WHEN IS A \( p \)-BLOCK A \( q \)-BLOCK?

GABRIEL NAVARRO AND WOLFGANG WILLEMS

(Communicated by Ronald Solomon)

Abstract. Let \( p \) and \( q \) be distinct prime numbers and let \( G \) be a finite group. If \( B_p \) is a \( p \)-block of \( G \) and \( B_q \) is a \( q \)-block, we study when the set of ordinary irreducible characters in the blocks \( B_p \) and \( B_q \) coincide.

1. Introduction

Let \( p \) and \( q \) be distinct prime numbers and let \( G \) be a finite group. Since the representation theory of \( G \) in every prime characteristic is connected to that in characteristic zero via the decomposition map, there is a relationship between the theories in characteristics \( p \) and \( q \). However, one hardly finds anything in the literature devoted to this connection.

In this note, we investigate the situation when the set \( \text{Irr}(B_p) \) of irreducible ordinary characters of a \( p \)-block \( B_p \) of \( G \) coincides with \( \text{Irr}(B_q) \), where \( B_q \) is a \( q \)-block. This happens, for instance, if the group \( G \) has an irreducible character \( \chi \) which is of \( p \)- and \( q \)-defect zero, that is, with \( |G|_p |G|_q \) dividing \( \chi(1) \). This is equivalent to the fact that \( \text{Irr}(B_p) = \text{Irr}(B_q) \) consists only of one character \( \chi \).

Here we shall give some evidence that this is the only possibility for \( p \)- and \( q \)-blocks to coincide. We state the

Conjecture. If \( \text{Irr}(B_p) = \text{Irr}(B_q) \), then \( |\text{Irr}(B_p)| = 1 \).

In what follows, we give an affirmative answer in the case that \( G \) is “solvable for one prime” or that one of the blocks contains only one Brauer character.

We would like to remark that one should be able to prove the conjecture for principal blocks. In this case, the problem reduces to simple groups and then a matter of checking will give the answer. We have not attempted to do this, however.

Finally, we are indebted to M. Isaacs for the shortened proof we present here.

2. Proofs

Our objective is to prove the following result.

(2.1) Theorem. Let \( p \) and \( q \) be distinct primes and let \( G \) be a finite group. Let \( B_p \) be a \( p \)-block and let \( B_q \) be a \( q \)-block of \( G \) with \( \text{Irr}(B_p) = \text{Irr}(B_q) \). If \( G \) is \( p \)-solvable or \( q \)-solvable, then \( |\text{Irr}(B_p)| = 1 \).
We need a result of independent interest. As a consequence of it, notice that the
characters in a block of positive defect cannot all be conjugate under the action of
\(\text{Aut}(G)\).

\textbf{(2.2) Proposition.} Let \(N\) be a normal subgroup of \(G\) and let \(b\) be a \(p\)-block of \(N\).
If \(\text{Irr}(b)\) consists of \(G\)-conjugates of some \(\theta \in \text{Irr}(N)\), then \(|\text{Irr}(b)| = 1\).

\textbf{Proof.} By replacing \(\theta\) by some conjugate, certainly we may assume that \(\theta \in \text{Irr}(b)\).
Write \(|\text{Irr}(b)| = k\) and first of all, observe that \(\theta^g \in \text{Irr}(b)\) if and only if \(g\) is an
element of the stabilizer \(I(b)\) of \(b\) in \(G\). Therefore,
\[ k = |\text{Irr}(b)| = |I(b) : I(\theta)|, \]
where \(I(\theta)\) denotes the stabilizer of \(\theta\) in \(G\). Hence, notice that we may assume that
\(I(b) = G\).

Now, by the Weak Block Orthogonality (3.6.21 of [2]), we have that
\[ \sum_{\alpha \in \text{Irr}(b)} \alpha(1)\alpha(n) = 0 \]
for all \(p\)-elements \(1 \neq n \in N\). Since all characters in \(b\) have the same degree \(\theta(1)\),
we conclude that
\[ \sum_{\alpha \in \text{Irr}(b)} \alpha \]
is a character of \(N\) which vanishes on its \(p\)-elements, so we conclude that \(|N|_p\)
divides its degree \(k\theta(1)\). Now, by Theorem (5.5.17) of [2], we know that there is a
character \(\xi \in \text{Irr}(b)\) such that \(|G : I(\xi)|\) is not divisible by \(p\). By the hypothesis, \(\xi\) is
some conjugate of \(\theta\) and we conclude that \(k = |G : I(\theta)|\) is a \(p^2\)-number. Therefore,
\(|N|_p\) divides \(\theta(1)\), as required. \(\square\)

We will derive Theorem (2.1) from the following.

\textbf{(2.3) Theorem.} Let \(G\) be a finite group and let \(\pi\) be a set of primes such that for
every chief factor of \(G\) there exists a prime in \(\pi\) not dividing the order of the factor.
Let \(\mathcal{X} \subseteq \text{Irr}(G)\) be such that for every \(p \in \pi\), there is a \(p\)-block \(B_p\) of \(G\) satisfying
\(\text{Irr}(B_p) = \mathcal{X}\). Then \(|\mathcal{X}| = 1\).

