KEEPING ADDITIVITY OF THE NULL IDEAL SMALL

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Abstract. We shall show that various statements are consistent with additivity of the null ideal equal to \(\aleph_1\); for example, “all branchless trees of size \(\aleph_1\) are special”, (S) conjecture and “there are only five cofinal types of directed posets of size \(\aleph_1\)”.

0. Introduction

In this paper we provide machinery for proving that a certain large class of forcings has a certain regularity property. The class in question includes posets used for

(1) specializing branchless trees [S],
(2) (S) conjecture [T2],
(3) embedding the poset for adding \(\aleph_1\) Cohen reals into a given poset of uniform density \(\aleph_1\) [SZ],
(4) classification of directed posets [T1] or transitive relations [T3] of size \(\aleph_1\),
(5) other “side condition” combinatorics on \(\aleph_1\); e.g. shooting an uncountable set through a coherent sequence on \(\omega_1\) [T2].

The regularity property we obtain implies preservation of additivity of the ideal of Lebesgue null sets. As a corollary, it is consistent with ZFC set theory that additivity of the null ideal is \(\aleph_1\) and all statements obtainable through (1)–(5) above hold, that is, all branchless trees of size \(\aleph_1\) are special, (S) conjecture holds etc. Thus a definite limitation on a canonical variation of the powerful “side condition” method has been exacted for the first time.

Our notation follows the set-theoretical standard as set forth in [J]. In a forcing notion, \(p \leq q\) means “\(p\) is more informative than \(q\)”. A tree \(T\) of height \(\omega_1\) is special if there is a function \(f : T \to \omega\) with \(s < t\) in \(T\) implying \(f(s) \neq f(t)\). Trees grow upwards. (S) conjecture is the statement “every hereditarily separable Hausdorff space is hereditarily Lindelöf”. The symbol \(\mathcal{N}\) denotes the collection of Borel null sets, often confused with their Borel codes. \(H_\kappa\) is the set of all sets of hereditary cardinality < \(\kappa\). If \(N\) is an elementary submodel of \(H_\kappa\) and \(P \in N\) is a forcing, a condition \(p \in P\) is called \(N\)-master if for every dense set \(D \subset P\) in \(N\), the set \(D \cap N\) is predense below \(p\).

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1. Localization

**Definition 1.** Let $\mathfrak{F} \subset \omega^\omega$ and let $e$ be a positive integer. The family $\mathfrak{F}$ is said to be $e$-localized if there exists a function $h : \omega \to [\omega]^{<\aleph_0}$ such that

1. $|h(n)| \leq n^e$ for every integer $n$,
2. for every $f \in \mathfrak{F}$ there is an integer $n \in \omega$ so that for every $m > n$, $f(m) \in h(m)$ holds.

If $e = 1$ then the family $\mathfrak{F}$ is said to be localized.

The relevance of the above definition is revealed in the result of Bartoszyński [B, BJ Section 2.3.A] saying that for a transitive model $M$ of ZFC the following are equivalent:

1. $M \cap \omega^\omega$ is localized;
2. $M \cap \omega^\omega$ is $e$-localized for some positive integer $e$;
3. the union of all Lebesgue measure zero Borel sets coded in $M$ has measure zero.

There is a natural c.c.c. forcing for making the set of ground model reals localized [Tr] and there are some preservation theorems for “the set of ground model reals is not localized” [JS], [BJ]. We shall prove that a large class of forcings preserves unlocalized families in a strong sense.

**Definition 2.** (1) [JS] Let $\kappa$ be a large regular cardinal and $N \prec H_\kappa$. We say that a function $f \in \omega^\omega$ is $N$-big if for every $h : \omega \to [\omega]^{<\aleph_0}$ in the model $N$ with $|h(n)| \leq n$ the set $\{m \in \omega : f(m) \notin h(m)\}$ is infinite.

2. Let $P$ be a forcing. We say that $P$ is friendly if for every $p \in P$, every large enough regular cardinal $\kappa$, every countable elementary submodel $N \prec H_\kappa$ with $p, P \in N$ and every $N$-big function $f \in \omega^\omega$ there is an $N$-master condition $q \leq p$ such that $q \Vdash \"f is $N[G]$-big\".

The important point is that it is possible to iterate friendly forcings preserving the statement “the family of ground model reals is not localized” or equivalently, “$\bigcup (N \cap V) \notin N^\omega$” –Lemma 13. Obviously, a finite iteration of friendly forcings is friendly and friendliness is inherited by regular subposets. In [JS] it is proved that the random algebra as well as every $\sigma$-centered forcing is friendly. We considerably extend these results.

