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ON p-SUMMABLE SEQUENCES IN THE RANGE OF A VECTOR MEASURE

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ABSTRACT. Let $p > 2$. Among other results, we prove that a Banach space X has the property that every sequence $(x_n) \in \ell_u^p(X)$ lies inside the range of an X-valued measure if and only if, for all sequences (x_n^*) in X^* satisfying that the operator $x \in X \to (\langle x, x_n^* \rangle) \in \ell_1$ is 1-summing, the operator $x \in X \to (\langle x, x_n^* \rangle) \in \ell_q$ is nuclear, being q the conjugate number for p. We also prove that, if X is an infinite-dimensional $\mathcal{L}_p$-space for $1 \leq p < 2$, then X can't have the above property for any $s > 2$.

INTRODUCTION

Let X be a Banach space and let $p > 2$. In [Pi2] it is proved that every sequence (x_n) in X satisfying $\sum_n |\langle x_n, x^\star \rangle|^p < +\infty$ for all $x^\star \in X^\star$ lies inside the range of an X-valued measure with bounded variation if and only if X is finite-dimensional. On the other hand, in [AD] the authors proved that for every Banach space and for every sequence $(x_n) \in \ell_w^2(X)$ there is an X-valued countably additive measure whose range contains (x_n).

The purpose of this paper is to characterize, given a real number $p > 2$, the Banach spaces X in which every sequence $(x_n) \in \ell_u^p(X)$ lies inside the range of an X-valued countably additive measure.

NOTATION

We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [T] and [DU]. We only consider real Banach spaces. If X is a such space, B_X will denote its closed unit ball. The phrase "range of an X-valued measure" always means a set of the form $rg(F) = \{F(A) : A \in \Sigma\}$, where Σ is a σ-algebra of subsets of a set Ω and $F : \Sigma \to X$ is countably additive. We denote by $\|F\|$ its total semivariation

$$\|F\| = \sup\{|x^* \circ F|(\Omega) : x^* \in B_{X^*}\}.$$

Given $p \geq 1$, $\ell_w^p(X)$ will denote the vector space of all sequences (x_n) in X such that $\sum_{n=1}^{\infty} |\langle x_n, x^\star \rangle|^p < +\infty$ for all $x^\star \in X^\star$. It is easy to see that if $(x_n) \in \ell_w^p(X)$

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then

$$\epsilon_p((x_n)) = \sup\{\big(\sum_{n=1}^{\infty} |\langle x_n, x^\star\rangle|^p\big)^{\frac{1}{p}} : x^* \in B_{X^*}\} < +\infty$$

and $\big(\ell_w^p(X), \epsilon_p\big)$ is itself a Banach space.

If $\widehat{x} = (x_n) \in \ell_w^p(X)$ and P is a finite subset of $\mathbb{N}$, $\widehat{x}(P) = (x_n(P))$ is the sequence defined by

$$x_n(P) = \begin{cases} x_n, & \text{for } n \in P, \\ 0, & \text{for } n \notin P \end{cases}$$

for all $n \in \mathbb{N}$. $\ell_u^p(X)$ will denote the subspace of $\ell_w^p(X)$ consisting of the sequences $\widehat{x} = (x_n)$ such that the net $(\widehat{x}(P))_{P\in\mathcal{F}(\mathbb{N})}$ converges to (x_n) in $\ell_w^p(X)$, where $\mathcal{F}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$. Recall that $\ell_u^1(X)$ is formed by the unconditionally summable sequences in X. We need the following Proposition that lists some privileges that membership in $\ell_w^p(X)$ entails.

