ON \( p \)-SUMMABLE SEQUENCES IN THE RANGE OF A VECTOR MEASURE

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Abstract. Let \( p > 2 \). Among other results, we prove that a Banach space \( X \) has the property that every sequence \( (x_n) \in \ell_p(X) \) lies inside the range of an \( X \)-valued measure if and only if, for all sequences \( (x^*_n) \in X^* \) satisfying that the operator \( x \in X \rightarrow \langle x, x^*_n \rangle \in \ell_1 \) is 1-summing, the operator \( x \in X \rightarrow \langle x, x^*_n \rangle \in \ell_q \) is nuclear, being \( q \) the conjugate number for \( p \). We also prove that, if \( X \) is an infinite-dimensional \( \mathcal{L}_p \)-space for \( 1 \leq p < 2 \), then \( X \) can’t have the above property for any \( s > 2 \).

Introduction

Let \( X \) be a Banach space and let \( p > 2 \). In [Pi2] it is proved that every sequence \( (x_n) \) in \( X \) satisfying \( \sum_n |\langle x_n, x^* \rangle|^p < +\infty \) for all \( x^* \in X^* \) lies inside the range of an \( X \)-valued measure with bounded variation if and only if \( X \) is finite-dimensional. On the other hand, in [AD] the authors proved that for every Banach space and for every sequence \( (x_n) \in \ell_2(X) \) there is an \( X \)-valued countably additive measure whose range contains \( (x_n) \).

The purpose of this paper is to characterize, given a real number \( p > 2 \), the Banach spaces \( X \) in which every sequence \( (x_n) \in \ell_p(X) \) lies inside the range of an \( X \)-valued countably additive measure.

Notation

We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [T] and [DU]. We only consider real Banach spaces. If \( X \) is a such space, \( B_X \) will denote its closed unit ball. The phrase “range of an \( X \)-valued measure” always means a set of the form \( rg(F) = \{ F(A) : A \in \Sigma \} \), where \( \Sigma \) is a \( \sigma \)-algebra of subsets of a set \( \Omega \) and \( F : \Sigma \rightarrow X \) is countably additive. We denote by \( \| F \| \) its total semivariation

\[
\| F \| = \sup \{ |x^* \circ F|(\Omega) : x^* \in B_{X^*} \}.
\]

Given \( p \geq 1 \), \( \ell^p_w(X) \) will denote the vector space of all sequences \( (x_n) \) in \( X \) such that \( \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < +\infty \) for all \( x^* \in X^* \). It is easy to see that if \( (x_n) \in \ell^p_w(X) \)

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then
\[ \epsilon_p((x_n)) = \sup\{\left(\sum_{n=1}^{\infty}|(x_n, x^*)|^p\right)^{\frac{1}{p}} : x^* \in BX^*\} < +\infty \]
and \( (\ell_w^p(X), \epsilon_p) \) is itself a Banach space.

If \( \hat{x} = (x_n) \in \ell_w^p(X) \) and \( P \) is a finite subset of \( \mathbb{N} \), \( \hat{x}(P) = (x_n(P)) \) is the sequence defined by
\[ x_n(P) = \begin{cases} x_n, & \text{for } n \in P, \\ 0, & \text{for } n \notin P \end{cases} \]
for all \( n \in \mathbb{N} \). \( \ell_w^p(X) \) will denote the subspace of \( \ell_w^p(X) \) consisting of the sequences \( \hat{x} = (x_n) \) such that the net \( (\hat{x}(P))_{P \in \mathcal{F}(\mathbb{N})} \) converges to \( (x_n) \) in \( \ell_w^p(X) \), where \( \mathcal{F}(\mathbb{N}) \) is the set of all finite subsets of \( \mathbb{N} \). Recall that \( \ell_w^1(X) \) is formed by the unconditionally summable sequences in \( X \). We need the following Proposition that lists some privileges that membership in \( \ell_w^p(X) \) entails.