2. Specializing trees

The purpose of this section is to prove that the usual specializing forcing for a branchless tree of height $\omega_1$ is friendly. The technique will be of great use in the next section. For now, fix a tree $T$ of height $\omega_1$ and no branches of length $\omega_1$. There is no restriction on the size of levels of $T$.

**Definition 3.** (1) If $a, b$ are disjoint finite subsets of $T$ then we say that $a$ and $b$ fit together if for every $s \in a$ and $t \in b$, $s$ and $t$ are incompatible as elements of $T$.

2. The specialization forcing is $P = \{p : p$ is a finite function from $T$ to $\omega$ such that $s <_T t$ in $\text{dom}(p)$ implies $p(s) \neq p(t)\}$ ordered by reverse inclusion.

**Lemma 4.** Let $\{a_\alpha : \alpha \in \omega_1\}$ be a family of pairwise disjoint finite subsets of $T$. Then there is an infinite set $Y \subset \omega_1$ such that $a_\alpha : \alpha \in Y$ pairwise fit together.
Proof. The usual proof of c.c.c.-ness of $P$ [S, p. 103] shows that there are $\alpha \neq \beta$ with $a_\alpha, a_\beta$ fitting together. The lemma follows from the Erdős-Dushnik-Miller partition relation $\omega_1 \to (\omega_1, \omega)$ applied to the function $f : [\omega_1]^2 \to 2$ defined by $f(\alpha, \beta) = 0$ if $a_\alpha, a_\beta$ fit together.

Now assume that $\kappa$ is a large regular cardinal, $N, H_\kappa$ is a countable submodel with $T \in N$ and $f$ is $N$-big. We shall prove that $P \forces \#f = N[G]$-big. Then, since the forcing $P$ is c.c.c., any condition in it is $N$-master and it witnesses the friendliness of $P$ as desired.

For contradiction, let $p \in P, n \in \omega$ and $\dot{h} \in N$ be such that

1. $p \forces \text{"for all } m \in \omega, |\dot{h}(m)| \leq m",$
2. $p \forces \text{"for every } m > n, f(m) \in \dot{h}(m)".$

Let $p_0 = p \cap N \in N$. By c.c.c.-ness of $P$, by strengthening the condition $p$ if necessary one can arrange that $p_0 \forces \text{"for all } m \in \omega, |\dot{h}(m)| \leq m".$

Work in $N$. Fix an integer $m > n$ and by a tree induction construct a tree $X \subset {}^\omega(<\omega + 1)$, a partition $X = X_0 \cup X_1$ and a function $F : X \to P$ so that

1. the empty sequence 0 is in $X$ and $F(0) = p_0$,
2. $s \subset t$ in $X$ implies $F(s) \supseteq F(t)$ in $P$,
3. and for each $s \in X$ with $lth(s) = i$ exactly one of the following holds:
   - either, there is a sequence $\langle q_j : j \in \omega \rangle$ such that each $q_j \subseteq F(s)$ forces in $P$ that $i \in \dot{h}(m)$, and moreover, the sets $\text{dom}(q_j \setminus F(s)) : j \in \omega$ pairwise fit together. In this case, $s \in X_0$, the set of successors of $s$ in $X$ is exactly $
\text{dom}(q_j \setminus F(s)) = q_j$,
- or, no such sequence exists. Then $s \in X_1$, the only successor of $s$ in the tree $X$ is $s^\omega(\omega)$ and $F(s^\omega(\omega)) = F(s)$.

A set $o(X) \subset \omega$ is defined by $i \in o(X)$ iff there exists a function $G : X \to \omega$ such that for every sequence $s \in X$ of length $i$, if $\forall j \in i s(j) > G(s \upharpoonright j)$ then $s \in X_0$.

Claim 5. $|o(X)| \leq m$.

Proof. Suppose for a contradiction that there are $m + 1$ elements of $|o(X)|$, enumerated in the increasing order as $i_0$ through $i_m$. Pick witnesses $G_k : k \leq m$ for $i_k \in o(X)$. Then for every sequence $s \in X$ of length $i_m + 1$ such that $\forall j \leq i_m \forall k \leq m s(j) > G_k(s \upharpoonright j)$ (and there are plenty of these) the value $F(s)$ as an element of $P$ forces each one of the $m + 1$ distinct integers $i_k : k \leq m$ into the set $\dot{h}(m)$. But this is absurd, since $p \supseteq F(s)$ and $p \forces |\dot{h}(m)| \leq m$.