Proposition A. *Let $p > 1$ and X a Banach space. The following statements are equivalent:*

(i) $(x_n) \in \ell_w^p(X)$.
(ii) *The series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges unconditionally for every sequence $(\alpha_n) \in \ell_q$.*
(iii) *The map $(\alpha_n) \in \ell_q \to \sum_{n=1}^{\infty} \alpha_n x_n \in X$ defines a bounded operator.*

Recall that an operator $T : X \to Y$ is said to be ∞-nuclear if there are sequences (x_n^*) in X^* and (y_n) in Y such that

$$Tx = \sum_{n=1}^{\infty} \langle x, x_n^*\rangle y_n \ (\forall x \in X), \quad \lim_{n\to\infty} x_n^* = 0 \ \text{ and } \ \epsilon_1((y_n)) < +\infty.$$

A norm is then defined by taking the infimum of all admissible products

$$(\sup_n \|x_n^*\|)\ \epsilon_1((y_n)).$$

$\mathcal{N}_\infty(X, Y)$ will denote the Banach space of all ∞-nuclear operators from X into Y.

The spaces $\mathcal{R}(X)$ and $\mathcal{R}_c(X)$

We denote by $\mathcal{R}(X)$ the vector space of all sequences (x_n) in X so that there exists an X-valued measure F satisfying

(1) $$\{x_n : n \in \mathbb{N}\} \subset \text{rg}(F).$$

If (x_n) belongs to $\mathcal{R}(X)$, we put $\|(x_n)\| = \inf \|F\|$, where the infimum is taken over all vector measures F admissible in (1). Obviously, we have

(2) $$\|(x_n)\|_\infty \leq \|(x_n)\| \ \text{ for all } \ (x_n) \in \mathcal{R}(X).$$

Using a direct sum of vector measures [KK, p.35], it is easy to prove that any absolutely summable series in $\mathcal{R}(X)$ is convergent. So, $(\mathcal{R}(X), \|\cdot\|)$ is a Banach space.

Next we are going to consider sequences in X that lie inside the range of a vector measure with relatively compact range. We denote by $\mathcal{R}_c(X)$ the vector space of all such sequences (x_n) in X. By [PR, Proposition 1.4], if (x_n) belongs to $\mathcal{R}_c(X)$, there exists an unconditionally convergent series $\sum_m y_m$ satisfying $x_n \in \sum_m[-y_m, y_m]$ for all n. The set $\sum_m[-y_m, y_m]$ is the range of a vector measure F for which

$$\|F\| \leq 2 \sup\{\sum_{m=1}^{\infty} |\langle y_m, x^*\rangle| : x^* \in B_{X^*}\}.$$

If (x_n) belongs to $\mathcal{R}_c(X)$, we set

$$\|(x_n)\|_c = \inf \epsilon_1((y_m)),$$

the infimum being taken over all unconditionally convergent series $\sum_m y_m$ such that (x_n) is contained in $\sum_m[-y_m, y_m]$. Obviously, we have

$$\|(x_n)\|_\infty \leq \|(x_n)\| \leq 2\,\|(x_n)\|_c \text{ for all } (x_n) \in \mathcal{R}_c(X). \tag{3}$$

The next Proposition proves that $\mathcal{R}_c(X)$ can be isometrically identified to $\mathcal{N}_\infty(\ell_1, X)$.

Proposition 1. *The Banach spaces $R_c(X)$ and $\mathcal{N}_\infty(\ell_1, X)$ are isometric.*

Proof. If $(x_n) \in R_c(X)$, we define an operator $T : \ell_1 \to X$ by $T(\alpha_n) = \sum_n \alpha_n x_n$ for all $(\alpha_n) \in \ell_1$. Given $\epsilon > 0$, choose an unconditionally summable sequence (y_n) in X so that

$$x_n = \sum_{m=1}^{\infty} \alpha_m^n y_n \text{ and } \epsilon_1((y_m)) < \epsilon + \|(x_n)\|_c,$$

where $|\alpha_m^n| \leq 1$ for all $n, m \in \mathbb{N}$. Obviously, $T\alpha = \sum_m \langle \alpha, (\alpha_m^n)_n\rangle\, y_m$.

Claim. There exist $(\lambda_m) \in B_{c_o}$ and $(z_m) \in \ell_w^1(X)$, such that $y_m = \lambda_m z_m$ for all $m \in \mathbb{N}$ and $\epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m))$.

