ON SIMULTANEOUS EXTENSION OF CONTINUOUS PARTIAL FUNCTIONS

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Abstract. For a metric space \( X \) let \( C_{vc}(X) \) (that is, the set of all graphs of real-valued continuous functions with a compact domain in \( X \)) be equipped with the Hausdorff metric induced by the hyperspace of nonempty closed subsets of \( X \times \mathbb{R} \). It is shown that there exists a continuous mapping \( \Phi : C_{vc}(X) \to C_b(X) \) satisfying the following conditions:

(i) \( \Phi(f)|_{\text{dom} f} = f \) for all partial functions \( f \).

(ii) For every nonempty compact subset \( K \) of \( X \), \( \Phi|_{C_b(K)} : C_b(K) \to C_b(X) \) is a linear positive operator such that \( \Phi(1_K) = 1_X \).

1. Introduction

Let \( X \) be a topological space. Recently V.V. Filippov [4] and his students studied the space \( C_v(X) \) of partial real-valued continuous functions with closed domains in \( X \). (Let us mention that this space originates in [5, 6]):

Consider the set \( C_v(X) \) of all continuous functions \( f : A \to \mathbb{R} \), where \( A \) denotes a closed subspace of \( X \), and identify each such function with its graph. Equip now \( C_v(X) \) with the topology induced by the Vietoris topology on the hyperspace \( \exp(X \times \mathbb{R}) \) of all nonempty closed subsets of \( X \times \mathbb{R} \). In this paper we shall deal mainly with the subspace \( C_{vc}(X) \) of \( C_v(X) \) that consists of all partial continuous real-valued functions having a compact domain in \( X \). Observe that for a metric space \( X \) the space \( C_{vc}(X) \) is metrizable by the Hausdorff metric induced on \( X \times \mathbb{R} \) [7].

For an arbitrary topological space we denote by \( \exp_c X \) the set of nonempty compact subsets of \( X \) equipped with the Vietoris topology. Furthermore let \( \Pi : C_{vc}(X) \to \exp_c X \) be the natural projection. Note that for any \( K \in \exp_c X \) the equality \( \Pi^{-1}(K) = C_b(K) \) holds. In this paper \( C_b(X) \) denotes the set of all bounded continuous real-valued functions \( f \) on a topological space \( X \) endowed with the topology given by the sup-norm \( \| f \|_\infty = \sup_{x \in X} |f(x)| \).

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In the fifties J. Dugundji proved the following theorem.

**Theorem 1 ([2]).** Let $X$ be a metric space and $A$ a closed subspace of $X$: Then there exists a linear operator $\Phi$ which makes correspond to each $f \in C_b(A)$ an extension $\Phi(f) \in C_b(X)$.

For a discussion of some related positive and negative results we refer the reader to [11].

On the other hand E.N. Stepanova [10] proved recently that for a metric space $X$ it is possible to extend all elements of $C_{vc}(X)$ simultaneously to $X$ in the following sense.

**Theorem 2 ([10]).** For any metrizable space $X$ there exists a continuous mapping $\alpha : C_{vc}(X) \rightarrow C_b(X)$ such that for any function $f$, $\alpha(f) \mid \text{dom } f = f$ and $\sup_{x \in \text{dom } f} |f(x)| = \sup_{x \in X} |\alpha(f)(x)|$.

She showed moreover that among the Hausdorff paracompact $p$-spaces only metrizable spaces have such a mapping $\alpha$.

In this paper we want to prove the following theorem.

**Theorem 3.** Let $X$ be a metrizable space. Then there exists a continuous mapping $\Phi : C_{vc}(X) \rightarrow C_b(X)$ that satisfies the following conditions:

1. $\Phi(f) \mid \text{dom } f = f$ for each partial function $f$;
2. for any $K \in \exp_c X$ the restriction $\Phi \mid \Pi^{-1}(K)$ is a linear positive operator such that $\Phi(1_K) = 1_X$.