**Proposition A.** Let \( p > 1 \) and \( X \) a Banach space. The following statements are equivalent:

(i) \( (x_n) \in \ell_w^p(X) \).
(ii) The series \( \sum_{n=1}^{\infty} \alpha_n x_n \) converges unconditionally for every sequence \( (\alpha_n) \in \ell_q \).
(iii) The map \( (\alpha_n) \in \ell_q \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X \) defines a bounded operator.

Recall that an operator \( T : X \rightarrow Y \) is said to be \( \infty \)-nuclear if there are sequences \( (x_n^*) \) in \( X^* \) and \( (y_n) \) in \( Y \) such that
\[ Tx = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n \quad (\forall x \in X), \quad \lim_{n \to \infty} x_n^* = 0 \quad \text{and} \quad \epsilon_1((y_n)) < +\infty. \]
A norm is then defined by taking the infimum of all admissible products
\[ (\sup_n \|x_n^*\|) \epsilon_1((y_n)). \]

\( \mathcal{N}_\infty(X, Y) \) will denote the Banach space of all \( \infty \)-nuclear operators from \( X \) into \( Y \).

**The spaces \( \mathcal{R}(X) \) and \( \mathcal{R}_c(X) \)**

We denote by \( \mathcal{R}(X) \) the vector space of all sequences \( (x_n) \) in \( X \) so that there exists an \( X \)-valued measure \( F \) satisfying
\[ \{x_n : n \in \mathbb{N}\} \subset \text{rg}(F). \]
If \( (x_n) \) belongs to \( \mathcal{R}(X) \), we put \( \| (x_n) \| = \inf \|F\| \), where the infimum is taken over all vector measures \( F \) admissible in (1). Obviously, we have
\[ \| (x_n) \|_\infty \leq \| (x_n) \| \quad \text{for} \quad \text{all} \quad (x_n) \in \mathcal{R}(X). \]

Using a direct sum of vector measures [KK, p.35], it is easy to prove that any absolutely summable series in \( \mathcal{R}(X) \) is convergent. So, \( (\mathcal{R}(X), \| \cdot \|) \) is a Banach space.

Next we are going to consider sequences in \( X \) that lie inside the range of a vector measure with relatively compact range. We denote by \( \mathcal{R}_c(X) \) the vector space of all such sequences \( (x_n) \) in \( X \). By [PR, Proposition 1.4], if \( (x_n) \) belongs to \( \mathcal{R}_c(X) \), there
exists an unconditionally convergent series \( \sum_m y_m \) satisfying \( x_n \in \sum_m [-y_m, y_m] \) for all \( n \). The set \( \sum_m [-y_m, y_m] \) is the range of a vector measure \( F \) for which

\[
\|F\| \leq 2 \sup \{ \sum_{m=1}^{\infty} |\langle y_m, x^* \rangle| : x^* \in B_X \}.
\]

If \( (x_n) \) belongs to \( R_c(X) \), we set

\[
\|(x_n)\|_c = \inf \epsilon_1((y_m)),
\]

the infimum being taken over all unconditionally convergent series \( \sum_m y_m \) such that \( (x_n) \) is contained in \( \sum_m [-y_m, y_m] \). Obviously, we have

\[
\|(x_n)\|_c \leq \|(x_n)\| \leq 2 \|(x_n)\|_c \text{ for all } (x_n) \in R_c(X).
\]

The next Proposition proves that \( R_c(X) \) can be isometrically identified to \( N_\infty(\ell_1, X) \).

**Proposition 1.** The Banach spaces \( R_c(X) \) and \( N_\infty(\ell_1, X) \) are isometric.

**Proof.** If \( (x_n) \in R_c(X) \), we define an operator \( T : \ell_1 \to X \) by \( T(\alpha_n) = \sum_n \alpha_n x_n \) for all \( (\alpha_n) \in \ell_1 \). Given \( \epsilon > 0 \), choose an unconditionally summable sequence \( (y_n) \) in \( X \) so that

\[
x_n = \sum_{m=1}^{\infty} \alpha_m^n y_n \text{ and } \epsilon_1((y_m)) < \epsilon + \|(x_n)\|_c,
\]

where \( |\alpha_m^n| \leq 1 \) for all \( n, m \in \mathbb{N} \). Obviously, \( T\alpha = \sum_m \langle \alpha, (\alpha_m^n) \rangle y_m \).