To prove the claim, choose $p_o \in \mathbb{N}$ so that

$$\sum_{p \geq p_o} \frac{1}{p^2} < \epsilon.$$

Since the sequence (y_m) is unconditionally summable, we have

$$\lim_{m\to\infty} \sup\{\sum_{k=m}^{\infty} |\langle y_m, x^*\rangle| : \|x^*\| \leq 1\} = 0.$$

So we can determine a strictly increasing sequence of integers $(n_p)_p$, such that

$$\sup\{\sum_{m \geq n_p} |\langle y_m, x^*\rangle| : \|x^*\| \leq 1\} < \frac{1}{p^3}$$

for all $p \in \mathbb{N}$. Let (λ_m) the sequence defined by

$$\lambda_m = \begin{cases} 1, & \text{for } m \leq n_{p_o}, \\ \frac{1}{p}, & \text{for } n_p < m \leq n_{p+1}. \end{cases}$$

and put $z_m = (\lambda_m)^{-1} y_m$ for all $m \in \mathbb{N}$. It is obvious that $(\lambda_m) \in B_{c_o}$. Hence, if we prove the inequality $\epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m))$, the claim will be established. Given

$x^* \in B_{X^*}$, we have

$$\begin{aligned}\sum_{m=1}^{\infty} |\langle z_m, x^*\rangle| &= \sum_{m=1}^{n_{p_o}} |\langle y_m, x^*\rangle| + \sum_{p=p_o}^{\infty} \sum_{k=n_p+1}^{n_{p+1}} |\langle z_k, x^*\rangle| \\ &\le \epsilon_1((y_m)) + \sum_{p=p_o}^{\infty} p \sum_{k=n_p+1}^{n_{p+1}} |\langle y_k, x^*\rangle| \\ &\le \epsilon_1((y_m)) + \sum_{p\ge p_0} \frac{1}{p^2} \\ &< \epsilon + \epsilon_1((y_m)).\end{aligned}$$

Now, using the claim we can write T in the form $T\alpha = \sum_m \langle \alpha, \lambda_m (\alpha_m^n)_n\rangle z_m$. Then $T \in \mathcal{N}_\infty(\ell_1, X)$ and

$$\nu_\infty(T) \le \left(\sup_m \|(\lambda_m(\alpha_m^n)_n\|_\infty\right) \epsilon_1((z_m)) = \epsilon_1((z_m)) < \epsilon_1((y_m)) + \epsilon < 2\epsilon + \|(x_n)\|_c,$$

for all $\epsilon > 0$. Therefore, we have prove that $\nu_\infty(T) \le \|(x_n)\|_c$ for all sequences (x_n) belonging to $R_c(X)$.

Conversely, let (x_n) a sequence in X such that the operator

$$T : (\alpha_n) \in \ell_1 \to \sum_{n=1}^{\infty} \alpha_n x_n \in X$$

is ∞-nuclear. Given $\epsilon > 0$, choose a null sequence (β_m) in ℓ_∞ and $(y_m) \in \ell_w^1(X)$ so that

$$T\alpha = \sum_{m=1}^{\infty} \langle \alpha, \beta_m\rangle y_m \text{ for all } \alpha \in \ell_1$$

and

$$\left(\sup_m \|\beta_m\|\right) \epsilon_1((y_m)) < \nu_\infty(T) + \epsilon.$$

Since the sequence $(\|\beta_m\|_\infty\, y_m)$ is unconditionally summable, it follows easily that (x_n) belongs to $R_c(X)$ and

$$\|(x_n)\|_c < \epsilon + \nu_\infty(T),$$

for all $\epsilon > 0$. This concludes the proof.

It is well-known that with the trace duality $\mathcal{N}_\infty(X, Y)^*$ and $\Pi_1(Y, X^{**})$ can be isometrically identified, if X^* has the approximation property. In fact, for any $S \in \Pi_1(Y, X^{**})$ the map $\phi_S : \mathcal{N}_\infty(X, Y) \to \mathbb{R}$ defined by $\phi_S(T) = \mathrm{tr}(S \circ T)$ is a continuous linear form on $\mathcal{N}_\infty(X, Y)$ and every linear form on $\mathcal{N}_\infty(X, Y)$ can be obtained in this way. Accordingly, $\mathcal{R}_c(X)^*$ and $\Pi_1(X, \ell_1^{**})$ are isometric. The next Proposition shows that every operator $T \in \Pi_1(X, \ell_1)$ also defines a continuous linear form on $\mathcal{R}(X)$.