Observe that for compact domains our result generalizes simultaneously the theorems of Dugundji and Stepanova.

**Remark 1.** It is readily seen that each one-to-one (continuous) preimage of a space having an extension operator as described in Theorem 3 also admits such an operator. Therefore every one-to-one preimage of a metric space (i.e. a submetrizable space) admits an operator of this kind. Note that the class of one-to-one preimages of metric spaces contains all paracompact $\sigma$-spaces (compare [3, Exercise 5.5.7]). We deduce in particular that each stratifiable space admits an extension operator as described in Theorem 3. Observe however that the one-point-Lindelöfication of an uncountable discrete space admits such an operator, although it is not the one-to-one preimage of a metric space.

2. Selections

Our proof of Theorem 3 will be based on the following well-known selection theorem due to E. Michael [8, Theorem 3.2”]. Let $F : X \rightarrow Y$ be a lower semi-continuous multi-valued mapping from a paracompact Hausdorff space to a Banach space such that for each $x \in X$, $F(x)$ is a nonempty closed convex subset of $Y$. Then there exists a continuous mapping $f : X \rightarrow Y$ such that $f(x) \in F(x)$ whenever $x \in X$.

We recall that a multi-valued mapping $F : X \rightarrow Y$ is called lower semi-continuous provided that $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in $X$ whenever $U$ is open in $Y$.

For any metric space $(X, d)$, any $\delta > 0$ and $A \subseteq X$ we set $B(A, \delta) = \{y \in X : d(a, y) < \delta \text{ for some } a \in A\}$.

The proof of Theorem 3 consists of the verification of three lemmas.
Lemma 1. Let $K$ and $L$ be compact subsets of a metric space $(X,d)$, $r : X \to K$ a continuous mapping and $\epsilon$ a positive real number such that $\max_{x \in L} d(x,r(x)) < \epsilon$. Then there exist a locally finite open cover $\gamma = \{V_t : t \in T\}$ of $X \setminus L$ and $\delta > 0$ such that for any $t \in T$ with $V_t \cap B(L, \delta) \neq \emptyset$ we can choose $a_t \in K$, $b_t \in L$ and $y_t \in X \setminus L$ satisfying the following conditions:

(i) $V_t \subseteq r^{-1}(B(a_t, 2\epsilon)) \cap B(y_t, \frac{1}{2}d(y_t, L))$;
(ii) $d(a_t, b_t) < 2\epsilon$; and
(iii) $d(y_t, b_t) \leq \frac{3}{2}d(y_t, L)$.

Proof. Note that $C := \{B(y_t, \frac{1}{2}d(y_t, L)) \cap r^{-1}(B(r(y_t), 2\epsilon)) : y_t \in X \setminus L\}$ is an open cover of $X \setminus L$. Since the subspace $X \setminus L$ of $X$ is paracompact, there is a locally finite open refinement $\gamma = \{V_t : t \in T\}$ of $C$. We shall show that $\gamma$ satisfies the condition formulated in Lemma 1. Assume the contrary. Then for each $n \in \omega$ there is an element $V_{n} \in \gamma$ that hits $B(L, 2^{-n})$, but we cannot find a triple $(a_{n}, b_{n}, y_{n})$ as described in Lemma 1.

Fix $n \in \omega$. There is $y_{n} \in X \setminus L$ such that $V_{n} \subseteq B(y_{n}, \frac{1}{4}d(y_{n}, L))$.

Next we want to verify that $d(y_{n}, L) < \frac{3}{2} \cdot 2^{-n}$. Indeed, let $y \in V_{n} \cap B(L, 2^{-n})$. Thus $d(y_n, y) < \frac{1}{4} d(y_n, L)$. Let $z \in L$ be such that $d(y, z) = d(y, L)$. We have $d(y_n, L) \leq d(y_n, z) \leq d(y_n, y) + d(y, z) < \frac{1}{4} d(y_n, L) + 2^{-n}$.

