SPLITTING NUMBER

TOMEK BARTOSZYŃSKI

(Communicated by Andreas R. Blass)

Abstract. We show that it is consistent with ZFC that every uncountable set can be continuously mapped onto a splitting family.

1. Introduction

A family $A \subseteq [\omega]^{\omega}$ is called a splitting family if for every infinite set $B \subseteq \omega$ there exists $A \in A$ such that
$$|A \cap B| = |(\omega \setminus A) \cap B| = \aleph_0.$$ We denote by $s$ the least size of a splitting family. It is well-known that $\aleph_1 \leq s \leq 2^{\aleph_0}$. Let
$$S = \{X \subseteq 2^{\omega} : \text{no Borel image of } X \text{ is a splitting family}\}.$$ By “Borel image” we mean image by a Borel function. It is easy to see that $S$ is a $\sigma$-ideal containing all countable sets. The purpose of this paper is to show that one cannot prove in ZFC that $S$ contains an uncountable set.

Recall ([5] or [1]) that a forcing notion $(P, \leq)$ is Suslin if
(1) $P$ is ccc,
(2) $P$ is a $\Sigma^1_1$ set of reals,
(3) relations $\leq, \perp$ are $\Sigma^1_1$.

Let $\text{MA}(\text{Suslin})$ denote Martin’s Axiom for Suslin partial orders. It is well known that $\text{MA}(\text{Suslin})$ implies that many cardinal invariants, most notably additivity of measure, are equal to $2^{\aleph_0}$.

Notation used in this paper is standard. In particular, for $s, t \in 2^{<\omega}$, $[s] = \{x \in 2^{\omega} : x|\text{dom}(s) = s\}$ and $s \upharpoonright t$ denotes the concatenation of $s$ and $t$. For $A, B \subseteq \omega$ define $A \subseteq^* B$ if $|A \setminus B| < \aleph_0$.

To simplify notation, throughout this paper we will identify elements of $[\omega]^{\omega}$ with elements of $2^{\omega}$ via characteristic functions.

2. Consistency result

The goal of this paper is to show $\text{MA}(\text{Suslin})$ is consistent with $S = [\mathbb{R}]^{<\aleph_1}$. This is a generalization of a result from [5], where it was proved that $\text{MA}(\text{Suslin}) + s = \aleph_1$ is consistent.

Received by the editors December 11, 1995 and, in revised form, January 18, 1996.
1991 Mathematics Subject Classification. Primary 04A20.
Key words and phrases. Splitting family, cardinal invariants.
Research partially supported by NSF grant DMS 95-05375.

©1997 American Mathematical Society
**Theorem 2.1.** There exists a model of $V' \models ZFC$ such that:

1. $V' \models S = [\mathbb{R}]^{<\aleph_1}$,
2. $V' \models MA(Suslin) + 2^{\aleph_0} = \aleph_2$.

The rest of this section is devoted to the proof of this theorem.

We start with a definition of a forcing notion, due to Hechler (see [4]), that will be crucial for our construction.

Let $\text{Seq}^* \subseteq \omega^{<\omega}$ be the set of strictly increasing finite sequences. Let

$$D = \{(s, f) : s \in \text{Seq}^*, s \subseteq f, \text{ and } f \text{ is strictly increasing}\}.$$ 

For $(s, f), (t, g) \in D$ define

$$(s, f) \geq (t, g) \iff s \supseteq t \& \forall n \in \omega \ f(n) \geq g(n).$$

Define a rank function on $D$:

**Definition 2.2 ([2]).** Suppose that $D \subseteq D$ is a dense open set. For $s \in \text{Seq}^*$ define the rank of $s$ as follows:

1. $\text{rank}_D(s) = 0$ if there exists a function $f$ such that $(s, f) \in D$.
2. If $\text{rank}_D(s) \neq 0$, then

$$\text{rank}_D(s) = \min \left\{ \alpha : \exists m \ \exists \{s_k : k \in \omega\} \subseteq \text{Seq}^* \cap \omega^m \right\}.$$ 

**Lemma 2.3 ([2]; [1], Lemma 3.5.6).** For every $s \in \text{Seq}^*$, $\text{rank}_D(s)$ is defined. \hfill \square

Let $(\mathcal{P}_\alpha, \hat{\mathcal{Q}}_\alpha : \alpha < \omega_2)$ be a finite support iteration such that

1. $\Vdash_{\mathcal{P}_\alpha} \hat{\mathcal{Q}}_\alpha$ is Suslin,
2. if $\alpha$ is a limit ordinal then $\hat{\mathcal{Q}}_\alpha \simeq D$.