Proposition 2. *Let X be a Banach space. If (x_n^*) is a sequence in X^* so that the operator $S : x \in X \to (\langle x, x_n^*\rangle) \in \ell_1$ is 1-summing, then the linear map $\psi_S : \mathcal{R}(X) \to \mathbb{R}$ defined by $\psi_S((x_n)) = \sum_n \langle x_n, x_n^*\rangle$ for all $(x_n) \in \mathcal{R}(X)$ is well-defined, continuous and $\|\psi_S\| \le \pi_1(S)$.*

Proof. Let $(x_n) \in \mathcal{R}(X)$. Given $\epsilon > 0$, choose a vector measure $F : \Sigma \to X$ so that

$$\{x_n : n \in \mathbb{N}\} \subset \mathrm{rg}(F) \quad \text{and} \quad \|F\| < \epsilon + \|(x_n)\|.$$

Let μ a control measure for F and consider the integration operator

$$I : f \in L^\infty(\mu) \to \int f dF \in X.$$

The operator $S \circ I$ is 1-summing and, hence, integral. Since ℓ_1 has the Radon-Nikodym property, it follows that $S \circ I$ is nuclear and $\pi_1(S \circ I) = \nu_1(S \circ I)$ (see [DU, p. 174]). So is $(S \circ I)^* : \ell_\infty \to L^\infty(\mu)$. Then

$$\sum_{n=1}^{\infty} \|(S \circ I)^*(e_n)\| < +\infty, \tag{4}$$

where (e_n) is the unit basis of c_o. On the other hand, we have

$$\langle (S \circ I)f, e_n \rangle = \langle f, (S \circ I)^*(e_n) \rangle.$$

Then

$$\|(S \circ I)^*(e_n)\| = \sup\{|\langle f, (S \circ I)^*(e_n)\rangle| : \|f\|_\infty \leq 1\}$$

$$= \sup\{|\int f d(x_n^* \circ F)| : \|f\|_\infty \leq 1\}.$$

This shows that (4) can be written in the form

$$\sum_{n=1}^{\infty} \sup\{|\int f d(x_n^* \circ F)| : \|f\|_\infty \leq 1\} < +\infty. \tag{5}$$

For every $n \in \mathbb{N}$, we choose $A_n \in \Sigma$ such that $F(A_n) = x_n$. Then, it follows from (5) that

$$\sum_{n=1}^{\infty} |\langle x_n, x_n^* \rangle| = \sum_{n=1}^{\infty} |\int \chi_{A_n} \, d(x_n^* \circ F)| < +\infty. \tag{6}$$

So, the linear form ψ_S is well-defined. To conclude the proof we need to show that

$$\sup\{|\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle| : \|(x_n)\| \leq 1\} \leq \pi_1(S).$$

If B denotes the restriction map of $(S \circ I)^*$ to c_o, we have

$$\begin{aligned} |\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle| \leq \sum_{n=1}^{\infty} \|(S \circ I)^*(e_n)\| &= \nu_1(B) \leq \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) \\ &= \pi_1(S \circ I). \end{aligned} \tag{7}$$

Since $S \circ F$ is the representing measure of an 1-summing operator,$S \circ I$, it follows from [DU, p. 162] that $\pi_1(S \circ I) = |S \circ F|$ (here, $|S \circ F|$ denotes the total variation of $S \circ F$). Finally, we can deduce from (7) that

$$|\sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle| \leq |S \circ F| \leq \pi_1(S) \, \|F\| < \pi_1(S) \, (\epsilon + \|(x_n)\|),$$

