. *$D^-(\Gamma_a) = D^+(\Gamma_a) = a$.*

5. Interpolation sets and classical interpolation sequences

Looking at the proof of Theorem S1 in [10], one may extract the result that there is a constant K such that the upper inequality in (1),

$$\sum_{z\in\Gamma}(1-|z|^2)^2|f(z)|^p \leq K\|f\|_p^p,$$

holds for all $f \in A^p$ if and only if Γ is a finite union of uniformly discrete sets. (A different proof of this is given in [16].) Therefore, by the closed graph theorem, $T_p(A^p) \subseteq \ell^p$ if and only if Γ is a finite union of uniformly discrete sets. Since every set of interpolation is uniformly discrete,

$$T_p(A^p) \supseteq \ell^p \Rightarrow T_p(A^p) = \ell^p.$$

For the Hardy spaces, we define the linear operator T'_p which maps an H^p function f to the sequence $\{f(z)(1-|z|^2)^{\frac{1}{p}}\}_{z\in\Gamma}$. We say that Γ is a *classical interpolation sequence* if $T'_p(H^p) \supseteq \ell^p$. It is a well-known theorem of Shapiro and Shields [12] that $T'_p(H^p) = \ell^p$ if and only if Γ is uniformly separated, i.e. if $\Gamma = \{z_j\}$, then there is a $\delta > 0$ such that

$$\prod_{\substack{j=1\\ j\neq k}}^{\infty} \left| \frac{z_k - z_j}{1 - \overline{z_j} z_k} \right| \geq \delta, \qquad k = 1, 2, \ldots.$$

Looking at the proof of the main result in [12], we see that the condition $T'_p(H^p) \supseteq \ell^p$ is actually sufficient to ensure that Γ be uniformly separated. A consequence is the fact that $T'_p(H^p) \supseteq \ell^p \Rightarrow T'_p(H^p) = \ell^p$.

A natural question concerns the relationship between the classical interpolation sequences and the sets of interpolation for the Bergman spaces.

Theorem 6. *Every classical interpolation sequence is a set of interpolation for A^p $(0 < p < \infty)$. For each p, there is a sequence which is a set of interpolation for A^p but which is not even a zero set for H^p (i.e. it is not a Blaschke sequence).*

Kehe Zhu [15] stated the first half of the theorem for $p = 2$. In a private communication with the author he sketched a proof of the following lemma, of which the result is an immediate consequence.

Lemma 5. *If Γ is uniformly separated, then $D^+(\Gamma) = 0$.*

Proof. Let $z \in \mathbb{D}$ and $0 < r < 1$. Since $-\log x \leq 1 - x^2$ for $1/2 < x < 1$,

$$\begin{aligned}\log\left(\frac{1}{1-r}\right) D(\Gamma_z, r) &= \sum_{\frac{1}{2} < |\phi_z(z_k)| < r} \log\left(\frac{1}{|\phi_z(z_k)|}\right) \leq \sum_{\frac{1}{2} < |\phi_z(z_k)| < r} (1 - |\phi_z(z_k)|^2) \\ &\leq \sum_{k=1}^{\infty} (1 - |\phi_z(z_k)|^2) = \sum_{k=1}^{\infty} \frac{(1-|z|^2)(1-|z_k|^2)}{|1 - \overline{z} z_k|^2}.\end{aligned}$$

In [12] it is shown that the last sum is bounded by a constant if Γ is uniformly separated, with the bound depending only on the degree of separation. Therefore,

$$D(\Gamma_z, r) \leq \frac{C}{\log(\frac{1}{1-r})} \qquad \text{for all } z \in \mathbb{D} \text{ and all } r < 1$$

and so $D^+(\Gamma) = 0$.

Now we choose $a > 1$ and $b > 0$ so that $\Gamma(a, b)$ is an A^p interpolation set. It is clear that $\Gamma(a, b)$ is not a Blaschke sequence, as then it would not be an A^q sampling set for any q, a contradiction.

Note that the examples $\Gamma(a, b)$ illustrate that if the only restriction on a sequence Γ is uniform discreteness, then $D^-(\Gamma)$ and $D^+(\Gamma)$ can take on any positive value. They must both be finite, however, since every uniformly discrete sequence is a

finite union of A^p interpolation sets, by a result of Amar [1]. Theorem S2 thus tells us that any uniformly discrete sequence is an A^p interpolation set, for some p.