By careful bookkeeping we can ensure that $V^{\mathcal{P}_{<\omega}} \models MA(Suslin) + 2^{\aleph_0} = \aleph_2$.

We will show that $V^{\mathcal{P}_{<\omega}} \models S = [\mathbb{R}]^{<\aleph_1}$. The following construction is a modification of a construction from [3].

Suppose that $A \subseteq \omega$ is an infinite set. Let $A_-, A_+$ be two, canonically chosen, disjoint infinite sets such that $A_- \cup A_+ = A$. Assume that $g \in \omega^\omega$ is an increasing function such that $\text{range}(g) \cap A = \{x_n : n \in \omega\}$ is infinite ($x_n < x_{n+1}$ for all $n$).

Define a real $z_{A,g} \in 2^\omega$ as follows:

$$z_{A,g}(n) = \begin{cases} 1 & \text{if } x_n \in A_+ \\ 0 & \text{if } x_n \in A_- \end{cases} \text{ for } n \in \omega.$$ 

Fix a bijection $\ell : 2^{<\omega} \to \omega$ and for $x \in 2^\omega$ define $\ell(x) = \{\ell(x|n) : n \in \omega\}$. Define $S_g : \text{dom}(S_g) \to 2^\omega$ as

$$S_g(x) = z_{\ell(x),g} \text{ for } x \in 2^\omega.$$ 

The following lists some easy properties of the function defined above:

**Lemma 2.4.**

1. $\text{dom}(S_g)$ is a $G_\delta$ subset of $2^\omega$,
2. $S_g$ is continuous on its domain,
3. $S_g$ extends to a Borel function on $2^\omega$.

**Proof.**

1. Note that $\text{dom}(S_g) = \{x \in 2^\omega : |\text{range}(g) \cap \ell(x)| = \aleph_0\}$, which is a $G_\delta$ set (possibly empty).
2. (2) is easy to see and (3) is well-known. \hfill \square
Lemma 2.6. Every Luzin set is a splitting family.

Proof. Suppose that \( X \subseteq 2^\omega \) is a non-meager set. Since we identify elements of \([\omega]^{\omega}\) with elements of \(2^\omega\) via characteristic functions we can assume that \( X \subseteq [\omega]^{\omega}\). Let \( A \in [\omega]^{\omega}\). Consider the set

\[
F = \{ z \in [\omega]^{\omega} : z \subseteq^* (\omega \setminus A) \text{ or } A \subseteq^* z \}.
\]

It is easy to see that \( F \) is a meager set, and that any element of \( X \setminus F \) splits \( A \).

Lemma 2.7. Suppose that \( d \) is a \( D \)-generic real over \( V \). If \( Z \subseteq 2^\omega \cap V \) is uncountable then \( S_d(Z) \) is a Luzin set in \( V[d] \).

Proof. Observe first that by genericity \( V \cap 2^\omega \subseteq \text{dom}(S_d) \) and \( S_d \) is one-to-one on \( V \cap 2^\omega \). In particular, \( S_d(Z) \) is an uncountable set.

Suppose that \( F \in V[d] \) is a closed nowhere dense subset of \( 2^\omega \). To show that \( S_d(Z) \) is a Luzin set it is enough to show that \( S_d(Z) \cap F \) is countable. Let \( f \in (2^{<\omega})^\omega \cap V[d] \) be a function defined as follows:

\[
f(n) = \min \{ s \in 2^{<\omega} : \forall t \in 2^{<n} [t s] \cap F = \emptyset \}.
\]

(The minimum is taken with respect to some canonical enumeration of \( 2^{<\omega} \).) It is well-known that such an \( s \) exists.

Let \( \dot{f} \) be a \( D \)-name for \( f \) and define for \( n \in \omega \),

\[
D_n = \{ p \in D : \exists s \in 2^{<\omega} . p \Vdash \dot{f}(n) = s \}.
\]

Let \( N < H(\lambda) \) be a countable model containing \( \dot{f} \) and \( Z \), where \( \lambda \) is a sufficiently large regular cardinal.

Lemma 2.8. If \( x \in V \cap 2^\omega \) but \( x \notin N \cap 2^\omega \) then \( S_d(x) \notin F \).