\textbf{Proof.} If \(N < G\) and \(\mathcal{Y} \subseteq \text{Irr}(N)\) is the set of irreducible constituents of \(\chi_N\), where \(\chi\)
runs through \(\mathcal{X}\), we prove by induction on \(|N|\) that every member of \(\mathcal{Y}\) has \(p\)-defect
zero for all \(p \in \pi\). Notice that if this is true, the theorem follows by putting \(N = G\).

Certainly, we may assume that \(N > 1\). We fix \(p \in \pi\) and \(\theta \in \mathcal{Y}\) and we prove that
\(\theta\) has \(p\)-defect zero. Let \(N/M\) be a chief factor of \(G\). By the inductive hypothesis,
we have that if \(Z \subseteq \text{Irr}(M)\) is the set of irreducible constituents of \(\chi_M\), where \(\chi\)
runs through \(\mathcal{X}\), every member of \(Z\) has defect zero for every prime in \(\pi\).

Now, let \(\eta \in \text{Irr}(M)\) under \(\theta\) and notice that \(\eta\) has \(p\)-defect zero, since \(\eta \in Z\).
Therefore, we may assume that \(N/M\) is divisible by \(p\) because otherwise
\[ \theta(1)_p = \eta(1)_p = |M|_p = |N|_p \]
and this would prove the theorem. By hypothesis, choose \(q \in \pi\) to be a prime not
dividing \(|N/M|\) and let \(b_q\) be the \(q\)-block of \(\theta\). Since \(\eta\) has \(q\)-defect zero, then \(\theta\) has
\(q\)-defect zero and then \(b_q = \{\theta\}\).

Now, let \(b_p\) be the \(p\)-block of \(\theta\) and notice that, by Proposition (2.2), it is enough
to show that \(b_p\) consists of \(G\)-conjugates of \(\theta\).
Let $\chi \in \mathcal{X}$ lying over $\theta$ and let $B_p$ and $B_q$ be the $p$- and $q$-block of $\chi$, respectively. Therefore, $B_p$ covers $b_p$, $B_q$ covers $b_q$ and $\text{Irr}(B_p) = \text{Irr}(B_q)$. Let $\gamma \in \text{Irr}(b_p)$. Since $B_p$ covers the block $b_p$, by (5.5.8.ii) of [2], we may find $\xi \in \text{Irr}(b_p)$ lying over $\gamma$. Now, $\xi \in \text{Irr}(B_p)$ and then the $q$-block of $\gamma$ is covered by $B_q$. But, since $B_q$ covers $b_q = \{\theta\}$, the $q$-block of $\gamma$ is just $\{\gamma\}$ and by (5.5.3) of [2], $\gamma$ and $\theta$ are $G$-conjugate. This completes the proof of the theorem.

Proof of Theorem (2.1). If $\mathcal{X} = \text{Irr}(B_p) = \text{Irr}(B_q)$, the hypotheses of Theorem (2.3) are satisfied with $\pi = \{p, q\}$.

(2.4) Theorem. Let $p$ and $q$ be distinct primes and let $B_p$ and $B_q$ be a $p$-block and a $q$-block of $G$, respectively. Suppose again that $\text{Irr}(B_p) = \text{Irr}(B_q)$. If one of the blocks contains only one Brauer character, then $|\text{Irr}(B_p)| = 1$.

Proof. Assume, for instance, that $|\text{IBr}(B_p)| = 1$ and consider the character

$$\rho = \sum_{\chi \in \text{Irr}(B_p)} \chi(1)\chi = \sum_{\chi \in \text{Irr}(B_q)} \chi(1)\chi.$$ 

If $\rho^0$ denotes the restriction of $\rho$ to the to $p$-regular elements of $G$ and $\phi$ is the unique Brauer character of $B_p$, then

$$\rho^0 = e\phi$$

for some integer $e$.

As $\rho$ is the regular character of the $q$-block $B_q$, we have in particular that $\rho^0(x) = 0$ for all $q$-elements $x \in G$. Thus $\phi(x) = 0$ for all $q$-elements $1 \neq x \in G$, and consequently, $\chi(x) = 0$ for all $q$-elements $1 \neq x \in G$ and $\chi \in \text{Irr}(B_p) = \text{Irr}(B_q)$. This means that $\chi$ has $q$-defect zero ([2], 3.6.27), and we are done.

Remark. To attack the conjecture for arbitrary groups and arbitrary blocks, Brauer’s dimensional formula (see [1])

$$\dim(B_p) = p^{2a-d}u_{B_p}^2v_{B_p},$$

where $|G|_p = p^a$, $d$ is the defect of $B_p$, $u_{B_p} = \gcd\{\chi(1)p^r \mid \chi \in \text{Irr}(B_p)\}$ and $p$ does not divide $v_{B_p}$, might be the key.

If $\text{Irr}(B_p) = \text{Irr}(B_q)$, one has to show that $q$ does not divide the mysterious invariant $v_{B_p}$. This seems to be difficult, however. Actually, if $B_p$ contains only one Brauer character, then $v_{B_p} = 1$ (see [1]) and this gives another proof of (2.4).

References


Departament d’Algebra, Facultat de Matematiques, Universitat de Valencia, 46100 Burjassot, Valencia, Spain
E-mail address: gabriel@uv.es

Fachbereik Mathematik, Universitat Mainz, 55099 Mainz, Germany
E-mail address: willems@mat.mathematik.uni-mainz.de