**Claim.** There exist \( (\lambda_m) \in B_{c_0} \) and \( (z_m) \in \ell_1^p(X) \), such that \( y_m = \lambda_m z_m \) for all \( m \in \mathbb{N} \) and \( \epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m)) \).

To prove the claim, choose \( p_o \in \mathbb{N} \) so that

\[
\sum_{p \geq p_o} \frac{1}{p^2} < \epsilon.
\]

Since the sequence \( (y_m) \) is unconditionally summable, we have

\[
\lim_{m \to \infty} \sup \{ \sum_{k=m}^{\infty} |\langle y_m, x^* \rangle| : \|x^*\| \leq 1 \} = 0.
\]

So we can determine a strictly increasing sequence of integers \( (n_p)_p \), such that

\[
\sup \{ \sum_{m \geq n_p} |\langle y_m, x^* \rangle| : \|x^*\| \leq 1 \} < \frac{1}{p^2}
\]

for all \( p \in \mathbb{N} \). Let \( (\lambda_m) \) the sequence defined by

\[
\lambda_m = \begin{cases} 
1, & \text{for } m \leq n_{p_o}, \\
\frac{1}{p}, & \text{for } n_p < m \leq n_{p+1}.
\end{cases}
\]

and put \( z_m = (\lambda_m)^{-1} y_m \) for all \( m \in \mathbb{N} \). It is obvious that \( (\lambda_m) \in B_{c_0} \). Hence, if we prove the inequality \( \epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m)) \), the claim will be established. Given
Let \( \epsilon > 0 \) for all \( m \in \mathbb{N} \). Therefore, we have prove that \( \nu_\infty(T) \leq \| (x_n) \|_c \) for all sequences \( (x_n) \) belonging to \( R_c(X) \).

Conversely, let \( (x_n) \) a sequence in \( X \) such that the operator

\[
T : (\alpha_n) \in \ell_1 \rightarrow \sum_{n=1}^\infty \alpha_n x_n \in X
\]

is \( \infty \)-nuclear. Given \( \epsilon > 0 \), choose a null sequence \( (\beta_n) \) in \( \ell_\infty \) and \( (y_m) \in \ell_1^w(X) \) so that

\[
T\alpha = \sum_{m=1}^\infty (\alpha, \beta_m) y_m \quad \text{for all } \alpha \in \ell_1
\]

and

\[
\left( \sup_m \| (y_n) \| \right) \epsilon_1((y_m)) < \nu_\infty(T) + \epsilon.
\]

Since the sequence \( (\| \beta_n \|_\infty, y_m) \) is unconditionally summable, it follows easily that \( (x_n) \) belongs to \( R_c(X) \) and

\[
\| (x_n) \|_c < \epsilon + \nu_\infty(T),
\]

for all \( \epsilon > 0 \). This concludes the proof.

It is well-known that with the trace duality \( \mathcal{N}_\infty(X,Y)^* \) and \( \Pi_1(Y, X^{**}) \) can be isometrically identified, if \( X^* \) has the approximation property. In fact, for any \( S \in \Pi_1(Y, X^{**}) \) the map \( \phi_S : \mathcal{N}_\infty(X,Y) \rightarrow \mathbb{R} \) defined by \( \phi_S(T) = \text{tr}(S \circ T) \) is a continuous linear form on \( \mathcal{N}_\infty(X,Y) \) and every linear form on \( \mathcal{N}_\infty(X,Y) \) can be obtained in this way. Accordingly, \( R_c(X)^* \) and \( \Pi_1(X, \ell_1^{*}) \) are isometric. The next Proposition shows that every operator \( T \in \Pi_1(X, \ell_1^1) \) also defines a continuous linear form on \( \mathcal{R}(X) \).

