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KEEPING ADDITIVITY OF THE NULL IDEAL SMALL

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ABSTRACT. We shall show that various statements are consistent with additivity of the null ideal equal to $\aleph_1$; for example, "all branchless trees of size $\aleph_1$ are special", (S) conjecture and "there are only five cofinal types of directed posets of size $\aleph_1$".

0. INTRODUCTION

In this paper we provide machinery for proving that a certain large class of forcings has a certain regularity property. The class in question includes posets used for

(1) specializing branchless trees [S],
(2) (S) conjecture [T2],
(3) embedding the poset for adding $\aleph_1$ Cohen reals into a given poset of uniform density $\aleph_1$ [SZ],
(4) classification of directed posets [T1] or transitive relations [T3] of size $\aleph_1$,
(5) other "side condition" combinatorics on $\aleph_1$; e.g. shooting an uncountable set through a coherent sequence on ω_1 [T2].

The regularity property we obtain implies preservation of additivity of the ideal of Lebesgue null sets. As a corollary, it is consistent with ZFC set theory that additivity of the null ideal is $\aleph_1$ and all statements obtainable through (1)–(5) above hold, that is, all branchless trees of size $\aleph_1$ are special, (S) conjecture holds etc. Thus a definite limitation on a canonical variation of the powerful "side condition" method has been exacted for the first time.

Our notation follows the set-theoretical standard as set forth in [J]. In a forcing notion, $p \leq q$ means "p is more informative than q". A tree T of height ω_1 is special if there is a function $f : T \to \omega$ with $s < t$ in T implying $f(s) \neq f(t)$. Trees grow upwards. (S) conjecture is the statement "every hereditarily separable Hausdorff space is hereditarily Lindelöf". The symbol $\mathcal{N}$ denotes the collection of Borel null sets, often confused with their Borel codes. H_κ is the set of all sets of hereditary cardinality $< \kappa$. If N is an elementary submodel of H_κ and $P \in N$ is a forcing, a condition $p \in P$ is called N-master if for every dense set $D \subset P$ in N, the set $D \cap N$ is predense below p.

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1. Localization

Definition 1. Let $\mathfrak{F} \subset {}^\omega\omega$ and let e be a positive integer. The family $\mathfrak{F}$ is said to be e-localized if there exists a function $h : \omega \to [\omega]^{<\aleph_0}$ such that

1. $|h(n)| \leq n^e$ for every integer n,
2. for every $f \in \mathfrak{F}$ there is an integer $n \in \omega$ so that for every $m > n$, $f(m) \in h(m)$ holds.

If $e = 1$ then the family $\mathfrak{F}$ is said to be localized.

The relevance of the above definition is revealed in the result of Bartoszyński [B, BJ Section 2.3.A] saying that for a transitive model M of ZFC the following are equivalent:

1. $M \cap {}^\omega\omega$ is localized;
2. $M \cap {}^\omega\omega$ is e-localized for some positive integer e;
3. the union of all Lebesgue measure zero Borel sets coded in M has measure zero.

There is a natural c.c.c. forcing for making the set of ground model reals localized [Tr] and there are some preservation theorems for "the set of ground model reals is not localized"[JS], [BJ]. We shall prove that a large class of forcings preserves unlocalized families in a strong sense.

Definition 2. (1) [JS] Let κ be a large regular cardinal and $N \prec H_\kappa$. We say that a function $f \in {}^\omega\omega$ is N-big if for every $h : \omega \to [\omega]^{<\aleph_0}$ in the model N with $|h(n)| \leq n$ the set $\{m \in \omega : f(m) \notin h(m)\}$ is infinite.

(2) Let P be a forcing. We say that P is friendly if for every $p \in P$, every large enough regular cardinal κ, every countable elementary submodel $N \prec H_\kappa$ with p, P in N and every N-big function $f \in {}^\omega\omega$ there is an N-master condition $q \leq p$ such that $q \Vdash$"$\check{f}$ is $N[G]$-big".

The important point is that it is possible to iterate friendly forcings preserving the statement "the family of ground model reals is not localized" or equivalently, "$\bigcup(\mathcal{N} \cap V) \notin \mathcal{N}$" –Lemma 13. Obviously, a finite iteration of friendly forcings is friendly and friendliness is inherited by regular subposets. In [JS] it is proved that the random algebra as well as every σ-centered forcing is friendly. We considerably extend these results.

2. Specializing trees

The purpose of this section is to prove that the usual specializing forcing for a branchless tree of height ω_1 is friendly. The technique will be of great use in the next section. For now, fix a tree T of height ω_1 and no branches of length ω_1. There is no restriction on the size of levels of T.

Definition 3. (1) If a, b are disjoint finite subsets of T then we say that a and b fit together if for every $s \in a$ and $t \in b$, s and t are incompatible as elements of T.

(2) The specialization forcing is $P = \{p : p$ is a finite function from T to ω such that $s <_T t$ in $dom(p)$ implies $p(s) \neq p(t)\}$ ordered by reverse inclusion.