Claim 6. $f(m) \in o(X)$.

Proof. The proof of this fact takes place outside of the model $N$. To define the witness $G : X \to \omega$ for $f(m) \in o(X)$, consider two cases:

1. $s \in X_1$. Then let $G(s) = 0$.
2. $s \in X_0$. By (3) above, $\{a_j = \text{dom}(F(s^\omega(\omega)) \setminus F(s)) : j \in \omega \}$ is a family of pairwise disjoint fitting finite subsets of the tree $T$. There is an integer $j_0$ such that for every $j > j_0$, the sets $a_j$ and $\text{dom}(p \setminus p_0)$ fit together; set $G(s) = j_0$.

The existence of an integer $j_0$ as in (2) above can be demonstrated as follows. By elementarity of the model $N$, if $u \in N \cap T$ and $t \in \text{dom}(p \setminus p_0)$ are compatible as elements of the tree $T$, then necessarily $u \prec_T t$, for if $t \prec_T u$ then $t \in N$, contradicting the definition of the condition $p_0$. By the mutual fitting of the $a_j$’s, only finitely many of the sets $a_j \subset N \cap T$ can have nonempty intersection with the
Theorem 6. Let \( X \) is valid at \( \beta \).

Remark. The result becomes rather trivial if the tree \( T \) is supposed to have countable levels. In such a case, it is easy to prove that every real added by the specialization forcing comes from a Cohen-generic extension. Thus the specialization forcing for \( T \) must necessarily be friendly by results of [JS].

Remark. The same technology can be used to demonstrate friendliness of a number of finite condition forcings, whose c.c.c. is proved in a certain canonical manner. For example, let \( \langle f_\alpha : \alpha \in \omega_1 \rangle \subset {}^\omega \omega \) be a modulo finite increasing unbounded sequence of increasing functions. Let the partition \( H : [\omega_1]^2 \rightarrow 2 \) be defined as \( H(\alpha, \beta) = 0 \) if \( \alpha \prec \beta \) and there is an integer \( n \in \omega \) with \( f_\alpha(n) > f_\beta(n) \). It is known [T2] that the poset \( P \) of finite 0-homogeneous sets ordered by reverse inclusion is c.c.c. and destroys the unboundedness of the sequence. Using the same method as above, it is possible to show that \( P \) is friendly. The following is open:

Question 7. Is OCA [T2] consistent with additivity of the null ideal equal to \( \aleph_1 \)?

3. Side condition forcings

The purpose of this section is to define the class of ideal-based forcings and to prove friendliness of elements of this class. Our scheme is designed to comprehend many side condition forcings as used in the work of S. Todorcevic [T1], [T2], [T3] and others. Let \( A \) be a set of finite subsets of \( \omega_1 \) ordered by \( \subseteq \) and let \( \mathcal{F} \) be an ideal on \( \omega_1 \) such that the following axioms are satisfied:

(A) \( \subseteq \) refines the inclusion, for each \( a \in A \) and \( \beta \in \omega_1 \) \( a \cap \beta \subseteq a \) holds and if \( a, b \) are both in \( A \) and \( \subseteq \)-compatible then \( a \cup b \in A \) is their \( \subseteq \)-upper bound.
(B) $\mathcal{I}$ contains singletons, every $\mathcal{I}$-positive set has a countable $\mathcal{I}$-positive subset and the $\sigma$-ideal $\sigma\mathcal{I}$ generated by $\mathcal{I}$ is proper.

Moreover, for each $a \in A$ there are

(C) a $\mathcal{I}$-positive set $Z \subset \omega_1$ such that $a \subseteq a \cup \{\beta\} \in A$ holds for every $\beta \in Z$,

(D) an $\mathcal{I}$-large set $Y \subset \omega_1$ such that for every $\beta \in Y$ the implication $a \cap \beta \subseteq (a \cap \beta) \cup \{\beta\} \in A \rightarrow a \subseteq a \cup \{\beta\} \in A$ holds.