**Proposition 2.** Let \( X \) be a Banach space. If \( (x_n^*) \) is a sequence in \( X^* \) so that the operator \( S : x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1 \) is 1-summing, then the linear map \( \psi_S : \mathcal{R}(X) \rightarrow \mathbb{R} \) defined by \( \psi_S((x_n)) = \sum_n \langle x_n, x_n^* \rangle \) for all \( (x_n) \in \mathcal{R}(X) \) is well-defined, continuous and \( \| \psi_S \| \leq \pi_1(S) \).
Proof. Let \((x_n) \in \mathcal{R}(X)\). Given \(\epsilon > 0\), choose a vector measure \(F : \Sigma \to X\) so that
\[
\{x_n : n \in \mathbb{N}\} \subset \text{rg}(F) \quad \text{and} \quad \|F\| < \epsilon + \|(x_n)\|.
\]
Let \(\mu\) a control measure for \(F\) and consider the integration operator
\[
I : f \in L^\infty(\mu) \to \int f dF \in X.
\]
The operator \(S \circ I\) is 1-summing and, hence, integral. Since \(\ell_1\) has the Radon-Nikodym property, it follows that \(S \circ I\) is nuclear and \(\pi_1(S \circ I) = \nu_1(S \circ I)\) (see [DU, p. 174]). So is \((S \circ I)^* : \ell_\infty \to L^\infty(\mu)\). Then
\[
\sum_{n=1}^{\infty} \|(S \circ I)^*(e_n)\| < +\infty,
\]
where \((e_n)\) is the unit basis of \(c_0\). On the other hand, we have
\[
\langle (S \circ I)f, e_n \rangle = \langle f, (S \circ I)^*(e_n) \rangle.
\]
Then
\[
\|(S \circ I)^*(e_n)\| = \sup\{\|f, (S \circ I)^*(e_n)\| : \|f\|_{\infty} \leq 1\} = \sup\{\int f d(x_n^* \circ F) : \|f\|_{\infty} \leq 1\}.
\]
This shows that (4) can be written in the form
\[
\sum_{n=1}^{\infty} \sup\{\int f d(x_n^* \circ F) : \|f\|_{\infty} \leq 1\} < +\infty.
\]
For every \(n \in \mathbb{N}\), we choose \(A_n \in \Sigma\) such that \(F(A_n) = x_n\). Then, it follows from (5) that
\[
\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle = \sum_{n=1}^{\infty} \int_{A_n} d(x_n^* \circ F) | < +\infty.
\]
So, the linear form \(\psi_S\) is well-defined. To conclude the proof we need to show that
\[
\sup\{\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle : \|(x_n)\| \leq 1\} \leq \pi_1(S).
\]
If \(B\) denotes the restriction map of \((S \circ I)^*\) to \(c_0\), we have
\[
\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle \leq \sum_{n=1}^{\infty} \|(S \circ I)^*(e_n)\| = \nu_1(B) \leq \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) = \pi_1(S).
\]
Since \(S \circ F\) is the representing measure of an 1-summing operator \(S \circ I\), it follows from [DU, p. 162] that \(\pi_1(S \circ I) = |S \circ F|\) (here, \(|S \circ F|\) denotes the total variation of \(S \circ F\)). Finally, we can deduce from (7) that
\[
\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle \leq |S \circ F| \leq \pi_1(S) \|F\| < \pi_1(S) (\epsilon + \|(x_n)\|),
\]
for all \(\epsilon > 0\). Therefore, \(\|\psi_S\| \leq \pi_1(S)\).
\(\square\)
**P-summable sequences in the range of a vector measure**

