COMMUTATIVITY CRITERIA FOR BANACH $^*$-ALGEBRAS

L. C. LEINBACH AND BERTRAM YOOD

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $A$ be a Banach $^*$-algebra with an identity. Necessary and sufficient conditions are given for $A$ to be commutative modulo its $^*$-radical and for $A$ to be commutative if $A$ has a faithful $^*$-representation as operators on a Hilbert space.

1. Introduction

This paper deals with a Banach $^*$-algebra $A$ with an identity. The positive cone $P$ of $A$ is an important item in its study. In [7] Kelley and Vaught used $P$ in discussing $^*$-representations of $A$ as operators on a Hilbert space. More relevant to our present investigation is the use of $P$ by Curtis [3] which we now recall. Let $H$ be the set of all self-adjoint elements of $A$. A partial ordering of $H$ is induced by $P$ via the rule $h \geq k$ if and only if $h - k \in P$. Curtis showed that $A$ is commutative if $H$ is a lattice in that partial ordering. We take a different approach to commutativity using $P$.

We show that $A$ is commutative modulo its $^*$-radical if and only if $P$ is closed under Jordan multiplication (that is, $hk + kh \in P$ if $h, k \in P$). This result is used to help show the following. If $A$ has a faithful $^*$-representation, then $A$ is commutative if and only if, given $u, v \in H$, there exists $w \in H$ where $u^2 v + v^2 u = w^2$. In particular this criterion holds if $A$ is a semi-simple hermitian Banach $^*$-algebra and so for a $C^*$-algebra.

2. Preliminaries

Throughout, $A$ will denote a Banach $^*$-algebra with an identity $e$. Let $H$ be the set of self-adjoint elements of $A$. Let $P_0$ denote the set of all finite sums $\sum x_k^* x_k$ where each $x_k \in A$, and let $Q_0$ be the set of all finite sums $\sum h_j^2$ where each $h_j \in H$. Let $P(Q)$ denote the closure in $H$ of $P_0(Q_0)$. In [3] and [7] the positive cone $P$ is defined to be the closure of $P_0$ in $A$. But there the involution is assumed to be continuous. We take the closure of $P_0$ in $H$ to take care of the case where the involution is not continuous.

As in [11] a linear functional $f(x)$ on $A$ is said to be positive (weakly positive) if $f(x) \geq 0$ for all $x \in P_0(Q_0)$. As $A$ has an identity each such $f(x)$ is real-valued on $H$. In [4, Ch. XV, sec. 7] a positive linear functional $f(x)$ is said to be a trace if $f(xy) = f(yx)$ for all $x, y \in A$. We have the corresponding notion of a weak trace for weakly positive linear functionals.
A standard result [2, p. 198] is that any positive linear functional \( f(x) \) on \( A \) is continuous on \( A \) (and thus \( f(P) \geq 0 \)). This and the fact that a weakly positive linear functional on \( A \) is continuous are special cases of a neglected more general result [12, Th. 2.5] which reads as follows.

**Theorem.** Let \( T \) be a linear mapping of \( A \) into a complex normed linear space. Suppose that there is an integer \( n \geq 2 \) and a real number \( c > 0 \) such that
\[
\|T(v^n + w^n)\| \geq c\|T(w^n)\|
\]
for all \( v, w \in H \). Then \( T \) is continuous.

We consider \( P \) and \( Q \) as closed cones in the real normed linear space \( H \). It follows from Ford’s square root lemma [5] that \( e \) is an interior point of \( Q \) and hence of \( P \). Also (see [7, Lemma 1.2]) \( P(Q) = H \) if and only if 0 is the only positive (weakly positive) linear functional on \( A \).

Trivially \( Q \subset P \). For the case of a Banach \(^*\)-algebra \( B \) without an identity we have an example where \( Q \neq P \). Let \( B \) be the algebra of all triples \((\lambda, \mu, \nu)\) of complex numbers with the non-standard multiplication \((\lambda_1, \mu_1, \nu_1)(\lambda_2, \mu_2, \nu_2) = (0, 0, \lambda_1 \mu_2 - \mu_1 \lambda_2)\) and involution \((\lambda, \mu, \nu)^* = (-\lambda, -\mu, -\nu)\) where the norm \( \|\| \|/(\lambda, \mu, \nu)\| = (|\lambda|^2 + |\mu|^2 + |\nu|^2)^{1/2} B \) is a Banach \(^*\)-algebra. Here \( Q = (0) \) and \( P \) is the set of purely imaginary multiples of \((0, 0, 1)\). We do not have an example of \( A \) with identity where \( P \neq Q \). We believe such exists but the following shows that an example may not be easily found. In §4 we shall need to have \( Q = P \) in the cases there at hand.

