ON THE BOUNDED CLOSURE OF THE RANGE
OF AN OPERATOR

ROBIN HARTE AND WOO YOUNG LEE

(Communicated by Palle E. T. Jorgensen)

Abstract. The "bounded closure of the range" of an operator between two normed spaces is a linear subspace lying between the range and its closure. The induced concept of "almost onto" is a sort of first draft of the concept of "almost open".

If \( T \in BL(X,Y) \) is bounded and linear between normed spaces then the range of \( T \) is a linear subspace of \( Y \):

\[
T(X) = \{ Tx : x \in X \}. \tag{0.1}
\]

Since \( T(X) \) may or may not be closed we must also consider its closure:

\[
\text{cl} \, T(X) = \{ \lim_n T x_n : x \in X^N \text{ with } T x \in c_1(Y) \}. \tag{0.2}
\]

Here we write \( c_1(Y) \) for the convergent sequences in \( Y \), and \( c_0(Y) \) for the null sequences:

\[
c_0(X) \subseteq c_1(X) \subseteq \ell_\infty(X) = \{ x \in X^N : \sup_n ||x_n|| < \infty \}. \tag{0.3}
\]

1. Definition. If \( T \in BL(X,Y) \) is bounded and linear between normed spaces then the bounded closure, or almost closure, of its range is the set

\[
\text{cl}^\sim(T,X) = \{ \lim_n T x_n : x \in \ell_\infty(X) \text{ with } T x \in c_1(Y) \}. \tag{1.1}
\]

We shall describe \( T \) as having bounded closure, or being relatively almost onto, if there is equality

\[
\text{cl}^\sim(T,X) = \text{cl} \, T(X), \tag{1.2}
\]

and as being boundedly closed, or almost closed, if there is equality

\[
T(X) = \text{cl}^\sim(T,X). \tag{1.3}
\]

Evidently the bounded closure is a linear subspace, with

\[
T(X) \subseteq \text{cl}^\sim(T,X) \subseteq \text{cl} \, T(X). \tag{1.4}
\]

Like the range and its closure, the bounded closure is unchanged when the operator is made to be one-one:

Received by the editors August 22, 1995 and, in revised form, February 1, 1996.
1991 Mathematics Subject Classification. Primary 47A05; Secondary 47B07, 46B06.
Key words and phrases. Bounded closure, almost onto, almost open.
The second author was supported by KOSEF grants 941-0100-028-2 and 94-0701-02-01-3.

©1997 American Mathematical Society

2313
2. Theorem. If $T \in BL(X,Y)$ there is equality
\begin{equation}
\text{cl}^\sim (T^\wedge, X/T^{-1}(0)) = \text{cl}^\sim (T, X).
\end{equation}

Proof. Inclusion one way is obvious: if $Tx_n \to y$ with $\sup_n ||x_n|| < \infty$ then certainly
\begin{equation}
T x_n \to y \text{ with } \sup_n \text{dist}(x_n, T^{-1}(0)) < \infty;
\end{equation}
conversely if (2.2) holds we may find $(x'_n)$ with $x'_n - x_n \in T^{-1}(0)$ and $||x'_n|| \leq \text{dist}(x_n, T^{-1}(0)) + 1$.

We shall describe $T \in BL(X,Y)$ as almost onto if there is equality
\begin{equation}
\text{cl}^\sim (T, X) = Y.
\end{equation}

3. Theorem. If $T \in BL(X,Y)$ there is implication
\begin{align}
T \text{ onto } & \implies T \text{ almost onto } \implies T \text{ dense}, \\
T \text{ almost open } & \implies T \text{ almost onto } \implies T \text{ dense},
\end{align}
and hence also
\begin{equation}
T \text{ relatively almost open } \implies T \text{ has bounded closure}.
\end{equation}