**Proposition 3.** Let $X$ be a Banach space and let $p > 2$. $X$ has the property that every sequence $(x_n) \in \ell_p^n(X)$ lies inside the range of a vector measure if and only if there exists a constant $c > 0$ so that, for all finite sets $\{x_1, ..., x_n\} \subset X$ satisfying $\epsilon_p((x_i)_n^{i=1}) \leq 1$, there is a vector measure $F : \Sigma \to X$ satisfying

$$\|F\| \leq c \text{ and } \{x_1, ..., x_n\} \subset \text{rg}(F).$$

**Proof.** First, proceeding by contradiction, suppose $\ell_p^n(X) \subset \mathcal{R}(X)$ but there isn’t a such constant. Then there would exist a sequence $(H_n)$ of finite subsets of $X$ such that for every $n$ the conditions

$$H_n \subset \text{rg}(F) \text{ and } \sup\{|(x, x^*)|^p : \|x^*\| \leq 1\} \leq 1$$

would imply $\|F\| \geq n^2$. If $H_n = \{x_1^n, ..., x_k(n)\}$, the sequence

$$\{x_1^n, ..., x_k(n) : 1/2 x_1^n, ..., 1/2 x_2(n), ..., 1/n x_1^n, ..., 1/n x_k(n), \ldots\}$$

belongs to $\ell_p^n(X)$. So, there exists a vector measure $F$ with $(1/n)H_n \subset \text{rg}(F)$ for all $n \in \mathbb{N}$. Thus $H_n \subset \text{rg}(nF)$ and this would yield $\|F\| \geq n$ for every $n$, a contradiction with the fact that $F$ has bounded semivariation since it is countably additive.

Conversely, let $c > 0$ be a constant such that, for every finite set $\{x_1, ..., x_n\} \subset X$ satisfying $\epsilon_p((x_i)_n^{i=1}) \leq 1$, there exists a vector measure $F$ with $\|F\| \leq c$ and whose range contains $\{x_1, ..., x_n\}$. Then, the natural map from $(X^{(n)}, \epsilon_p)$ to $(\mathcal{R}(X), \|\cdot\|)$ is linear and continuous. Therefore, it has a unique continuous linear extension to $\ell_p^n(X)$.

Now we are ready to face our main result.

**Theorem 4.** Let $X$ be a Banach space and let $p > 2$. The following are equivalent:

(i) Every sequence $(x_n) \in \ell_p^n(X)$ lies inside the range of a vector measure.

(ii) For all sequence $(x_n^n)$ in $X^*$ satisfying that $x \in X \to (\langle x, x_n^n \rangle) \in \ell_1$ is 1-summing, the operator $x \in X \to (\langle x, x_n^n \rangle) \in \ell_q$ is nuclear.

(iii) There is a constant $c > 0$ such that, for all $n \in \mathbb{N}$, for all $\{x_1, ..., x_n\} \in X$ and all $\{x_1^n, ..., x_n^n\}$ in $X^*$, we have

$$\sum_{i=1}^n |\langle x_i, x_i^n \rangle| \leq c \sum_{i=1}^n x_i^n \otimes e_i : X \to \ell_p^n \epsilon_p((x_i)_n^{i=1}).$$

**Proof.** (i) $\Rightarrow$ (ii) We have seen in the proof of Proposition 3 that the natural map $\ell_p^n(X) \to \mathcal{R}(X)$ is continuous. So, given a sequence $(x_n^n)$ in $X^*$ such that the map

$$x \in X \to (\langle x, x_n^n \rangle) \in \ell_1$$

is 1-summing, it follows from Proposition 2 that the linear form $\phi$ on $\ell_p^n(X)$ defined by $\phi(x_n) = \sum_n \langle x_n, x_n^n \rangle$ is 1-summing. Then the linear map $x \in X \to (\langle x, x_n^n \rangle) \in \ell_q$ is integral [DU, p. 232]. Now recall that nuclear and integral operators into a reflexive space are the same [DU].