The pair $\langle A, \subseteq \rangle$ is to be understood as a problematic finite-condition forcing construction for which c.c.c. cannot be proved, or which collapses $\aleph_1$ outright. The existence of the ideal ensures that there is a way to add a $\subseteq$-filter which meets many dense subsets of $\langle A, \subseteq \rangle$. Fix a large regular cardinal $\kappa$. The ideal-based forcing $P$ derived from $A, \subseteq, \mathcal{I}$ has the following form:

$$P = \{f : f \text{ is a finite function from } \omega_1 \text{ to } H_\kappa, \text{ for } \alpha \in \text{dom}(f) \, f(\alpha) = \langle M_\alpha, \xi_\alpha \rangle \text{ and} \}$$

(E) the set $body(f) = \{\xi_\alpha : \alpha \in \text{dom}(f)\}$ is in $A$,

(F) every $M_\alpha$ is a countable elementary submodel of $H_\kappa$ containing $A, \subseteq, \mathcal{I}, f \upharpoonright \alpha$,

(G) $\xi_\alpha \notin \bigcup(M_\alpha \cap \mathcal{I})$.

The order on $P$ is defined by $f \leq g$ if $g \subset f$ and $body(g) \subseteq body(f)$.

The forcing $P$ adds a $\subseteq$-filter $\{a \in A : a = body(p) \text{ for some } p \in G\}$ which meets all dense subsets of $\langle A, \subseteq \rangle$ which are in some sense large as measured by $\mathcal{I}$.

Frequently, for the sake of preservation of $\aleph_2$ one needs to consider an amended variation of $P$ which has matrices of models as side conditions instead of just an $\varepsilon$-chain of models as above [T2]. We call such forcings amended ideal-based: since our proofs carry over to the class of amended ideal-based forcings with only more complicated notation, we concentrate on the class of ideal-based forcings proper. The point of course is that this class is reasonably wide; indeed, our scheme includes many of the side condition posets used in the literature. The following fact provides a by no means complete list.

**Fact 8.** The forcings for the following problems are (amended) ideal-based:

1. (S) conjecture [T2],
2. making a poset of uniform density $\aleph_1$ add $\aleph_1$ Cohen reals [SZ],
3. classification of transitive relations on $\aleph_1$ [T1], [T3],
4. shooting an uncountable set through a coherent sequence on $\omega_1$ [T2].

**Proof.** We consider the case of (S) conjecture. As in [T2], it is only necessary to cope with the following problem. Let $\omega_1$ be equipped with topology $\mathcal{I}$ so that the space $(\omega_1, \mathcal{I})$

1. is hereditarily separable, that is, for every $X \subset \omega_1$ there is a countable subset $Y \subset X$ with the same closure,
2. is not hereditarily Lindelöf, and it is even right separated, that is, for each $\alpha \in \omega_1$ there is an open set $O_\alpha$ such that $\alpha \in O_\alpha$ and the closure of $O_\alpha$ is a subset of $\alpha + 1$.

We wish to violate the hereditary separability of the space $(\omega_1, \mathcal{I})$ by introducing an uncountable discrete subset to it. The forcing for doing that [T2] can be cast as an ideal-based forcing derived from $A = [\omega_1]^{<\aleph_0}$, $a \subseteq b$ just in case $a \subset b$ and for every $\xi \in (b \setminus a)$ and every $\zeta \in a \xi \notin O_\zeta$; furthermore, $\mathcal{J} = \{X \subset \omega_1 : \text{the closure of } X \text{ is countable}\}$. It is not difficult to check the axioms (A) through (D) in the definition of ideal-based. (B) follows from hereditary separability and (D) from $\mathcal{I}$-smallness of every $O_\alpha$. 

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The intended uncountable discrete set will be $\bigcup\{\text{body}(f) : f \in G\}$, where $G \subset P$ is a generic filter.

**Theorem 9.** Any ideal-based forcing $P$ is proper and friendly.

**Proof.** Let $A, \subseteq, J, \kappa$ be the parameters from which $P$ is defined. To prove the properness, let $p_0 \in P, \lambda$ be a large regular cardinal, let $N \prec H_\lambda$ be a countable elementary submodel with $p_0, A \subseteq J, \kappa \in N$, and let $\delta = N \cap \omega_1$. By (C) there is a countable ordinal $\xi$ such that $\xi \notin (J \cap N)$ and $\text{body}(p_0) \sqsubseteq \text{body}(p_0) \cup \{\xi\} \in A$. Let $p_1 = p_0 \cup \{\langle \delta, (N \cap H_\kappa, \xi)\rangle\}$. Obviously, $p_1 \leq p_0$ and we shall show that $p_1$ is a master condition for the model $N$. Thus, for any dense set $D$ of $P$ which happens to be in $N$, the set $D \cap N$ must be proved predense below $p_1$. Fix $p_2 \leq p_1$ and a dense set $D \in N$; we shall produce conditions $p_5 \leq p_2$ and $q \in D \cap N$ with $p_5 \leq q$, completing the proof of properness. By strengthening $p_2$ if necessary, it can be assumed that there is an element of $D$ above $p_2$.