Proof. The common second implication is clear. If $T \in BL(X,Y)$ is onto then
\begin{equation}
y \in Y \implies y = Tx = \lim_n Tx_n \text{ with } x_n = x \ (n \in \mathbb{N}),
\end{equation}
finishing the proof of (3.1). If $T \in BL(X,Y)$ is almost open then there is $k > 0$ for which
\begin{equation}
y \in Y \implies y \in \text{cl}\{Tx : ||x|| \leq k||y||\},
\end{equation}
giving the condition (1.1). This proves (3.2); for (3.3) we apply (3.2) with $Y$ replaced by cl $T(X)$, noting that if $T^\vee : X \to T(X)$ is almost open then so is the related operator $T^{\vee\vee} : X \to \text{cl} T(X)$. To make this last observation note that if $Z \subseteq Y$ is a linear subspace and $y \in \text{cl} Z$ then ([5]; [6] Theorem 1.5.1) there is $(z_n)$ in $Z$ for which
\begin{equation}
z_n \to y \text{ with } ||z_n|| \leq ||y||.
\end{equation}

For example if $T$ is bounded below then it has bounded closure; in particular the “bounded closure” of a subspace $X \subseteq Y$ is just the closure.

It is familiar that the first implications of neither (3.1) nor (3.2) can be reversed: if for example $T : X \to Y$ is the embedding of a dense proper subspace then ([5] Theorem 2.1; [6] Theorem 4.7.2) $T$ is almost open but not onto, while if $T : X \to Y$ presides over the change of norm from a complete space $X$ to the same space with a strictly weaker norm then ([5] Theorem 2.2; [6] Theorem 4.7.3) $T$ is onto but not almost open. For an example in which $T$ is dense but not almost onto take $X = Y = c_0$ and $T = W : (x_n) \mapsto (x_n/n)$: then if $y_n = 1/\sqrt{n}$ we find
\begin{equation}
||y - Tx_n|| \leq \frac{1}{n} \implies \sqrt{n} - 1 \leq |x_{nn}| \leq \sqrt{n} + 1,
\end{equation}
so that $||x_n|| \geq \sqrt{n} - 1$ and $x = (x_n) \not\in \ell_\infty(c_0)$. More generally, the first implication in (3.1) cannot be reversed, even when the space $X$ is complete:
4. Example. If \( e = (e_n) \in c_0 \) and \( T : X \to Y \) is given by setting
\[
X = Y = c_0 \text{ and } (T x)_n = e_n x_n \quad (n \in \mathbb{N}, \ x \in c_0),
\]
then
\[
T(X) = \{ (e_n u_n) : u = (u_n) \in c_0 \},
\]
\[
\text{cl}^\sim(T, X) = \{ (e_n u_n) : u = (u_n) \in \ell_\infty \}
\]
and
\[
\text{cl}(T X) = \{ y \in c_0 : \forall n \in \mathbb{N}, e_n = 0 \implies y_n = 0 \}.
\]
Proof. Equality (4.2) is clear, and trivial. For inclusion one way in (4.3) suppose \( y_n = e_n u_n \) with \( u = (u_n) \in \ell_\infty \) and define \( x = (x_m) = (x_{mn}) \) by setting
\[
x_{mn} = u_n - \frac{1}{m} \text{ if } |u_n| \geq \frac{1}{m}, = 0 \text{ if } |u_n| < \frac{1}{m}:
\]
then
\[
|u_n| \geq \frac{1}{m} \implies |y_n - e_n x_{mn}| = |e_n(u_n - x_{mn})| = |e_n| \frac{1}{m} |e|_\infty,
\]
while
\[
|u_n| < \frac{1}{m} \implies |y_n - e_n x_{mn}| = |e_n u_n| \leq \frac{1}{m} |e|_\infty;
\]
in both cases
\[
|x_{mn}| \leq |u_n| \leq ||u||_\infty.
\]
Conversely if \( x = (x_m) = (x_{mn}) \) exists for which
\[
\sup_n |y_n - e_n x_{mn}| \leq \frac{1}{m} \text{ with } \sup_{m,n} |x_{mn}| = k < \infty
\]
then \( |y_n| \leq k |e_n| + \frac{1}{m} \) for all \( m, n \) and hence \( |y_n| \leq k |e_n| \) for all \( n \), giving
\[
y_n = e_n u_n \text{ for all } n \text{ with } ||u||_\infty \leq k.
\]
Inclusion one way in (4.4) follows from the continuity of the functionals \( x \mapsto x_n \) on \( c_0 \); conversely if \( e_n = 0 \implies y_n = 0 \) then \( y - P_m y \to 0 \) with \( P_m y = (y_1, y_2, \ldots, y_m, 0, \ldots) \in T(X) \).