for all $\epsilon > 0$. Therefore, $\|\psi_S\| \leq \pi_1(S)$. □

p-Summable sequences in the range of a vector measure

Proposition 3. *Let X be a Banach space and let $p > 2$. X has the property that every sequence $(x_n) \in \ell_u^p(X)$ lies inside the range of a vector measure if and only if there exists a constant $c > 0$ so that, for all finite sets $\{x_1, .., x_n\} \subset X$ satisfying $\epsilon_p((x_i)_1^n) \le 1$, there is a vector measure $F : \Sigma \to X$ satisfying*

$$\|F\| \le c \ \text{ and } \ \{x_1, .., x_n\} \subset rg(F).$$

Proof. First, proceeding by contradiction, suppose $\ell_u^p(X) \subset \mathcal{R}(X)$ but there isn't a such constant. Then there would exist a sequence (H_n) of finite subsets of X such that for every n the conditions

$$H_n \subset \mathrm{rg}(F) \ \text{ and } \ \sup\{\Big(\sum_{x \in H_n} |\langle x, x^* \rangle|^p\Big)^{\frac{1}{p}} : \|x^*\| \le 1\} \le 1$$

would imply $\|F\| \ge n^2$. If $H_n = \{x_1^n, .., x_{k(n)}^n\}$, the sequence

$$\{x_1^1, .., x_{k(1)}^1, \frac{1}{2}x_1^2, .., \frac{1}{2}x_{k(2)}^2, .., \frac{1}{n}x_1^n, .., \frac{1}{n}x_{k(n)}^n, ..\}$$

belongs to $\ell_u^p(X)$. So, there exists a vector measure F with $(1/n)H_n \subset \mathrm{rg}(F)$ for all $n \in \mathbb{N}$. Thus $H_n \subset \mathrm{rg}(nF)$ and this would yield $\|F\| \ge n$ for every n, a contradiction with the fact that F has bounded semivariation since it is countably additive.

Conversely, let $c > 0$ be a constant such that, for every finite set $\{x_1, .., x_n\} \subset X$ satisfying $\epsilon_p((x_i)_{i=1}^n) \le 1$, there exists a vector measure F with $\|F\| \le c$ and whose range contains $\{x_1, .., x_n\}$. Then, the natural map from $(X^{(\mathbb{N})}, \epsilon_p)$ to $(\mathcal{R}(X), \|\cdot\|)$ is linear and continuous. Therefore, it has a unique continuous linear extension to $\ell_u^p(X)$.

Now we are ready to face our main result. □

Theorem 4. *Let X be a Banach space and let $p > 2$. The following are equivalent:*

(i) *Every sequence $(x_n) \in \ell_u^p(X)$ lies inside the range of a vector measure.*
(ii) *For all sequence (x_n^*) in X^* satisfying that $x \in X \to (\langle x, x_n^* \rangle) \in \ell_1$ is 1-summing, the operator $x \in X \to (\langle x, x_n^* \rangle) \in \ell_q$ is nuclear.*
(iii) *There exists a constant $c > 0$ such that, for all $n \in \mathbb{N}$, for all $\{x_1, .., x_n\}$ in X and all $\{x_1^*, .., x_n^*\}$ in X^*, we have*

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \le c\, \pi_1(\sum_i^n x_i^* \otimes e_i : X \to \ell_1^n)\ \epsilon_p((x_i)_{i=1}^n).$$

Proof. (i) ⇒ (ii) We have seen in the proof of Proposition 3 that the natural map $\ell_u^p(X) \to \mathcal{R}(X)$ is continuous. So, given a sequence (x_n^*) in X^* such that the map

$$x \in X \to (\langle x, x_n^* \rangle) \in \ell_1$$

is 1-summing, it follows from Proposition 2 that the linear form ϕ on $\ell_u^p(X)$ defined by $\phi(x_n) = \sum_n \langle x_n, x_n^* \rangle$ is continuous. Then the linear map $x \in X \to (\langle x, x_n^* \rangle) \in \ell_q$ is integral [DU, p. 232]. Now recall that nuclear and integral operators into a reflexive space are the same [DU].