Therefore $\frac{1}{2}d(y_n, L) < 2^{-n}$ and thus $d(y_n, L) < \frac{1}{2} \cdot 2^{-n}$.

It follows that $L \cup \{y_n : n \in \omega\}$ is a compact subset of $X$. Without loss of generality we can suppose that $(y_n)_{n \in \omega}$ is a convergent sequence and that $y := \lim_{n \to \infty} y_n \in L$. For each $n \in \omega$ let $z_{n} \in L$ be such that $d(y_{n}, z_{n}) = d(y_{n}, L)$. Then $y = \lim_{n \to \infty} z_{n}$, because $d(y_{n}, z_{n}) \to 0$. Since $d(y, r(y)) < \epsilon$ and $r$ is continuous, there exists $\delta > 0$ such that $\delta < \epsilon$, $B(y, \delta) \subseteq B(r(y), 2\epsilon)$ and $r(B(y, \delta)) \subseteq B(r(y), 2\epsilon)$.

Furthermore there exists $m \in \omega$ such that $B(y_{m}, \frac{1}{2}d(y_{m}, L)) \subseteq B(y, \delta)$ and $z_{m} \in B(y, \delta)$. Consequently for $V_{m}$ we can define the triple $(a_{m} = r(y), b_{m} = z_{m}, y_{m})$. By our choice $V_{m} \subseteq B(y_{m}, \frac{1}{4}d(y_{m}, L)) \subseteq B(y, \delta) \subseteq r^{-1}(B(r(y), 2\epsilon)) = r^{-1}(B(a_{m}, 2\epsilon))$. Moreover $d(a_{m}, b_{m}) = d(r(y), z_{m}) \leq d(r(y), y) + d(y, z_{m}) < \epsilon + \delta < 2\epsilon$. Finally $d(y_{m}, b_{m}) = d(y_{m}, z_{m}) = d(y_{m}, L) \leq \frac{3}{2}d(y_{m}, L)$.

Hence the defined triple satisfies conditions (i)–(iii) of Lemma 1. We have reached a contradiction and conclude that the cover $\gamma$ possesses all the properties described in Lemma 1.

Proof. Let $R : \text{exp}_c X \to \mathcal{C}_b(X, X)$ be the multi-valued mapping defined by $K \mapsto \{r \in \mathcal{C}_b(X, X) : r(X) \subseteq \overline{\text{co}}K \text{ and } r|K = \text{id}_K\}$.

Note that for each $K \in \text{exp}_c X$, $R(K)$ is a closed convex subset of $\mathcal{C}_b(X, X)$. It is nonempty by Dugundji’s theorem [2, Theorem 4.1].
Since $\exp_c X$ is metrizable, it remains to be shown that $R$ is lower semi-continuous; then we can apply Michael’s selection theorem in order to obtain the continuous mapping $\mu$ described in Lemma 2.

Let $U$ be an open subset of $C_{\delta}(X, X)$ and let $K \in \exp_c X$ be such that $R(K) \cap U \neq \emptyset$. If $K$ is not an interior point of $\{K \in \exp_c X : R(K) \cap U \neq \emptyset\}$, then there exists a sequence $(K_n)_{n \in \omega}$ in $\exp_c X$ converging to $K$ such that $R(K_n) \cap U = \emptyset$ whenever $n \in \omega$. Fix $r \in R(K) \cap U$ and let $\epsilon > 0$ be such that $B(r, 2\epsilon) \subseteq U$. For each $n \in \omega$ set $d_n = \max_{x \in K_n} \|r(x) - x\|.

Since $r$ is continuous, $K_n \to K$ and $r|K = \text{id}_K$, it is readily seen that the sequence $(d_n)_{n \in \omega}$ converges to $0$. Set $\delta = \frac{\epsilon}{2}$ and find $N \in \omega$ such that for any $n \in \omega$ with $n > N$ we have $\max\{|\text{dist}(\overline{\exp K}, \overline{\exp K_n}), d_n|\} < \delta$.