As soon as we have found an operator for which \( T(X) \neq \text{cl}^\sim(T, X) \) we have an operator which is almost onto but not onto:

5. Theorem. If \( T = W^\sim : c_0 \to \text{cl}^\sim(W, c_0) \) is induced by the weight operator \( W: (x_n) \mapsto (\frac{1}{m} x_n) \) on \( c_0 \) then \( T \) is almost onto but not onto.

Proof. By construction \( T = W^\sim \) is almost onto; it fails to be onto because the sequence \( \frac{1}{n} \) is in \( Y = \text{cl}^\sim(W, c_0) \) and not in the range \( W(c_0) \).

The first implication of (3.2) does reverse when the space \( Y \) is complete. It is tempting to try a Baire theorem argument with sets of the form
\[
R_k^\sim(T) = \{ y \in Y : y \in \text{cl} \{ T x : ||x|| \leq k ||y|| \} \};
\]
this appears to founder on the lack of additivity among the \( R_k^\sim(T) \). However the uniform boundedness principle and the Hahn-Banach theorem are available:

6. Theorem. If \( T \in BL(X, Y) \) there is implication
\[
Y \text{ complete, } T \text{ almost onto } \implies T \text{ almost open}.
\]
Proof. If \( T : X \to Y \) is almost onto then
\[
\{ g(y) : \|gT\| \leq 1 \} \text{ is bounded for all } y \in Y : \tag{6.2}
\]
for if \( y \in Y \) there is \( x \in \ell_\infty(X) \) with \( Tx_n \to y \), giving
\[
\|g(y)\| = \lim_n \|gTx_n\| \leq \|gT\||x||\infty. \tag{6.3}
\]
By uniform boundedness ([6] Theorem 4.9.1), using the completeness of \( Y \), there is \( k > 0 \) for which
\[
\|gT\| \leq 1 \implies \|g\| \leq k, \tag{6.4}
\]
so that \( \|g\| \leq k\|gT\| \). This makes the dual operator \( T^\dagger : Y^\dagger \to X^\dagger \) bounded below, and hence by Hahn-Banach separation ([6] Theorem 5.5.2) \( T : X \to Y \) almost open.

The bounded closure of the range can be obtained from the range of the “enlargement”:

7. Theorem. If \( T \in BL(X,Y) \) then
\[
\text{cl}\,^\sim(T,X) = \{ y \in Y : q(y) \in Q(T)Q(X) \}. \tag{7.1}
\]

Proof. Here ([5]; [6] Definition 1.9.2)
\[
Q(X) = \ell_\infty(X)/c_0(X), \tag{7.2}
\]
with \( q : X \to Q(X) \) the natural embedding and \( Q(T) \) the operator induced naturally by \( T \); then, abusing notation,
\[
y \in \text{cl}\,^\sim(T,X) \iff y \in T\ell_\infty(X) + c_0(Y) \iff q(y) \in Q(T)Q(X). \tag{7.3}
\]
From Theorem 7 it follows that if \( M \subseteq Y \) is a linear subspace then
\[
M \subseteq \text{cl}\,^\sim(T,X) \iff M \subseteq T\ell_\infty(X) + c_0(Y) : \tag{7.4}
\]
this contrasts with the result of Harte and Shannon [8] that, if the spaces are complete,
\[
\text{cl } M \subseteq T(X) \iff \ell_\infty(M) \subseteq T\ell_\infty(X); \tag{7.5}
\]
in the special case \( M = TX \) Albrecht and Mehta [1] find
\[
\text{cl } M \subseteq T(X) \iff \ell_\infty(M) \subseteq T\ell_\infty(X) + c_0(Y). \tag{7.6}
\]
The Albrecht/Mehta argument ([6] Theorem 5.7.1) solves Problem (4.1.3) of [5], possibly more directly than the argument of Harte and Mathieu [7]:

8. Theorem. If \( T \in BL(X,Y) \) there is implication
\[
\ell_\infty(Y) \subseteq T\ell_\infty(X) + c_0(Y) \iff T \text{ almost open.} \tag{8.1}
\]