(ii) ⇒ (iii) The linear map

$$T \in \Pi_1(X, \ell_1) \to I_{1q} \circ T \in \mathcal{N}(X, \ell_q)$$

has closed graph (here I_{1q} is the inclusion map from ℓ_1 into ℓ_q). So, it is continuous. Then there is a constant $c > 0$ such that

$$\nu_1(I_{1q}^n \circ T) \leq c\, \pi_1(T) \tag{8}$$

for all $n \in \mathbb{N}$ and all $T \in \mathcal{L}(X, \ell_1^n)$, being $I_{1q}^n : \ell_1^n \to \ell_q^n$ the identity map. Now, given $\{x_1, .., x_n\}$ in X and $\{x_1^*, .., x_n^*\}$ in X^*, we consider the operators $u : X \to \ell_q^n$ and $v : \ell_q^n \to X$ defined by

$$u(x) = (\langle x, x_i^* \rangle)_{i=1}^n \text{ and } v((\alpha_i)) = \sum_{i=1}^n \alpha_i x_i.$$

Since $\operatorname{tr}(u \circ v) = \sum_1^n \langle x_i, x_i^* \rangle$, we have

$$|\sum_{i=1}^n \langle x_i, x_i^* \rangle| \leq \nu_1^o(u \circ v) = \nu_1(u \circ v) \leq \nu_1(u)\, \|v\|,$$

being ν_1^o the finite-nuclear norm [P]. Then, by (8) we obtain

$$|\sum_{i=1}^n \langle x_i, x_i^* \rangle| \leq \pi_1(\sum_i^n x_i^* \otimes e_i : X \to \ell_1^n)\, \|v\|.$$

This yields (iii) because of the equality $\|v\| = \epsilon_p((x_i)_{i=1}^n)$.

(iii) $\Rightarrow$ (i) For every $n \in \mathbb{N}$, we define a linear map

$$U_n : (X^n, \epsilon_p) \to \mathcal{N}_\infty(\ell_1^n, X)$$

in the following way: Given $\widehat{x} = (x_i)_1^n \in X^n$, we denote by $T_{\widehat{x}}$ the operator from ℓ_1^n into X defined by $T_{\widehat{x}}((\alpha_i)_1^n) = \sum_1^n \alpha_i x_i$. Then we put $U_n(\widehat{x}) = T_{\widehat{x}}$. To show that U_n is continuous, we are going to compute $\nu_\infty(T_{\widehat{x}})$:

$$\nu_\infty(T_{\widehat{x}}) = \sup\{|\langle T_{\widehat{x}}, S \rangle| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n)\}$$
$$= \sup\{|\operatorname{tr}(S \circ T_{\widehat{x}})| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n)\}$$
$$= \sup\{|\sum_{i=1}^n \langle x_i, S^*(e_i^*) \rangle| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n)\},$$

where $(e_i^*)_1^n$ is the unit basis of ℓ_∞^n. By (iii), we obtain

$$\nu_\infty(T_{\widehat{x}}) \leq c\, \epsilon_p((x_i)_1^n).$$

So, we have

$$\|U_n\| \leq c \quad \text{for every } n \in \mathbb{N}. \tag{9}$$

Now we consider the linear map

$$U : (X^{(\mathbb{N})}, \epsilon_p) \to \mathcal{N}_\infty(\ell_1, X)$$

defined by $U(\widehat{x}) = T_{\widehat{x}}$ with $T_{\widehat{x}}((\alpha_n)) = \sum_n \alpha_n x_n$, for all $\widehat{x} = (x_n) \in X^{(\mathbb{N})}$ and all $(\alpha_n) \in \ell_1$. From (9) it follows that $\|U\| \leq c$. Finally, Proposition 1 implies that $\ell_u^p(X)$ is contained in $\mathcal{R}_c(X)$.