Fix $n \in \omega$ such that $n > N$. We want to define $r_n \in R(K_n)$ such that

$$\sup_{x \in X} \|r(x) - r_n(x)\| \leq \epsilon.$$

To this end we are going to apply Lemma 1. The role of $L$ is now played by $K_n$, the role of $K$ is played by $\overline{\exp K}$ and the role of $\epsilon$ is played by $\delta$.

By Lemma 1 there exist a locally finite open cover $\gamma = \{V_t : t \in T\}$ of $X \setminus K_n$ and a neighborhood $V$ of $K_n$ in $X$ such that for each $V_t$ that hits $V$ we can find a triple $(a_t, b_t, y_t) \in \overline{\exp K} \times X \setminus K_n \times (X \setminus K_n)$ satisfying

(i) $V_t \subseteq r^{-1}(B(a_t, 2\delta)) \cap B(y_t, \frac{1}{2}\|y_t - K_n\|);$

(ii) $\|a_t - b_t\| < 2\delta$; and

(iii) $\|y_t - b_t\| < \frac{\delta}{2}\|y_t - K_n\|.$

If $V_t \cap V = \emptyset$, then in the light of the definition of the cover $C$ in the proof of Lemma 1 we can choose $a_t \in \overline{\exp K}$ such that $V_t \subseteq r^{-1}(B(a_t, 2\delta))$. Furthermore, since $\text{dist}(\overline{\exp K}, \overline{\exp K_n}) < \delta$ we find $b_t \in \overline{\exp K_n}$ such that $\|a_t - b_t\| < 2\delta$.

Now let $\{g_t : t \in T\}$ be a partition of unity on $X \setminus K_n$ subordinated to $\{V_t : t \in T\}$. Define $r_n : X \to X$ by $r_n(x) = x$ if $x \in K_n$ and $r_n(x) = \Sigma_{t \in T} g_t(x)b_t$ if $x \in X \setminus K_n$.

Note that $r_n(X) \subseteq \overline{\exp K_n}$ and $r_n|K_n = \text{id}_{K_n}$. Clearly $r_n$ is continuous at any $x \in X \setminus K_n$. Since inside $V \supseteq K_n$ our construction coincides with the one used by Dugundji as outlined by R. Engelking in [3, Exercise 4.5.20], we see that $r_n$ is also continuous at any $x \in K_n$.

Let us estimate $\sup_{x \in X} \|r(x) - r_n(x)\|.$

**Case 1.** Suppose that $x \in K_n$. Then $\|r(x) - r_n(x)\| = \|r(x) - x\| \leq d_n < \delta = \frac{\epsilon}{2}$.

**Case 2.** Suppose that $x \in X \setminus K_n$ and that $\{V_t : i = 1, \ldots, m\}$ is the collection of all the members $V_t \subseteq X$ such that $x \in V_t$. Then $r_n(x) = \Sigma_{i=1}^m g_{t_i}(x)b_{t_i}$. Since $r(\bigcap_{i=1}^m V_{t_i}) \subseteq \bigcap_{i=1}^m B(a_{t_i}, 2\delta)$, we have $\|r(x) - a_{t_i}\| < 2\delta$ whenever $i \leq m$.

Thus

$$\|r(x) - r_n(x)\| = \|r(x) - \Sigma_{i=1}^m g_{t_i}(x)b_{t_i}\|$$

$$\leq \|r(x) - \Sigma_{i=1}^m g_{t_i}(x)a_{t_i}\| + \|\Sigma_{i=1}^m g_{t_i}(x)a_{t_i} - \Sigma_{i=1}^m g_{t_i}(x)b_{t_i}\|$$

$$= \|\Sigma_{i=1}^m g_{t_i}(x)(r(x) - a_{t_i})\| + \|\Sigma_{i=1}^m g_{t_i}(x)a_{t_i} - \Sigma_{i=1}^m g_{t_i}(x)b_{t_i}\|$$

$$\leq (\Sigma_{i=1}^m g_{t_i}(x))\|r(x) - a_{t_i}\| + (\Sigma_{i=1}^m g_{t_i}(x))\|a_{t_i} - b_{t_i}\| < 2\delta + 2\delta = 2\epsilon.$$

We conclude that $\sup_{x \in X} \|r(x) - r_n(x)\| \leq \epsilon$. It follows that $r_n \in R(K_n) \cap U$.