**Theorem 2.1.** Under any of the following conditions on \( A \) we have \( P = Q \): (1) if \( A \) is commutative, (2) if \( A \) is a hermitian \(^*\)-algebra, (3) if \( A \) is finite-dimensional, (4) if there exists some \( x \in A \) where \( x^*x = -e \).

**Proof.** (1) If \( A \) is commutative, then \( P_0 = Q_0 \) so \( P = Q \). (2) Let \( x \in A \). By the Shirali-Ford Theorem ([2, p. 226]) we have \( sp(x^*x) \subset [0, \infty) \). Take any \( \varepsilon > 0 \). Then \( sp(\varepsilon x + x^*x) \subset (0, \infty) \). Hence, by [12, Lemma 2.4], there is some \( w \in H \) where \( \varepsilon x + x^*x = w^2 \). It follows that \( P = Q \). (3) If \( A \) is finite-dimensional, then every weakly positive linear functional is a positive linear functional, by [10, Cor. 7.4]. Then, by [7, Lemma 1.2] we see that \( P = Q \). (4) Let \( f(x) \) be a weakly positive linear functional. Calculations in [10, Lemma 7.2] show that \( f(e) = 0 \). As \( |f(h)|^2 \leq f(e)f(h^2) \) for all \( h \in H \) ([11, p. 231]) we see that \( f(x) = 0 \) for all \( x \in A \). It follows that \( Q = H \) so again \( P = Q \).

An instance of (4) occurs for the algebra \( A \) of all two-by-two matrices over the complexes with a non-standard involution. We define
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix}.
\]
This is an involution on \( A \) and \( x^*x = -e \) for
\[
x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

3. **On cones and traces**

We set forth our notation. Let \( \Sigma (\Sigma_w) \) be the set of all positive (weakly positive) linear functionals \( f(x) \) on \( A \) where \( f(e) = 1 \). Let \( \Phi \) denote the set of all non-zero...
Theorem 3.1. The following statements are equivalent.

\[ f \text{ then } v \text{ there is some } x, y \text{ morphism.} \]

It follows that \( h \in t > \text{real number} \), we see that (1) also holds for all \( x, y \in H \).

Note that \( P(Q) = H \) if and only if \( \Sigma (\Sigma_w) \) is empty. In that case \( \Phi \) is also empty. In the sequel we may suppose that \( P(Q) \neq H \).

Lemma 3.1. Each \( \gamma \in \Phi \) is an extreme point of \( \Sigma (\Sigma_w) \).

Proof. Suppose that \( \gamma = af + (1 - a)g \) where \( f \) and \( g \) are in \( \Sigma (\Sigma_w) \) and \( 0 < a < 1 \). Let \( h \in H \). Note that \( f(h)^2 \leq f(h^2) \) and \( g(h)^2 \leq g(h^2) \). Then

\[ \gamma(h^2) = af(h^2) + (1 - a)g(h^2) = \gamma(h)^2 \]

so that

\[ af(h^2) + (1 - a)g(h^2) \leq a^2f(h^2) + 2a(1 - a)f(h)g(h) + (1 - a)^2g(h^2). \]

This implies that \( f(h^2) + g(h^2) \leq f(h)g(h) \). Consequently \( f(h)g(h) \geq 0 \) for all \( h \in H \).

We claim that \( f(h) = g(h) \) for all \( h \in H \) and so \( f = g \). Suppose otherwise. Then there is some \( v \in H \) where \( f(v) \neq g(v) \), say \( f(v) < g(v) \). Since \( f(e) = g(e) = 1 \) we can replace \( v \) by \( w = tv + v \) where \( t \) is real to obtain \( f(w) < 0 \) and \( g(w) > 0 \). But then \( f(w)g(w) < 0 \) which is impossible.

Theorem 3.1. The following statements are equivalent.

(a) \( P(Q) \) is closed under Jordan multiplication.

(b) \( \Phi \) is the set of all extreme points of \( \Sigma (\Sigma_w) \).

(c) Every positive (weakly positive) linear functional is a trace (weak trace).

Proof. Assume (a). To obtain (b) we must, using Lemma 3.1, show that each extreme point \( f_0 \) of \( \Sigma (\Sigma_w) \) is in \( \Phi \). Fix \( y \in P(Q) \) where \( e - y \in P(Q) \). We set

\[ g_y(x) = f_0(x \cdot y) - f_0(x)f_0(y) \]

for all \( x \in A \).