(ii) $\Rightarrow$ (iii) The linear map

$$T \in \Pi_1(X, \ell_1) \to I_{1q} \circ T \in \mathcal{N}(X, \ell_q)$$


has closed graph (here \(I_{1q}\) is the inclusion map from \(\ell_1\) into \(\ell_q\)). So, it is continuous. Then there is a constant \(c > 0\) such that

\[
\nu_1(I_{1q}^n \circ T) \leq c \pi_1(T)
\]

for all \(n \in \mathbb{N}\) and all \(T \in \mathcal{L}(X, \ell_1^n)\), being \(I_{1q}^n : \ell_1^n \rightarrow \ell_q^n\) the identity map. Now, given \(\{x_1, \ldots, x_n\} \in X\) and \(\{x_1^*, \ldots, x_n^*\}\) in \(X^*\), we consider the operators \(u : X \rightarrow \ell_q^n\) and \(v : \ell_q^n \rightarrow X\) defined by

\[
u_i(u) = \sum_{i=1}^n \alpha_i x_i.
\]

This yields (iii) because of the equality \(\|U\| = \nu_1^n = \nu_1^n\)

(iii) \(\Rightarrow\) (i) For every \(n \in \mathbb{N}\), we define a linear map

\[
U_n : (X^* \circ \nu_1^n, \pi_1^n) \rightarrow \mathcal{N}_\infty(\ell_1^n, X)
\]

in the following way: Given \(\hat{x} = (x_i)_{i=1}^n \in X^n\), we denote by \(T_{\hat{x}}^n\) the operator from \(\ell_1^n\) into \(X\) defined by \(T_{\hat{x}}^n((\alpha_i)^n) = \sum_{i=1}^n \alpha_i x_i\). Then we put \(U_n(\hat{x}) = T_{\hat{x}}^n\). To show that \(U_n\) is continuous, we are going to compute \(\nu_\infty^n(T_{\hat{x}}):\)

\[
\nu_\infty^n(T_{\hat{x}}) = \sup\{|(T_{\hat{x}}, S)| : \pi_1^n(S) \leq 1, S \in \Pi(X, \ell_1^n)\}
\]

\[
= \sup\{|\text{tr}(S \circ T_{\hat{x}})| : \pi_1^n(S) \leq 1, S \in \Pi(X, \ell_1^n)\}
\]

\[
= \sup\{|\sum_{i=1}^n \langle x_i, S^*(\epsilon_i^*) \rangle| : \pi_1^n(S) \leq 1, S \in \Pi(X, \ell_1^n)\},
\]

where \((\epsilon_i^*)^n\) is the unit basis of \(\ell_\infty^n\). By (iii), we obtain

\[
\nu_\infty^n(T_{\hat{x}}) \leq c \epsilon_p((\epsilon_i^*)^n).
\]

So, we have

\[
(9) \quad \|U_n\| \leq c \quad \text{for every} \quad n \in \mathbb{N}.
\]

Now we consider the linear map

\[
U : (X^{\mathbb{N}} \circ \epsilon_p, \mathcal{N}_\infty(\ell_1, X)
\]

defined by \(U(\hat{x}) = T_{\hat{x}}^n\) with \(T_{\hat{x}}^n((\alpha_n)) = \sum_n \alpha_n x_n\), for all \(\hat{x} = (x_n) \in X^{\mathbb{N}}\) and all \((\alpha_n) \in \ell_1^n\). From (9) it follows that \(\|U\| \leq c\). Finally, Proposition 1 implies that \(\ell_\infty^n(X)\) is contained in \(\mathcal{R}_c(X)\).

Using the methods introduced in [Pi2], we can prove that an operator \(T : X \rightarrow Y\) takes every sequence \((x_n) \in \ell_\infty^n(X)\) into a sequence \((T x_n)\) lying in the range of an \(X\)-valued measure with bounded variation if and only if \(T^*: X^* \rightarrow X^*\) is \((1, 1, p)\)-summing. The next Proposition shows the relationship between these ideas and our problem.
Proposition 5. \( \ell^p_n(X) \) is contained in \( \mathcal{R}(X) \) if and only if \( T^*: Y^* \to X^* \) is \((1,1,p)\)-summing for all Banach space \( Y \) and for all \( T \in \Pi_1(X,Y) \).

Proof. Suppose \( X \) is a Banach space such that \( \ell^p_n(X) \subset \mathcal{R}(X) \). If \( T \in \Pi_1(X,Y) \), then \( T \) takes every sequence \( (x_n) \in \ell^p_n(X) \) into a sequence \( (Tx_n) \) lying in the range of a vector measure of bounded variation. So, \( T^*: Y^* \to X^* \) is \((1,1,p)\)-summing.