Proof. Suppose not and let \( x \notin N \cap 2^\omega \) be a counterexample. Choose \( (s, g) \in D \) such that

\[
(s, g) \Vdash D_n \subseteq (S_d(x)) \subseteq \dot{F}.
\]

Let \( \bar{k} = |\text{range}(s) \cap \ell(x)| \). In other words, \( S_d(x)|\bar{k} \) is determined by \( (s, g) \). Let

\[
U = \{ t \in \text{Seq}^* : s \subseteq t \land |\text{range}(t) \cap \ell(x)| = \bar{k} \land \forall j \in \text{dom}(t) \setminus \text{dom}(s) . t(j) \geq g(j) \}.
\]

Lemma 2.9. \( \min \{ \text{rank}_D(t) : t \in U \} = 0 \).

Proof. Suppose that the lemma is not true and let \( t \in U \) be an element of minimal rank. By the definition there exists \( m \) and a sequence \( \{ t_j : j \in \omega \} \subseteq \text{Seq}^* \cap \omega^m \) such that for every \( j \):

1. \( t \subseteq t_j \),
2. \( \text{rank}_D(t_j) < \text{rank}_D(t) \),
3. \( t_j(|t|) > j \).

Fix \( i \) such that \( |t| \leq i < m \) and let \( W_i = \{ t_j(i) : j \in \omega \} \). Note that every subsequence of \( \{ t_j : k \in \omega \} \) witnesses that \( \text{rank}_D(t) > 0 \) as well. Thus, by passing to a subsequence we can assume that there is a set \( \ell(x_i) \) such that \( W_i \subseteq \ell(x_i) \) or \( W_i \subseteq \ell(x) \) is finite for all \( x \). In particular, if such a real \( x \) exists it is a member of \( N \).
Since \( x \notin N \), \( W_i \cap \ell(x) \) is finite for all \( |t| \leq i < m \). Therefore, there exists \( j \) such that \( \text{range}(t_j) \cap \ell(x) = \text{range}(t) \cap \ell(x) \). In particular, \( t_j \in U \) and \( \text{rank}_{D_k}(t_j) < \text{rank}_{D_k}(t) \), which is a contradiction. \( \square \)

Let \( t \in U \) be such that \( \text{rank}_{D_k}(t) = 0 \). There exists \( h \in \omega^\omega \) such that \( (t, h) \in D_k \). Therefore, \( (t, \max(h, g)) \geq (s, g) \) and \( (t, \max(h, g)) \) decides the value of \( \hat{f}(\bar{k}) \). Denote this value by \( \bar{s} \). However, \( (t, \max(h, g)) \) does not put any restrictions on values of \( S_d(x)(j) \) for \( j \geq \bar{k} \). Extend \( t \) to \( t' \) such that

\[
(t', \max(h, g)) \Vdash D \left( S_d(x)[\bar{k}] \right) \, \bar{s} \subseteq S_d(x).
\]

It is clear that

\[
(t', \max(h, g)) \Vdash D S_d(x) \not\subseteq \check{F}.
\]

This contradiction ends the proof of Lemma 2.7. \( \square \)

Let \( G \subseteq \mathcal{P}_{\omega_2} \) be a generic filter over \( V \). Suppose that \( Z \subseteq 2^\omega \cap V[G] \) is a set of cardinality \( \aleph_1 \). First we find a limit ordinal \( \alpha \) such that \( Z \subseteq V[G \cap \mathcal{P}_\alpha] \). We will work in the model \( V_1 = V[G \cap \mathcal{P}_{\alpha+1}] = V[G \cap \mathcal{P}_\alpha][d] \), where \( d \) is a \( D \)-generic real over \( V[G \cap \mathcal{P}_\alpha] \).

To finish the proof it is enough to show that \( S_d(Z) \) is a splitting family in \( V[G] \). Note however that \( S_d(Z) \) is not a Luzin set in \( V[G] \). In fact, \( S_d(Z) \) is meager in \( V[G] \).

**Lemma 2.10.** \( \{ S_d(x) : x \in Z \} \) is a splitting family in \( V[G] \).

**Proof.** We will work in \( V_1 \). By 2.7, we know that \( S_d(Z) \) is a Luzin set in \( V_1 \). Note that \( V[G] \) is a generic extension of \( V_1 \) via finite support iteration of Suslin forcings \( \mathcal{P}_{\alpha+1,\omega_2} \).

Let \( \dot{A} \) be a \( \mathcal{P}_{\alpha+1,\omega_2} \)-name for a set \( A \in [\omega]^\omega \). We will need the following lemma:

**Lemma 2.11.** For every \( p \in \mathcal{P}_{\alpha+1,\omega_2} \), the set

\[
Z_p = \left\{ z \in Z : p \Vdash_{\alpha+1,\omega_2} \text{"}S_d(z) \subseteq^* (\omega \setminus \dot{A}) \text{ or } \dot{A} \subseteq^* S_d(z) \text{"} \right\}
\]

is countable.