We have reached a contradiction and deduce that $R : \exp_c X \to C_{\delta}(X, X)$ is lower semi-continuous.
3. Measures

Before proving Lemma 3 let us recall some facts established in [1]. Let \((X, d)\) be a metric space. Denote the \(\sigma\)-algebra of Borel sets of \(X\) by \(\mathcal{B}(X)\) and let \(\mathcal{M}(X)\) be the set of all finite, real-valued, countably additive functions defined on \(\mathcal{B}(X)\).

A bounded real-valued function \(f\) on \(X\) is called Lipschitzian if
\[
\|f\|_L = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} : d(x, y) \neq 0 \right\} < \infty.
\]

On the set \(BL(X, d)\) of all such functions a norm is defined by setting \(\|f\|_{BL} = \|f\|_\infty + \|f\|_L\). Then \((BL(X, d), \| \cdot \|_{BL})\) is a Banach algebra according to [1].

We shall denote its dual space by \(BL(X, d)\) and the corresponding norm by \(\|T\|_{BL} = \sup\{|(Tf)(x) : \|f\|_{BL} = 1\}\). By \(\mu \in \mathcal{M}(X) \mapsto \mu(f) = \int fd\mu \in BL(X, d)^*\) an injection of \(\mathcal{M}(X)\) into \(BL(X, d)^*\) is defined.

In this way we can consider \(\mathcal{M}(X)\) a subset of \(BL(X, d)^*\). The topology induced on \(\mathcal{M}(X)\) by the norm on \(BL(X, d)^*\) will be denoted by \(TBL^*\). The weak* topology \(TW^*\) on \(\mathcal{M}(X)\) is determined by \(\mu_n \to \mu\) iff \(\int fd\mu_n \to \int fd\mu\) whenever \(f\) in \(C_0(X)\). It coincides with \(TBL^*\) on the set \(\mathcal{M}^*_+(X)\) (of nonnegative measures with separable supports) [1, Theorem 18].

Let us denote by \(\mathcal{P}_c(X)\) the set of all finite, real-valued, countably additive functions defined on \(\sigma\)-algebra of Borel sets of \(X\). Then \(\mathcal{P}_c(X) \subseteq \mathcal{M}^*_+(X)\) and the weak* topology on \(\mathcal{P}_c(X)\) coincides with the topology induced on \(\mathcal{P}_c(X)\) by \(\| \cdot \|_{BL}^*\).

It is known that \(x \mapsto \delta_x\) defines a topological embedding of \(X\) into \(BL(X, d)^*\).

Here, as usual, \(\delta_x\) denotes the Dirac measure.

**Lemma 3.** Let \(\mu : \exp_c X \to C_0(X, BL(X, d)^*)\) be a continuous mapping such that for each \(K \in \exp_c X\) we have that \(\mu(K)(X) \subseteq \mathcal{P}(K)\) (where \(\mathcal{P}(K)\) denotes the set of probability measures on \(K\)) and \(\mu(K)[K = \text{id}_K\) (modulo the identification \(x \mapsto \delta_x\)).

Then the mapping \(\Phi : C_{vc}(X) \to C_0(X)\) given by the formula
\[
\Phi(f)(x) = \int fd\mu(\text{dom } f)(x)
\]
for every partial function \(f\) and every \(x \in X\) satisfies the conditions of Theorem 3.

**Remark 2.** Let us note that Theorem 3 is an immediate consequence of Lemma 2 (applied to the Banach space \(BL(X, d)^*\)) and Lemma 3, because \(\mathcal{P}(K)\) is a closed convex set in \(BL(X, d)^*\).