Clearly \( g_y(e) = 0 \). Next we set \( w_1(x) = f_0(x) + g_y(x) \) and \( w_2(x) = f_0(x) - g_y(x) \) and note that \( w_1(e) = w_2(e) = 1 \). Now we use (a).

For \( x_0 \in P(Q) \) we have

\[ w_1(x_0) = f_0(x_0)(1 - f_0(y)) + f_0(x_0 \cdot y) \geq 0. \]

Note that \( x_0 \cdot (e - y) = x_0 - (x_0 \cdot y) \in P(Q) \). Then

\[ w_2(x_0) = f_0(x_0 - (x_0 \cdot y)) + f_0(x_0)f_0(y) \geq 0. \]

Therefore \( w_1 \) and \( w_2 \) are in \( \Sigma (\Sigma_w) \). Inasmuch as \( f_0 = (w_1 + w_2)/2 \) and \( f_0 \) is an extreme point of \( \Sigma (\Sigma_w) \) we see that \( g_y = 0 \). Hence

\[ f_0(x \cdot y) = f_0(x)f_0(y) \]

for all \( x \in A \) and for all \( y \in P(Q) \) where also \( e - y \in P(Q) \). Now we consider an arbitrary \( z \in P(Q) \) as \( e \) is in the interior of \( P(Q) \) as a subset of \( H \) there is a real number \( t > 0 \) such that \( e - tz \in P(Q) \) and \( tz \in P(Q) \). Therefore, from (1), we see that (1) also holds for all \( x \in A \) and \( y \in P(Q) \). However, by [8, p. 208], \( H = P - P = Q - Q \). Also \( A = H + iH \) so that we see that (1) is valid for all \( x, y \in A \).

A theorem of Jacobson and Rickart [6, Th. 2] asserts that a Jordan homomorphism of a ring into an integral domain is either a homomorphism or an antihomomorphism. It follows that \( f_0 \) is a multiplicative linear functional and thus lies in \( \Phi \).
Now we assume (b). Let \( f \in \Sigma(\Sigma_w) \). By the Krein-Milman theorem \( f \) is the \( w^* \)-limit in \( \mathbb{A}^* \) of elements of the form \( \sum_{k=1}^{n} a_k \gamma_k \) where each \( a_k \geq 0 \), \( \sum_{k=1}^{n} a_k = 1 \) and each \( \gamma_k \in \Phi \). As each \( \gamma_k \) is a trace (weak trace) it follows that so is \( f \).

Next we assume (c). We treat the positive and weakly positive cases separately. Suppose first that every positive linear functional \( f(x) \) is a trace. Let \( w_1, w_2 \in P_0 \). We show that \( f(w_1 \cdot w_2) \geq 0 \). Write \( w_1 = \sum_{j=1}^{n} h_j^2 \) and \( w_2 = \sum_{k=1}^{n} z_k^2 \) where each \( h_j \) and \( z_k \) is in \( H \). To see that \( g(w_1 \cdot w_2) \geq 0 \) it is enough to show that \( g(h^2 z^2 + z^2 h^2) \geq 0 \) for all \( h, z \in H \). First note that

\[
g(h^2 z^2 + z^2 h^2) = g(h z^2 h + z h^2 z)
\]

However

\[
(h z + z h)^2 + [i(h z - z h)]^2 = 2(h z^2 h + z h^2 z)
\]

where \( h z + z h \in H \) and \( i(h z - z h) \in H \). Thus \( h z^2 h + z h^2 z \in Q_0 \) and so \( g(h^2 z^2 + z^2 h^2) \geq 0 \). We then see, by the argument used above for \( P \), that \( Q \) is closed under Jordan multiplication. Hence (c) implies (a).

**Corollary 3.1.** If \( Q \) is closed under Jordan multiplication, then \( P = Q \).

**Proof.** If \( Q \) is closed under Jordan multiplication, then, by Theorem 3.1, every weakly positive linear functional is a weak trace. Hence every positive linear functional is a trace. By Theorem 3.1, \( \Phi \) is the set of extreme points of each of \( \Sigma \) and \( \Sigma_w \). Hence \( \Sigma = \Sigma_w \). Then, by [7, Lemma 1.2], we have \( P = Q \).

4. **Commutativity criteria**

**Theorem 4.1.** \( A \) is commutative modulo its \(*\)-radical if and only if \( P \) is closed under Jordan multiplication.