Let $p_3 = p_2 \cap N$. Obviously $p_3 \in P \cap N$ and $p_2 \leq p_3$, by (A). The whole point of the proof is to find a way of carefully extending $p_3$ within $N$ while preserving compatibility with $p_2$. Let $k = |p_2 \setminus p_3|$ and $\xi_0 \ldots \xi_{k-1}$ enumerate $\text{body}(p_2) \setminus \text{body}(p_3)$ in the increasing order. By induction on $l < k$ define sets $S(t)(l) \subset \omega_1$ for all $t \in \omega_1$ simultaneously by

1. $S(t)(0) = \{\zeta \in \omega_1 \setminus \text{max(\text{body}(p_3) \cup \text{rng}(t))} : \exists p_4 \leq p_3$ such that $p_4$ has an element of $D$ above it and $\text{body}(p_3) \cup \text{rng}(t) \sqsubseteq \text{body}(p_4) = \text{body}(p_3) \cup \text{rng}(t) \cup \{\zeta\} \in A\}$.
2. $S(t)(l+1) = \{\zeta \in \omega_1 \setminus \text{max(\text{body}(p_3) \cup \text{rng}(t))} : \text{body}(p_3) \cup \text{rng}(t) \sqsubseteq \text{body}(p_3) \cup \text{rng}(t) \cup \{\zeta\} \in A \text{ and the set } S(t^\langle \zeta \rangle)(l) \text{ is } J\text{-positive}\}$.

**Claim 10.** The set $S(t^\langle \zeta \rangle)(k - 1)$ is $J$-positive.

**Proof.** Note that the system $\{S(t)(l) : t \in \omega_1, l < k\}$ belongs to all the models mentioned in $p_2$ above $\delta$ since it is in $N \cap H_{k \omega_1}$. The claim will be proved by contradiction. If $S(t^\langle \zeta \rangle)(k - 1)$ were an element of $J$, by induction on $l < k$ one could show that $\xi \notin S(t^\langle \zeta \rangle)(k - 1)$ if $\xi, p_2 \setminus p_3$, and $\xi \notin S(t^\langle \zeta \rangle)(k - 1)$ (1) of the definition of the system $\{S(t)(l) : t \in \omega_1, l < k\}$, since $\xi_{k-1} \in S(t^\langle \zeta \rangle)(l)$ as witnessed by the condition $p_2$.

Now by induction on $l < k$ build $\zeta_l, T_l, X_l$ so that

1. for $l \leq k$, $a_l = \langle \zeta_l : l' < l \rangle$ is an increasing sequence of countable ordinals larger than $\text{max}(\text{body}(p_3))$ in $N$,
2. $T_l \in N$ is a $J$-positive countable subset of the $J$-positive set $S(a_l)(k - 1)$, by (B),
3. $\text{body}(p_2)$ and $\text{body}(p_3) \cup \text{rng}(a_l)$ are $\subseteq$-compatible and an $J$-large set $X_l \subset \omega_1$ is a witness to (D) for $\text{body}(p_2) \cup \text{rng}(a_l)$,
4. $\zeta_l \in T_l \cap X_l$.

By the construction, $a_k \in N$, $\text{body}(p_2)$ and $\text{body}(p_3) \cup \text{rng}(a_k)$ are $\subseteq$-compatible and moreover, there is a condition $p_4 \leq p_3$ such that there is an element of $D$ above it and $\text{body}(p_4) = \text{body}(p_3) \cup \text{rng}(a_k)$. By the elementarity of the model $N$, there are such $p_4$ and $q \in D$ above it already in $N$. By the definition of the forcing $P$, $p_5 = p_4 \cup p_2$ is a lower bound of $p_4$ and $p_2$ and has $q \in D \cap N$ above it as desired.

The friendliness of $P$ is proved by a trick similar to the one in Section 2. Let us adopt the framework from the proof of properness of $P$, in particular, choose
it is necessary to define the witness collection; our candidate lies outside of $N$.