Conversely, let \( X \) be a Banach space with the property that, for all \( T \in \Pi_1(X,\ell_1) \), the operator \( T^* \) is \((1,1,p)\)-summing. Take a sequence \( (x^n) \) in \( X^* \) such that \( T: x \in X \to (\langle x, x^n \rangle) \in \ell_1 \) is 1-summing. By hypothesis, \( T^* \) is \((1,1,p)\)-summing. Now an appeal to [Pi2, Theorem 1] yields the nuclearity of the operator

\[
x^{**} \in X^{**} \to (\langle x^{**}, T^*e^n \rangle) \in \ell_q.
\]

Then \( x \in X \to (\langle x, x^n \rangle) \in \ell_q \) is also nuclear. Now Theorem 4 tells us that \( \ell^p_n(X) \subset \mathcal{R}(X) \). \( \square \)

Examples

If \( X \) is a Banach space, \( s(X) \) will denote the set of all real numbers \( s > 2 \) such that every sequence \( (x_n) \in \ell^p_n(X) \) lies inside the range of an \( X \)-valued measure. Obviously, \( s(X) \) is an interval whose bounds are 2 and \( \sup(s(X)) \). In this section we find out the form of the set \( s(X) \) for any \( L_p \)-space \( X \) (1 \( \leq \) \( p \) \( \leq \) \( +\infty \)).

a) If \( X \) is an \( L_p \)-space for \( p \geq 2 \), then \( s(X) = (2, +\infty) \).

By [LR, Theorem III], \( X^* \) is an \( L_q \)-space. So \( X^* \) is isomorphic to a subspace of an \( L_1(\mu) \)-space for some measure \( \mu \) [LP, Corollary 7.2]. Hence, every null sequence in \( X \) lies inside the range of an \( X \)-valued measure [PR].

Following [Ps2] we will say that a Banach space \( X \) satisfies Grothendieck’s Theorem (in short, \( X \) is a G. T. space) if \( \mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2) \). For some time the \( \mathcal{L}_1 \)-spaces remained the only known G. T. spaces. We have just seen that \( s(X) = (2, +\infty) \) for every Banach space \( X \) whose dual \( X^* \) is an \( L_1 \)-space. Nevertheless, there are spaces for which \( s(X) = \emptyset \) and \( X^* \) is a G. T. space. Next we will prove this assertion.

b) If \( X \) is an infinite-dimensional Banach space of cotype 2 and \( X^* \) is a G. T. space, then \( s(X) = \emptyset \).

Since \( X \) is a space of cotype 2 we have the identity \( \Pi_1(X, \ell_1) = \Pi_2(X, \ell_1) \) [Ps2, Theorem 5.16]. By [Ps2, Proposition 6.2] \( \Pi_2(X, \ell_1) = \mathcal{L}(X, \ell_1) \). So, it follows that

\[
\pi_1(u) \leq c \|u\| \quad \text{for all } u \in \mathcal{L}(X, \ell_1)
\]

and for some constant \( c > 0 \). Proceeding by contradiction, suppose \( p \in s(X) \). Then there exists a constant \( c' > 0 \) so that

\[
\sum_{i=1}^{n} |\langle x_i, x_i^* \rangle| \leq c' \|x_i\|_1 \quad \text{for all } \{x_i\}_{i=1}^{n} \in X \text{ and all } \{x_i^*\}_{i=1}^{n} \in X^*.
\]

(10) and (11) yield

\[
\sum_{i=1}^{n} |\langle x_i, x_i^* \rangle| \leq c' e_p((x_i^*{\|n\|}_1) |e_1((x_i^*{\|n\|}_1
\]