Lemma 4. *Let $\{a_\alpha : \alpha \in \omega_1\}$ be a family of pairwise disjoint finite subsets of T. Then there is an infinite set $Y \subset \omega_1$ such that $a_\alpha : \alpha \in Y$ pairwise fit together.*

Proof. The usual proof of c.c.c.-ness of P [S, p. 103] shows that there are $\alpha \neq \beta$ with a_α, a_β fitting together. The lemma follows from the Erdős-Dushnik-Miller partition relation $\omega_1 \to (\omega_1, \omega)$ applied to the function $f : [\omega_1]^2 \to 2$ defined by $f(\alpha, \beta) = 0$ iff a_α, a_β fit together. □

Now assume that κ is a large regular cardinal, $N \prec H_\kappa$ is a countable submodel with $T \in N$ and f is N-big. We shall prove that $P \Vdash$"$\check{f}$ is $N[G]$-big". Then, since the forcing P is c.c.c., any condition in it is N-master and it witnesses the friendliness of P as desired.

For contradiction, let $p \in P, n \in \omega$ and $\dot{h} \in N$ be such that

(1) $p \Vdash$"for all $m \in \omega$, $|\dot{h}(m)| \leq m$",
(2) $p \Vdash$"for every $m > n$, $\check{f}(m) \in \dot{h}(m)$".

Let $p_0 = p \cap N \in N$. By c.c.c.-ness of P, by strengthening the condition p if necessary one can arrange that $p_0 \Vdash$"for all $m \in \omega$, $|\dot{h}(m)| \leq m$".

Work in N. Fix an integer $m > n$ and by a tree induction construct a tree $X \subset {}^{<\omega}(\omega + 1)$, a partition $X = X_0 \cup X_1$ and a function $F : X \to P$ so that

(1) the empty sequence 0 is in X and $F(0) = p_0$,
(2) $s \subset t$ in X implies $F(s) \geq F(t)$ in P,

and for each $s \in X$ with $lth(s) = i$ exactly one of the following holds:

(3) either, there is a sequence $\langle q_j : j \in \omega \rangle$ such that each $q_j \leq F(s)$ forces in P that $i \in \dot{h}(m)$, and moreover, the sets $dom(q_j \setminus F(s)) : j \in \omega$ pairwise fit together. In this case, $s \in X_0$, the set of successors of s in X is exactly $\{s^\frown\langle j \rangle : j \in \omega\}$ and $F(s^\frown\langle j \rangle) = q_j$,
(4) or, no such sequence exists. Then $s \in X_1$, the only successor of s in the tree X is $s^\frown\langle \omega \rangle$ and $F(s^\frown\langle \omega \rangle) = F(s)$.

A set $o(X) \subset \omega$ is defined by $i \in o(X)$ iff there exists a function $G : X \to \omega$ such that for every sequence $s \in X$ of length i, if $\forall j \in i\ s(j) > G(s \restriction j)$ then $s \in X_0$.

Claim 5. $|o(X)| \leq m$.

Proof. Suppose for a contradiction that there are $m + 1$ elements of $|o(X)|$, enumerated in the increasing order as i_0 through i_m. Pick witnesses $G_k : k \leq m$ for $i_k \in o(X)$. Then for every sequence $s \in X$ of length $i_m + 1$ such that $\forall j \leq i_m\ \forall k \leq m\ s(j) > G_k(s \restriction j)$ (and there are plenty of these) the value $F(s)$ as an element of P forces each one of the $m + 1$ distinct integers $i_k : k \leq m$ into the set $\dot{h}(m)$. But this is absurd, since $p \geq F(s)$ and $p \Vdash |\dot{h}(m)| \leq m$. □

Claim 6. $f(m) \in o(X)$.

Proof. The proof of this fact takes place outside of the model N. To define the witness $G : X \to \omega$ for $f(m) \in o(X)$, consider two cases:

(1) $s \in X_1$. Then let $G(s) = 0$.
(2) $s \in X_0$. By (3) above, $\{a_j = dom(F(s^\frown\langle j \rangle) \setminus F(s)) : j \in \omega\}$ is a family of pairwise disjoint fitting finite subsets of the tree T. There is an integer j_0 such that for every $j > j_0$, the sets a_j and $dom(p \setminus p_0)$ fit together; set $G(s) = j_0$.

The existence of an integer j_0 as in (2) above can be demonstrated as follows. By elementarity of the model N, if $u \in N \cap T$ and $t \in dom(p \setminus p_0)$ are compatible as elements of the tree T, then necessarily $u <_T t$, for if $t <_T u$ then $t \in N$, contradicting the definition of the condition p_0. By the mutual fitting of the a_j's, only finitely many of the sets $a_j \subset N \cap T$ can have nonempty intersection with the finitely many linearly ordered subsets $\{u \in N \cap T : u <_T t\} : t \in dom(p \setminus p_0)$ of the tree T. Any j_0 larger than the indexes of these sets will do.