By results in [13] and [14], respectively, we see that a uniformly discrete sequence in a Stolz angle or on a convex curve tangent to the unit circle $\mathbb{T}$ at one point must be uniformly separated. Therefore, in those cases the sets of interpolation for A^p $(0 < p < \infty)$ and the classical interpolation sequences are precisely those sequences which are uniformly discrete.

6. Sets of sampling, canonical divisors and boundary behaviour

In §3 we discussed unions of sampling sets. Sets of sampling are "thick" in some sense, so we now consider the question of how much can be removed from a set of sampling and have it remain sampling. By Lemmas 1 and 5, if Γ is an A^p sampling set, Λ is an A^p interpolation set and S is a finite union of uniformly separated sequences, then $\Gamma \setminus S$ is again a set of sampling and $\Lambda \cup S$ is a set of interpolation, if it is uniformly discrete.

We now consider the problem in a different way, one which involves the notion of a canonical divisor. We list some of its properties below but for details we refer the reader to [3].

Let S be an A^p zero set. Then there is an A^p function G of unit norm, unique up to rotation, whose zero set is precisely S and which is a contractive divisor in A^p, that is,

$$\|f/G\|_p \leq \|f\|_p$$

for all $f \in A^p$ whose zero set contains S. We then obtain

Theorem 7. *Let $0 < p < \infty$ and let Γ be a set of sampling for A^p. Suppose S is an A^p zero set whose canonical divisor G is bounded. Then $\Gamma \setminus S$ is a set of sampling for A^p.*

Proof. Since G is bounded, it is a multiplier on A^p, that is, $f \in A^p \Rightarrow fG \in A^p$. An immediate corollary of this is the fact that any $f \in A^p$ can be written as $f = \dfrac{g}{G}$, where $g \in A^p$ and $g_{|S} = 0$. Suppose now that $|G(z)| \leq C$ for all $z \in \mathbb{D}$ and some constant C. Then

$$\begin{aligned}
\|f\|_p^p = \|\frac{g}{G}\|_p^p \leq \|g\|_p^p &\leq {K_1}^{-1} \sum_{z\in\Gamma} (1-|z|^2)^2 |g(z)|^p \\
&= {K_1}^{-1} \sum_{z\in\Gamma\setminus S} (1-|z|^2)^2 |g(z)|^p = {K_1}^{-1} \sum_{z\in\Gamma\setminus S} (1-|z|^2)^2 |G(z)|^p |f(z)|^p \\
&\leq {K_1}^{-1} C^p \sum_{z\in\Gamma\setminus S} (1-|z|^2)^2 |f(z)|^p.
\end{aligned}$$

It is clear that the property of being a finite union of uniformly discrete sets is unaffected by subtracting points, so we see that $\Gamma \setminus S$ is a set of sampling for A^p.

It actually turns out that Theorem 7 can be deduced from the remarks made at the beginning of this section and the following result, which is known but is included here for completeness.

Lemma 6. *Let S be an A^p zero set whose canonical divisor G is bounded. Then S is a finite union of uniformly separated sequences.*

Proof. It is clear that S is a Blaschke sequence with associated Blaschke product B. By Riesz' Theorem (see [2]), $G = Bk$, where k is a nonvanishing bounded analytic function. Now let $f \in A^p$ with f/B analytic. Then

$$\|f/B\|_p = \|kf/G\|_p \leq \|k\|_\infty \|f/G\|_p \leq \|k\|_\infty \|f\|_p$$

and so, by Theorem 2 in [7], S is a finite union of uniformly separated sequences.

Sets of sampling, as we can see from the definition, must be fairly dense near the boundary. In fact, we have:

Theorem 8. *Let $0 < p < \infty$. Every A^p sampling set accumulates non-tangentially at every point of $\mathbb{T}$.*

Proof. Let Γ be an A^p sampling set and Γ' a uniformly discrete subsequence satisfying $D^-(\Gamma') > 1/p$. If Γ' does not accumulate non-tangentially at some point $z_0 \in \mathbb{T}$, then there is a circle N, tangent to $\mathbb{T}$ at z_0, which contains no points of Γ'. Let $\{r_n\}$ be any sequence tending to 1. For each n, choose a $z_n \in \mathbb{D}$ such that the pseudo-hyperbolic disk $B(z_n, r_n)$ is contained in N. Therefore, $B(z_n, r_n) \cap \Gamma' = \emptyset$ and so $D(\Gamma'_{z_n}, r_n) = 0$, thus implying that $D^-(\Gamma') = 0$, a contradiction.

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1109

E-mail address: aschust@math.lsa.umich.edu