(12)

for all \( \{x_i\}_{i=1}^{n} \in X \) and all \( \{x_i^*\}_{i=1}^{n} \in X^* \). This proves that the identity map \( I_X \) is \((1,1,p)\)-summing [P]. Hence \( I_X^* \) is \((1,1,p)\)-summing [P, Theorem 17.1.5]. Since \( p \geq 2 \), from [Pi2] it follows that \( X^* \) is finite-dimensional, a contradiction which completes the proof.
In [Ps1, Theorem 3.2], it is proved that any Banach space $E$ of cotype 2 can be imbedded isometrically into a Banach space $X$ such that $X$ and $X^*$ are both of cotype 2 and both verify Grothendieck’s Theorem. So, if $X$ is the Banach space associated to $E = \ell_2$ in [Ps1, Theorem 3.2], for every $p > 2$ there are $p$-summable sequences in $X$ that do not lie inside the range of a $X$-valued vector measure. Nevertheless, the closed unit ball of $\ell_2$ is the range of a vector measure [AD].

c) $s(\ell_1) = \emptyset$.

In [AD] it is proved that the sequence $(\epsilon_n/\sqrt{n})$ does not lie inside the range of a measure. Nevertheless, $(\epsilon_n/\sqrt{n})$ belongs to $\ell_p^*(\ell_1)$ for any $s > 2$.

d) $s(\ell_p) = \emptyset$ for any $1 < p < 2$.

Given $s \in (2, q)$, we choose $\alpha \in (1/s - 1/q, 1/2 - 1/q)$. We are going to prove that the sequence $(\epsilon_n/n^\alpha)$ belongs to $\ell_p^*(\ell_p)$ but it isn’t in the range of a measure. If $x^* = (x^*_n) \in \ell_q$ with $\|x^*\|_q \leq 1$, we have for all $m \in \mathbb{N}$:

$$\sum_{n=m}^{\infty} |(\epsilon_n/n^\alpha, x^*)|^s = \sum_{n=m}^{\infty} \frac{1}{n^{s\alpha}} |x^*_n|^s \leq \left( \sum_{n=m}^{\infty} \frac{1}{n^{s\alpha}} \right)^{\frac{s}{q}} \left( \sum_{n=m}^{\infty} |x^*_n|^q \right)^{\frac{q}{s}}$$

where we have applied Holder’s inequality. Then

$$\sup \{ \sum_{n=m}^{\infty} |(\epsilon_n/n^\alpha, x^*)|^s : \|x^*\|_q \leq 1 \} \leq \left( \sum_{n=m}^{\infty} \frac{1}{n^{s\alpha}} \right)^{\frac{s}{q}}$$

for all $m \in \mathbb{N}$. Since $s\alpha q/(q - s) > 1$, it follows that $(\epsilon_n/n^\alpha)$ is plainly in $\ell_p^*(\ell_p)$. The sequence $(\epsilon_n/n^\alpha)$ is an unconditional basis of $\ell_p$; therefore, to prove that the sequence $(\epsilon_n/n^\alpha)$ is not in the range of a measure it suffices to show that it does not belong to $\ell_p^*(\ell_p)$ [AD, Theorem 5]. For this, choose $\beta > (1/q)$ and consider the scalar sequence $x^* = (1/n^\beta)$. It is plain that $x^* \in \ell_q$ and that viewed as a member of $(\ell_p)^*$, $x^*$ satisfies:

$$\sum_n |(\epsilon_n/n^\alpha, x^*)|^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2(\alpha + \beta)}}.$$ 

Therefore, if we choose $\beta \in (1/q, 1/2 - \alpha)$, the last series is divergent and we deduce that $s \notin s(\ell_p)$.

e) If $X$ is an infinite-dimensional $\mathcal{L}_p$-space for $1 \leq p < 2$, then $s(X) = \emptyset$.

By [LP, Proposition 7.3], $X$ has a complemented subspace $H$ isomorphic to $\ell_p$. Then $s(X) \subset s(H) = s(\ell_p) = \emptyset$.

In view of these results and [PR] a question arises: Are there Banach spaces $X$ such that $s(X) \neq \emptyset$ but $X^*$ is not isomorphic to a subspace of an $L_1(\mu)$-space for some measure $\mu$?

References


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