For contradiction, let $p_2 \leq p_1, h \in N$ and $n \in \omega$ be such that

1. $p_2 \models \text{“for all integers } m, |h(m)| \leq m\text{”}$,
2. $p_2 \models \text{“for all integers } m > n, f(m) \in \dot{h}(m)\text{”}$.

Let $p_3 = p_2 \cap N$. Then $p_3 \geq p_2$ is in $N$ and by strengthening the condition $p_2$ if necessary, we may assume that $p_3 \models \text{“for all integers } m, |\dot{h}(m)| \leq m\text{”}$.

Let $k = |p_2 \setminus p_3| \geq 1$.

Work in $N$. Fix an integer $m > n$. By a tree induction construct a tree $X \subset <\omega, \omega$, its subset $X_0$ and functions $F : X \to P, T : X_0 \to V$ so that:

1. the empty sequence $0$ is in $X$ and $F(0) = p_3$,
2. for every $s \subset t$ both in $X$, $F(t) \leq F(s)$ holds in $P$.

Moreover, at each $s \in X$, exactly one of the two following cases will hold.

Case 1. There is a tree $T(s)$ on $<\omega_\omega$ such that

(a) the empty sequence is in $T(s)$ and $T$ consists of increasing sequences of ordinals above $\max(\text{body}(F(s)))$,
(b) for every $t \in T(s)$ of length $< k$ the set $\{ \zeta \in \omega_1 : t^\frown \zeta \in T(s) \}$ is countable and $\mathfrak{I}$-positive; moreover, for each $\zeta$ in this set, $\text{body}(F(s)) \cup \text{rng}(t) \subseteq \text{body}(F(s)) \cup \text{rng}(t)$.
(c) for every $t \in T(s)$ of length $k$ (i.e., a terminal node) there is a condition $q_t \leq F(s)$ in $P$ such that $\text{body}(q_t) = \text{body}(F(s)) \cup \text{rng}(t)$ and $q_t \models \text{“length}(s) \in h(m)\text{”}$.

In this case, let $s \in X_0$, $T(s)$ will be the value of $T$ at $s$ and $s^\frown \langle l \rangle \in X$ for all $l \in \omega$. Also, $F(s^\frown \langle l \rangle) : n \in \omega$ enumerates the set $\{ q_t : t \in T(s) \text{ is a terminal node} \}$.

Case 2. No such tree exists. Then $s \notin X_0$, the only successor of the sequence $s$ in $X$ is $s^\frown \langle 0 \rangle$ and $F(s) = F(s^\frown \langle 0 \rangle)$.

This completes the inductive definition of the tree $X$ in $N$. Define a set $o(X) \subset \omega$ by $i \in o(X)$ if there exists a collection $\{ A(s, t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ of $\mathfrak{I}$-large sets so that for each sequence $u \in X$ of length $i$ if (*) below holds for all $j < i$ then $u \in X_0$.

(*) If $u \upharpoonright j \in X_0$ then $F(u \upharpoonright (j + 1)) = q_t$ for some terminal node $t \in T(u \upharpoonright j)$ such that $\forall l < k \in t \upharpoonright (l + 1) \in A(u \upharpoonright j, t \upharpoonright l)$.

Claim 11. $|o(X)| \leq m$.

Proof. For contradiction, suppose that there are $m + 1$ elements of $o(X)$ enumerated in the increasing order as $i_0$ through $i_m$. Pick witnesses $\{ A_i(s, t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ for $i_i \in o(X)$ and define $\{ A(s, t) : s \in X_0, t \in T(s) \text{ is not a terminal node} \}$ by $A(s, t) = \bigcap_{i \leq m} A_i(s, t)$. Now each $A(s, t)$ is still an $\mathfrak{I}$-large set and so it is possible to find a sequence $u \in X$ of length $i_m + 1$ such that (*) holds for each $j < i_1 \mathfrak{I} h(u)$. But then, from the construction of $X$ and $F$ it follows that $F(u) \models \text{“}\{ i_0, \ldots, i_m \} \subset \dot{h}(m)\text{”}$, which is absurd since $F(u) \leq p_3$ and $p_3 \models \text{“}|\dot{h}(m)| \leq m\text{”}$.

Claim 12. $f(m) \in o(X)$.