Before we prove the lemma notice that the theorem follows from it immediately – given \( p \in \mathcal{P}_{\alpha+1,\omega_2} \), \( \dot{A} \) and \( z \in Z \setminus Z_p \) we can find \( q \geq p \) such that \( q \Vdash \text{"}S_d(z) \text{ splits } \dot{A} \text{"} \).

**Proof of the lemma.** We will use the absoluteness properties of Suslin forcing (see [1] or [5]).

Fix a condition \( p \in \mathcal{P}_{\alpha+1,\omega_2} \). Let \( M \) be a countable elementary submodel of \( H(\lambda) \) containing \( \dot{A}, p \) and \( \mathcal{P}_{\alpha+1,\omega_2} \). Define a finite support iteration \( (\mathcal{P}_\alpha(M), \dot{Q}_\alpha(M) : \alpha < \omega_2) \) as follows:

\[
\mathcal{Q}_\alpha(M) = \begin{cases} \dot{Q}_\alpha & \text{if } \alpha \in M \\ \emptyset & \text{if } \alpha \notin M \end{cases} \text{ for } \alpha < \omega_2.
\]

Let \( \mathcal{P} = \lim \mathcal{P}_\alpha(M) \). \( \mathcal{P} \) is the part of the iteration that contains all information regarding \( \dot{A} \). \( \mathcal{P} \) is isomorphic to a countable iteration of Suslin forcings. In particular, \( \mathcal{P} \) has a definition that can be coded as a real number (essentially by encoding \( M \) as a real number).
From Suslinness it follows that $P \prec P_{\alpha+1, \omega_2}$ and that $\dot{A}$ is a $P$-name (see [1], Lemma 9.7.4, or [5]). Moreover, it is enough to show that

$$\left\{ z \in Z : p \Vdash \exists S_d(z) \subseteq^* (\omega \setminus \dot{A}) \text{ or } \dot{A} \subseteq^* S_d(z) \right\}$$

is countable.

Let $N < H(\lambda)$ be a countable model containing $M, \dot{A}$ and $P$. Since $S_d(Z)$ is a Luzin set in $V$, which will finish the proof. Fix $z \in Z \setminus Z_0$. In particular, for $z \in Z \setminus Z_0,$

$$P \nvdash S_d(z) \subseteq^* (\omega \setminus \dot{A}) \text{ or } \dot{A} \subseteq^* S_d(z),$$

which will finish the proof. Fix $z \in Z \setminus Z_0$ and let $Y = S_d(z)$ be a Cohen real over $N$. Without loss of generality we can assume that $P \nvdash Y \subseteq^* (\omega \setminus \dot{A}).$

Clearly, $N[Y][G \cap N[Y]] \models Y \subseteq^* (\omega \setminus \dot{A}[G \cap N[Y]])$ and therefore

$$N[Y] \models P \nvdash Y \subseteq^* (\omega \setminus \dot{A}),$$

since the last statement is absolute. Represent the Cohen algebra as $C = [\omega]^{<\omega}$ and let $\dot{Y}$ be the canonical name for a Cohen real. There is a condition $s \in C$ such that

$$N \models s \vdash C \text{ \text{“} } P \nvdash \dot{Y} \subseteq^* (\omega \setminus \dot{A}) \text{.”}$$

Let $Y' = s \cup \left( \omega \setminus \left( Y \setminus \text{max}(s) \right) \right)$. $Y'$ is also a a Cohen real over $N$ and since $s \subseteq Y'$ we get that $N[Y'] \models \text{“} P \nvdash Y' \subseteq^* (\omega \setminus \dot{A}) \text{.”}$ It follows that

$$N[Y'][G \cap N[Y']] \models Y' \subseteq^* (\omega \setminus \dot{A}[G \cap N[Y'])].$$

Note that $\dot{A}[G] = \dot{A}[G \cap N[Y']] = \dot{A}[G \cap N[Y]]$. Thus $V[G] \models Y \cup Y' \subseteq^* (\omega \setminus \dot{A}[G])$ which means that $\dot{A}[G]$ is finite. Contradiction.

The same argument shows that the assumption that $P \nvdash \dot{A} \subseteq^* Y$ leads to a contradiction. \hfill $\Box$

**Acknowledgement**

I would like to thank Andreas Blass for his helpful communication concerning the preparation of this paper.

**References**


Department of Mathematics, Boise State University, Boise, Idaho 83725

E-mail address: tomek@math.idbsu.edu