Proof. We claim, in the notation of (7.2), that
\[
Q(T) \text{ dense } \implies T \text{ almost open } \implies Q(T) \text{ open.} \tag{8.2}
\]
The second implication is straightforward ([5] Theorem 4.1; [6] Theorem 3.4.5); for the first we need the Hahn-Banach theorem. If \( T : X \to Y \) is not almost open then the dual \( T^\dagger : Y^\dagger \to X^\dagger \) is not bounded below, and hence there is \( g = (g_n) \) in \( Y^\dagger \) for which
\[
\|g_n\| = 1 \text{ and } \|g_nT\| \to 0, \tag{8.3}
\]
and then $y = (y_n)$ in $Y$ for which
\begin{equation}
||y_n|| = 1 \text{ and } |g_n(y_n)| \geq \frac{1}{2}.
\end{equation}
we claim that there is implication
\begin{equation}
x \in \ell_\infty(X) \implies \text{dist}(y - Tx, c_0(Y)) \geq \frac{1}{2}.
\end{equation}
Indeed if $x = (x_n)$ is bounded then
\begin{equation}
||y_n - Tx_n|| = ||g_n||||y_n - Tx_n|| \geq |g_n(y_n - Tx_n)| \geq \frac{1}{2} - ||g_nT||||x_n|| \to \frac{1}{2},
\end{equation}
and hence $\limsup_n||y_n - Tx_n|| \geq \frac{1}{2}$. This proves (8.2), and hence (8.1), if we note
that the left-hand side is the condition that $Q(T)$ is onto. \hfill \Box

The bounded closure of the range intervenes ([4] Theorem 11.3.2(c)) in the theory of compact operators:

8. Theorem. If $T$ is upper semi-Fredholm in the sense that
\begin{equation}
x \in \ell_\infty(X), Tx \in m_1(Y) \implies x \in m_1(X),
\end{equation}
where $m_1(X)$ is the space of sequences $x \in \ell_\infty(X)$ of which every subsequence has a convergent subsequence, then
\begin{equation}
T(X) = \overline{cl}(T), X) = \overline{cl} T(X).
\end{equation}
Proof. To verify the second equality in (9.2) we may by Theorem 2 replace $T$ by its one-one part $T^\wedge : X/T^{-1}(0) \to Y$, and hence assume that $T$ is one-one: we need to check that if (9.1) holds for $T$ then it also holds for $T^\wedge$. We claim that if $||y - Tx_n|| \to 0$ then the condition (9.1) forces $(x_n)$ to be bounded: for if $x' = (x_n')$ were a subsequence of $x = (x_n)$ for which $||x_n'|| \to \infty$ then
\begin{equation}
||T(x_n')|| = ||Tx_n'|| = ||x_n'|| \to 0,
\end{equation}
and hence by (9.1) there would be a subsequence $x'' = (x''_n)$ of $x'$ and an element $z_\infty \in X$ for which
\begin{equation}
||x''_n|| - z_\infty|| \to 0.
\end{equation}
But now $||Tz_\infty|| = 0$ while $||z_\infty|| = 1$, a contradiction.

To verify the first equality in (9.2) suppose $y = \lim Tx_n$ with bounded $x = (x_n)$: then
\begin{equation}
Tx \in c_1(Y) \subseteq m_1(Y),
\end{equation}
and hence by (9.1)
\begin{equation}
x \in m_1(X).
\end{equation}
Now if $x' \prec x$ is a convergent subsequence of $x$, with $x_n' \to x'_\infty \in X$, then
\begin{equation}
y = \lim Tx_n' = Tx'_\infty \in T(X).
\end{equation}
\hfill \Box

The “upper semi-Fredholm condition” (9.1) agrees with more traditional conditions ([3] Theorem 1.3.2; [2] Theorem 2), even for incomplete spaces ([6] Theorem 6.9.2): indeed when (9.1) holds then (1.2) can be replaced by the stronger condition, that $T$ is relatively open.
References


School of Mathematics, Trinity College, Dublin, Ireland
E-mail address: rharte@maths.tcd.ie

Department of Mathematics, Sung Kwan University, Suwon 440-746, Korea
E-mail address: wylee@yurim.skku.ac.kr