**Proof of Lemma 3.** Note first that \(\Phi(f)\) is an extension of \(f\), since for any \(x \in \text{dom } f\) we have \(\mu(\text{dom } f)(x) = \delta_x\) and therefore \(\Phi(f)(x) = \int fd\delta_x = f(x)\).

Clearly for any \(K \in \exp_c X, \Phi : C_0(K) \to C_0(X)\) is a linear positive operator such that \(\sup_{x \in X} |\Phi(f)(x)| = \sup_{x \in \text{dom } f} |f(x)|\) and \(\Phi(1_K) = 1_X\). Furthermore \(\Phi(f)\) is continuous, since \(\mathcal{P}(\text{dom } f)\) is endowed with the weak* topology.

It remains to check the continuity of \(\Phi\). To this end suppose that \(f_n \to f\) in \(C_{vc}(X)\), \(K = \text{dom } f\) and \(K_n = \text{dom } f_n\) whenever \(n \in \omega\). In the following the Hausdorff metric on \(C_{vc}(X)\) will be denoted by \(\text{dist}\).

Obviously \(K_n \to K\) in \(\exp_c X\), since \(f_n \to f\). Therefore \(\overline{K} = K \cup \bigcup \{K_n : n \in \omega\}\) is compact in \(X\) [7, Theorem 2.5]. Let us fix an arbitrary continuous real-valued function \(\overline{f}\) on \(\mathcal{P}(\overline{K})\) by setting \(\overline{f}(\mu) = \int fd\mu\) whenever \(\mu \in \mathcal{P}(\overline{K})\). Note that \(\overline{f}\) is uniformly continuous with respect to the norm \(\| \cdot \|_{BL}^*\) restricted to \(\mathcal{P}(\overline{K})\).
Define the partial continuous functions $g_n : K_n \to \mathbb{R}$ by setting $g_n = \tilde{f}|K_n$ where $n \in \omega$. It follows that $g_n \to f$ in $C_{vc}(X)$, because $K_n \to K$ (in $\exp_c X$). Then $	ext{dist}(f_n, g_n) \to 0$ in the metric space $C_{vc}(X)$. This implies that

$$\max_{x \in K_n} |f_n(x) - g_n(x)| = \max_{x \in K_n} |f_n(x) - \tilde{f}(x)| \to 0,$$

as it is readily verified.

Now let $\epsilon > 0$. We want to find $N \in \omega$ such that for all $n \in \omega$ with $n > N$ and all $x \in X$, $|\Phi(f)(x) - \Phi(f_n)(x)| < \epsilon$.

By uniform continuity of $\tilde{f}$ on $\mathcal{P}(\tilde{K})$ we first find $\delta > 0$ such that if $\mu, \nu \in \mathcal{P}(\tilde{K})$ and $\|\mu - \nu\|_{BL} < \delta$, then $|\int \tilde{f} d\mu - \int \tilde{f} d\nu| < \frac{\epsilon}{2}$.

Furthermore there exists $N_1 \in \omega$ such that for all $n \in \omega$ with $n > N_1$ we have $\max_{x \in K_n} |f_n(x) - \tilde{f}(x)| < \frac{\epsilon}{2}$. Moreover there exists $N_2 \in \omega$ such that for all $n \in \omega$ with $n > N_2$ and all $x \in X$ we have that $\|\mu(K) - \mu(K_n)(x)\|_{BL} < \delta$, because $\mu$ is continuous.