Using the methods introduced in [Pi2], we can prove that an operator $T : X \to Y$ takes every sequence $(x_n) \in \ell_u^p(X)$ into a sequence (Tx_n) lying in the range of an X-valued measure with bounded variation if and only if $T^* : Y^* \to X^*$ is $(1, 1, p)$-summing. The next Proposition shows the relationship between these ideas and our problem. □

Proposition 5. *$\ell_u^p(X)$ is contained in $\mathcal{R}(X)$ if and only if $T^*: Y^* \to X^*$ is $(1,1,p)$-summing for all Banach space Y and for all $T \in \Pi_1(X,Y)$.*

Proof. Suppose X is a Banach space such that $\ell_u^p(X) \subset \mathcal{R}(X)$. If $T \in \Pi_1(X,Y)$, then T takes every sequence $(x_n) \in \ell_u^p(X)$ into a sequence (Tx_n) lying in the range of a vector measure of bounded variation. So, $T^*: Y^* \to X^*$ is $(1,1,p)$-summing.

Conversely, let X be a Banach space with the property that, for all $T \in \Pi_1(X,\ell_1)$, the operator T^* is $(1,1,p)$-summing. Take a sequence (x_n^*) in X^* such that $T: x \in X \to (\langle x, x_n^*\rangle) \in \ell_1$ is 1-summing. By hypothesis, T^* is $(1,1,p)$-summing. Now an appeal to [Pi2, Theorem 1] yields the nuclearity of the operator

$$x^{**} \in X^{**} \to (\langle x^{**}, T^* e_n^*\rangle) \in \ell_q.$$

Then $x \in X \to (\langle x, x_n^*\rangle) \in \ell_q$ is also nuclear. Now Theorem 4 tells us that $\ell_u^p(X) \subset \mathcal{R}(X)$. □

Examples

If X is a Banach space, $s(X)$ will denote the set of all real numbers $s > 2$ such that every sequence $(x_n) \in \ell_u^s(X)$ lies inside the range of an X-valued measure. Obviously, $s(X)$ is an interval whose bounds are 2 and $\sup(s(X))$. In this section we find out the form of the set $s(X)$ for any $\mathcal{L}_p$-space X $(1 \le p \le +\infty)$.

a) *If X is an $\mathcal{L}_p$-space for $p \ge 2$, then $s(X) = (2, +\infty)$.*

By [LR, Theorem III], X^* is an $\mathcal{L}_q$-space. So X^* is isomorphic to a subspace of an $L_1(\mu)$-space for some measure μ [LP, Corollary 7.2]. Hence, every null sequence in X lies inside the range of an X-valued measure [PR].

Following [Ps2] we will say that a Banach space X satisfies Grothendieck's Theorem (in short, X is a G. T. space) if $\mathcal{L}(X,\ell_2) = \Pi_1(X,\ell_2)$. For some time the $\mathcal{L}_1$-spaces remained the only known G. T. spaces. We have just seen that $s(X) = (2,+\infty)$ for every Banach space X whose dual X^* is an $\mathcal{L}_1$-space. Nevertheless, there are spaces for which $s(X) = \emptyset$ and X^* is a G. T. space. Next we will prove this assertion.

b) *If X is an infinite-dimensional Banach space of cotype 2 and X^* is a G. T. space, then $s(X) = \emptyset$.*

Since X is a space of cotype 2 we have the identity $\Pi_1(X,\ell_1) = \Pi_2(X,\ell_1)$ [Ps2, Theorem 5.16]. By [Ps2, Proposition 6.2] $\Pi_2(X,\ell_1) = \mathcal{L}(X,\ell_1)$. So, it follows that

$$\pi_1(u) \le c\|u\| \quad \text{for all } u \in \mathcal{L}(X,\ell_1) \tag{10}$$

and for some constant $c > 0$. Proceeding by contradiction, suppose $p \in s(X)$. Then there exists a constant $c' > 0$ so that

$$\sum_{i=1}^{n} |\langle x_i, x_i^*\rangle| \le c'\pi_1\left(\sum_{i}^{n} x_i^* \otimes e_i\colon\ X \to \ell_1^n\right)\epsilon_p((x_i)_{i=1}^n) \tag{11}$$

for all $\{x_i\}_{i=1}^n$ in X and all $\{x_i^*\}_{i=1}^n$ in X^*. (10) and (11) yield

$$\sum_{i=1}^{n} |\langle x_i, x_i^*\rangle| \le c'c\epsilon_p((x_i)_{i=1}^n)\epsilon_1((x_i^*)_{i=1}^n) \tag{12}$$

for all $\{x_i\}_{i=1}^n$ in X and all $\{x_i^*\}_{i=1}^n$ in X^*. This proves that the identity map I_X is $(1,p,1)$-summing [P]. Hence I_{X^*} is $(1,1,p)$-summing [P, Theorem 17.1.5]. Since $p > 2$, from [Pi2] it follows that X^* is finite-dimensional, a contradiction which completes the proof.