**Proof.** Let \( P' \) denote the set of all positive linear functionals on \( A \). By Theorem 3.1, \( P \) is closed under Jordan multiplication if and only if each \( f \in P' \) is a trace. The latter is the case if and only if \( xy - yx \in \bigcap \{ f^{-1}(0) : f \in P' \} \) for all \( x, y \in A \). But \( \bigcap \{ f^{-1}(0) : f \in P' \} \) is the \(*\)-radical of \( A \); see [9, p. 265].

**Corollary 4.1.** Suppose that \( A \) has a faithful \(*\)-representation as bounded linear operators on a Hilbert space. Then \( A \) is commutative if and only if for each pair \( u, v \in H \) there exists \( w \in H \) so that \( u^2 v^2 + v^2 u^2 = w^2 \).

**Proof.** Suppose the condition on \( H \) holds. Let \( x, y \in Q_0 \) where \( x = \sum_{j=1}^{n} h_j^2 \) and \( y = \sum_{k=1}^{n} z_k^2 \). Then \( 2x \cdot y = \sum_{j} \sum_{k} h_j^2 \cdot z_k^2 \). Therefore \( x \cdot y \) is equal to the sum of squares of elements of \( H \). This shows that \( Q_0 \) is closed under Jordan multiplication and so is \( Q \). By Corollary 3.1, \( P \) is closed under Jordan multiplication. Applying Theorem 4.1 we see that \( A \) is commutative.
Theorem 4.2. A hermitian Banach *-algebra $A$ is commutative modulo its radical if and only if $sp(u^2v^2 + v^2u^2) \subset [0, \infty)$ for all $u, v \in H$.

Proof. In this situation $Q = P$ by Theorem 2.1. Let $\Gamma = \{h \in H: sp(h) \subset [0, \infty)\}$. It is known, [9, (5.6)], that $\Gamma$ is a cone in $H$. We show first that $\Gamma$ is closed in $H$. By Ford’s Lemma [5], if $\|e - h\| < 1$ for some $h \in H$, then $h = w^2$ for some $w \in H$. Consequently $e$ lies in $Int(\Gamma)$, the interior of $\Gamma$ as a subset of $H$. Thus $\Gamma$ is a convex set with interior. Let $y_0 \in Int(\Gamma)$. Therefore $sp(te + (1-t)y_0) \subset [0, \infty)$ and $sp(y_0) \subset [-t(1-t)^{-1}, \infty)$ for each $t$, $0 < t < 1$. Consequently $y_0 \in \Gamma$.

Clearly, [9, (5.6)], we have $Q_0 \subset \Gamma$, and, as $\Gamma$ is closed in $H$, we see that $P \subset \Gamma$. We show next that $\Gamma \subset P$. Let $y \in \Gamma$ and $0 < t < 1$. Then $sp(te + (1-t)y) \subset (0, \infty)$ so that, by [12, Lemma 2.4], there exists $w \in H$ where $w^2 = te + (1-t)y$. Letting $t \to 0$ we see that $y \in Q$.

Now we suppose that $u^2v^2 + v^2u^2 \in \Gamma$ for all $u, v \in H$. Let $x = \sum_{j=1}^n h_j^2, y = \sum_{k=1}^m z_k^2$ be two elements in $Q_0$. Then $x \cdot y$ is the sum of a finite number of summands of the form $u^2v^2 + v^2u^2$ so $x \cdot y \in \Gamma = P$. We see that $P$ is closed under Jordan multiplication. By Theorem 4.1, $A$ is commutative modulo its *-radical. However here the *-radical coincides with the radical by [9, Th. 6.6]. We point out that Theorem 4.2 does not follow from Corollary 4.1. It is true that if $A$ is a $C^\ast$-algebra and $w \in H$ with $sp(w) \subset [0, \infty)$, then $w = h^2$ for some $h \in H$. Easy examples show this is false for hermitian Banach *-algebras even though the required $h$ exists if $sp(w) \subset (0, \infty)$; see [12, Lemma 2.4].

Corollary 4.2. Let $A$ be a semi-simple hermitian Banach *-algebra. The following statements are equivalent:

(a) $A$ is commutative.
(b) $sp(u^2v^2 + v^2u^2) \subset [0, \infty)$ for all $u, v \in H$.
(c) Given $u, v \in H$ there exists $w \in H$ so that $u^2v^2 + v^2u^2 = w^2$.

Proof. Here the *-radical and the radical of $A$ are both (0) and $A$ has a faithful *-representation; see [9, (6.6)]. Then (c) is equivalent to (a) by Corollary 4.1 and (b) is equivalent to (a) by Theorem 4.2.

References


Department of Mathematics, Gettysburg College, Gettysburg, Pennsylvania 17325

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802