Let $N = \max\{N_1, N_2\}$. For all $n \in \omega$ such that $n > N$ and all $x \in X$, we obtain finally:

$$|\Phi(f)(x) - \Phi(f_n)(x)| = |\int f d\mu(K)(x) - \int f_n d\mu(K_n)(x)|$$

$$\leq |\int f d\mu(K)(x) - \int \tilde{f} d\mu(K_n)(x)| + |\int \tilde{f} d\mu(K_n)(x) - \int f_n d\mu(K_n)(x)|$$

$$= |\int \tilde{f} d\mu(K)(x) - \int \tilde{f} d\mu(K_n)(x)| + |\int \tilde{f} d\mu(K_n)(x) - \int f_n d\mu(K_n)(x)|$$

$$< \frac{\epsilon}{2} + \max_{x \in K_n} |\tilde{f}(x) - f_n(x)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown that $\Phi$ is continuous.

**Remark 3.** Let us finally remark that it is possible to obtain a variant of Theorem 3 by replacing the reals by an arbitrary Banach space over the reals. In fact we can prove the following result.

*If $E$ is a Hausdorff sequentially complete locally convex topological vector space over the reals, then there exists a continuous mapping $\Phi : C_{vc}(X, E) \to C(X, E)$ (where $C(X, E)$ is equipped with the topology of uniform convergence) that satisfies the following conditions:

(i) $\Phi(f)|\text{dom } f = f$ for any partial function $f \in C_{vc}(X, E)$;

(ii) $\text{range}(\Phi(f)) \subseteq \overline{\text{co}}(\text{range } f)$;

(iii) for any $K \in \exp_c X$ the restriction $\Phi|\Pi^{-1}(K)$ is a linear operator from $C(K, E)$ into $C(X, E)$.*

Indeed, according to Lemmas 2 and 3 let $\mu : \exp_c X \to C_0(X, BL(X, d)^*)$ be a continuous mapping such that for each $K \in \exp_c X$ we have that $\mu(K)(X) \subseteq \mathcal{P}(K)$ (where $\mathcal{P}(K)$ denotes the space of probability measures on $K$) and $\mu(K)|K = \text{id}_K$ (modulo the identification $x \mapsto \delta_x$).

First we assume that $E$ is a Banach space. Then the formula $\Phi(f)(x) = \int f d\mu(\text{dom } f)(x)$ gives the desired mapping. In order to check the statement observe that $\Phi(f)$ is a continuous mapping from $X$ into $E$: Since if $x_n \to x$ in $X$, then $\int f d\mu(\text{dom } f)(x_n)$ weakly converges to $\int f d\mu(\text{dom } f)(x)$; but

$$\int d\mu(\text{dom } f)(x_n) \in \overline{\text{co}}(\text{range } f).$$
and $\mathfrak{c}(\text{range } f)$ is a compact space. Therefore weak convergence coincides with the usual convergence in $E$. The proof of the continuity of $\Phi$ is the same as in the proof of Lemma 3 (instead of $\mathbb{R}$ we are working with $E$).

Now suppose that $E$ is a Hausdorff sequentially complete LCTVS. Then we can assume that $E$ is a subspace of a product of Banach spaces $\prod\{E_\alpha : \alpha \in A\}$.

Hence we set $\Phi(f)(x) := \int f \, d\mu(\text{dom } f)(x) = \{\int \text{pr}_\alpha \circ f \, d\mu(\text{dom } f)(x) : \alpha \in A\}$. Let us note that according to sequential completeness of $E$, $\int f \, d\mu(\text{dom } f)(x)$ belongs to $E$ : Clearly $\mu(\text{dom } f)(x)$ is a weak limit of a sequence $\{\mu_n : n \in \omega\}$ where $\mu_n \subseteq \text{dom } f$ and $|\text{supp } \mu_n| < \omega$. Consequently $\int f \, d\mu_n \in \text{co}(\text{range } f)$ and $\{\int f \, d\mu_n : n \in \omega\}$ is a Cauchy sequence in $E$. Thus $\int f \, d\mu(\text{dom } f)(x) \in E$ as a limit of a Cauchy sequence.

The continuity of $\Phi$ follows from the continuity of the mappings $\Phi_\alpha$, where $(\Phi_\alpha(f))(x) = \int \text{pr}_\alpha \circ f \, d\mu(\text{dom } f)(x)$.

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