Proof. It is necessary to define the witness collection; our candidate lies outside of $N$. For $s \in X_0$ and $t \in T(s)$ there will be two cases:
(1) Either body(p₂) and body(F(s)) ∪ rng(t) are ⊆-compatible. In such a case let A(s, t) be an Σ-large witness to (D) for body(p₂) ∪ body(F(s)) ∪ rng(t).

(2) Otherwise let A(s, t) = ω₁.

Now suppose that a sequence u ∈ X of length f(m) satisfies (*) for all j < f(m).

By the definition of A(s, t) and the tree X, necessarily body(F(u)) and body(p₂) are ⊆-compatible and therefore p₂ and F(u) are compatible conditions in P. The proof of u ∈ X₀ is essentially a repetition of the proof of the properness of P with the condition p₂ replaced with F(u) ∪ p₂ and the phrase “element of D above it” replaced with “forces f(m) into h(m)”.

Now the definition of X, o(X) was uniform for integers m > n. Thus within the model N there is a sequence Xₘ, o(Xₘ) : m > n such that |o(Xₘ)| ≤ m and f(m) ∈ o(Xₘ) for every m > n. But then the sequence o(Xₘ) : m > n in the model N, understood as a function of m, contradicts N-bigness of the function f.

It should be remarked that while ideal-based forcings preserve add(N), they can add dominating functions. If (f₂ : α ∈ ω₁) ⊆ ω is a modulo finite increasing unbounded sequence of increasing functions then it is possible to derive an S-space from it [T2]. Then the ideal-based forcing killing that space adds a function which modulo finite dominates all f₂ : α ∈ ω₁.

4. Conclusion

At last, we are in a position to construct some interesting models of set theory with the additivity of the null ideal equal to N₁. The classical iteration vehicle gives

**Lemma 13.** Let (Pₐ : α ≤ θ, Qₐ : α < θ) be a countable support iteration of forcings such that Pₐ ⊩ “Qₐ is friendly” for each α < θ. Then P₀ ⊩ “the union of the null sets coded in the ground model is not null”.

It seems likely that in fact P₀ is a friendly forcing, but we have no argument for that.

**Proof.** By induction on β ≤ θ we shall demonstrate that P₃ ⊩ “the union of the null sets coded in the ground model is not null”. The limit step is handled by [BJ, Theorem 6.3.41]. For the successor step, assume that P₃ ⊩ “the union of the null sets coded in the ground model is not null”; we shall prove the same statement for P₃₊₁. Choose a generic filter G ∈ P₃ and work in V[G]. It is enough to show that Q₃ ⊩ “V ∩ ω is not localized”. For contradiction, suppose q ∈ Q₃, ḥ such that q ⊩ Q₃ “for all n ∈ ω, h(n) ≤ n and ḥ localizes V ∩ ω”. Choose a large regular cardinal κ and a countable elementary submodel N ⊆ Hκ with q, Q₃, ḥ ∈ N.

**Claim 14.** There is an N-big function f ∈ V ∩ ω.

**Proof.** By an easy bookkeeping argument, there is a function ν : ω → [ω]_<ω₀ such that ∀u ∈ ω |k(u)| ≤ n² and for every function l ∈ N with l : ω → [ω]_<ω₀, ∀u ∈ ω |l(u)| ≤ u there is an integer m₀ such that for every m > m₀, l(m) ⊆ k(m).

By the induction hypothesis, V ∩ ω is not 2-localized and therefore there is a function f ∈ V ∩ ω such that the set {m ∈ ω : f(m) ̸∈ k(m)} is infinite. Obviously, the function f is N-big.

The postulated friendly master condition r ≤ q for N, f contradicts the assumption q ⊩ “∀m > n f(m) ∈ ḥ(m)”.

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Keeping additivity of the null ideal small

Therefore, starting from a model of the Continuum Hypothesis, any sufficiently generic iteration of length \( \omega_2 \) of proper, \( \aleph_2 \)-p.i.c. [S, pg. 262] and friendly forcings will provide for \( \Vdash \) \( \text{"add}(\mathcal{N}) = \aleph_1, c = \aleph_2 \), all branchless trees of size \( \aleph_1 \) are special, (S) conjecture holds, every poset of uniform density \( \aleph_1 \) adds \( \aleph_1 \) Cohen reals etc."

In the construction, it is necessary to use amended ideal-based forcings in order to ensure \( \aleph_2 \)-c.c. of the resulting iteration. The standard bookkeeping arguments are left to the reader.

References


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