In [Ps1, Theorem 3.2], it is proved that any Banach space E of cotype 2 can be imbedded isometrically into a Banach space X such that X and X^* are both of cotype 2 and both verify Grothendieck's Theorem. So, if X is the Banach space associated to $E = \ell_2$ in [Ps1, Theorem 3.2], for every $p > 2$ there are p-summable sequences in X that do not lie inside the range of a X-valued vector measure. Nevertheless, the closed unit ball of ℓ_2 is the range of a vector measure [AD].

c) $s(\ell_1) = \emptyset$.

In [AD] it is proved that the sequence $(e_n/\sqrt{n})$ does not lie inside the range of a measure. Nevertheless, $(e_n/\sqrt{n})$ belongs to $\ell_u^s(\ell_1)$ for any $s > 2$.

d) $s(\ell_p) = \emptyset$ *for any* $1 < p < 2$.

Given $s \in (2, q)$, we choose $\alpha \in (1/s - 1/q, 1/2 - 1/q)$. We are going to prove that the sequence (e_n/n^α) belongs to $\ell_u^s(\ell_p)$ but it isn't in the range of a measure. If $x^* = (x_n^*) \in \ell_q$ with $\|x^*\|_q \leq 1$, we have for all $m \in \mathbb{N}$:

$$\sum_{n=m}^{\infty} |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^s = \sum_{n=m}^{\infty} \frac{1}{n^{\alpha s}} |x_n^*|^s \leq \Big(\sum_{n=m}^{\infty} \frac{1}{n^{\frac{\alpha s q}{q-s}}} \Big)^{\frac{q-s}{q}} \Big(\sum_{n=m}^{\infty} |x_n^*|^q \Big)^{\frac{s}{q}}$$

where we have applied Holder's inequality. Then

$$\sup\{ \sum_{n=m}^{\infty} |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^s : \|x^*\|_q \leq 1\} \leq \Big(\sum_{n=m}^{\infty} \frac{1}{n^{\frac{\alpha s q}{q-s}}} \Big)^{\frac{q-s}{q}}$$

for all $m \in \mathbb{N}$. Since $\alpha s q/(q - s) > 1$, it follows that (e_n/n^α) is plainly in $\ell_u^s(\ell_p)$. The sequence (e_n/n^α) is an unconditional basis of ℓ_p; therefore, to prove that the sequence (e_n/n^α) is not in the range of a measure it suffices to show that it does not belong to $\ell_w^2(\ell_p)$ [AD, Theorem 5]. For this, choose $\beta > (1/q)$ and consider the scalar sequence $x^* = (1/n^\beta)$. It is plain that $x^* \in \ell_q$ and that viewed as a member of $(\ell_p)^*$, x^* satisfies:

$$\sum_n |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2(\alpha+\beta)}}.$$

Therefore, if we choose $\beta \in (1/q, 1/2 - \alpha)$, the last series is divergent and we deduce that $s \notin s(\ell_p)$.

e) *If X is an infinite-dimensional $\mathcal{L}_p$-space for $1 \leq p < 2$, then $s(X) = \emptyset$.*

By [LP, Proposition 7.3], X has a complemented subspace H isomorphic to ℓ_p. Then $s(X) \subset s(H) = s(\ell_p) = \emptyset$.

In view of these results and [PR] a question arises: *Are there Banach spaces X such that $s(X) \neq \emptyset$ but X^* is not isomorphic to a subspace of an $L_1(\mu)$-space for some measure μ?*

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