Why does the function G have the desired properties? Well, choose an arbitrary sequence $s \in X$ of length $f(m)$ with $\forall j \in f(m)\ s(j) > G(s \restriction j)$. It is necessary to verify that $s \in X_0$. By the definition of G and X, the condition $F(s)$ is compatible with p. By induction on $\alpha \in \omega_1$ construct a sequence $\langle a_\alpha : \alpha \in \omega_1\rangle$ so that

(1) a_α are finite pairwise disjoint subsets of the tree T,
(2) for every $\alpha \in \omega_1$ there is $q \leq F(s)$ such that $q \Vdash$"$f(m) \in \dot{h}(m)$" and $a_\alpha = dom(q \setminus F(s))$.

Note that any sequence $\langle a_\alpha : \alpha \in \beta\rangle$ of countable length β satisfying (1,2) above for $\alpha \in \beta$ can be prolonged further to some $\langle a_\alpha : \alpha \in \beta+1\rangle$ such that (1,2) continue to hold. For if this were not the case, by elementarity there would be a witness $\langle a_\alpha : \alpha \in \beta\rangle \in N$ which cannot be prolonged. However, such a sequence from the model N *can* be prolonged using $a_\beta = dom(p \setminus p_0)$, with the condition $q = F(s) \cup p$ witnessing the property (2) at β.

By elementarity of the model N, there is a sequence $\langle a_\alpha : \alpha \in \omega_1\rangle$ satisfying (1,2) above in N. It follows from Lemma 4 that the case (3) of definition of the tree X is valid at s and $s \in X_0$. □

Now within the model N, for each integer $m > n$ construct a tree X_m and a set $o(X_m)$ as above. The function $g : \omega \setminus n \to [\omega]^{<\aleph_0}$ defined by $g(m) = o(X_m)$ is in the model N and contradicts the assumption of f being N-big. Therefore, we have proved

Theorem 6. *Let T be a tree of height ω_1 without branches of height ω_1. Then the standard T-specialization forcing is friendly.*

Remark. The result becomes rather trivial if the tree T is supposed to have countable levels. In such a case, it is easy to prove that every real added by the specialization forcing comes from a Cohen-generic extension. Thus the specialization forcing for T must necessarily be friendly by results of [JS].

Remark. The same technology can be used to demonstrate friendliness of a number of finite condition forcings, whose c.c.c. is proved in a certain canonical manner. For example, let $\langle f_\alpha : \alpha \in \omega_1\rangle \subset {}^\omega\omega$ be a modulo finite increasing unbounded sequence of increasing functions. Let the partition $H : [\omega_1]^2 \to 2$ be defined as $H(\alpha, \beta) = 0$ if $\alpha < \beta$ and there is an integer $n \in \omega$ with $f_\alpha(n) > f_\beta(n)$. It is known [T2] that the poset P of finite 0-homogeneous sets ordered by reverse inclusion is c.c.c. and destroys the unboundedness of the sequence. Using the same method as above, it is possible to show that P is friendly. The following is open:

Question 7. Is OCA [T2] consistent with additivity of the null ideal equal to $\aleph_1$?

3. Side condition forcings

The purpose of this section is to define the class of ideal-based forcings and to prove friendliness of elements of this class. Our scheme is designed to comprehend many side condition forcings as used in the work of S. Todorcevic [T1], [T2], [T3] and others. Let A be a set of finite subsets of ω_1 ordered by $\sqsubseteq$ and let $\mathfrak{I}$ be an ideal on ω_1 such that the following axioms are satisfied:

(A) $\sqsubseteq$ refines the inclusion, for each $a \in A$ and $\beta \in \omega_1$ $a \cap \beta \sqsubseteq a$ holds and if a, b are both in A and $\sqsubseteq$-compatible then $a \cup b \in A$ is their $\sqsubseteq$-upper bound.

(B) $\mathfrak{I}$ contains singletons, every $\mathfrak{I}$-positive set has a countable $\mathfrak{I}$-positive subset and the σ-ideal $\sigma\mathfrak{I}$ generated by $\mathfrak{I}$ is proper.

Moreover, for each $a \in A$ there are

(C) a $\sigma\mathfrak{I}$ positive set $Z \subset \omega_1$ such that $a \sqsubseteq a \cup \{\beta\} \in A$ holds for every $\beta \in Z$,
(D) an $\mathfrak{I}$-large set $Y \subset \omega_1$ such that for every $\beta \in Y$ the implication $a \cap \beta \sqsubseteq (a \cap \beta) \cup \{\beta\} \in A \to a \sqsubseteq a \cup \{\beta\} \in A$ holds.

The pair $\langle A, \sqsubseteq \rangle$ is to be understood as a problematic finite-condition forcing construction for which c.c.c. cannot be proved, or which collapses $\aleph_1$ outright. The existence of the ideal ensures that there is a way to add a $\sqsubseteq$-filter which meets many dense subsets of $\langle A, \sqsubseteq \rangle$. Fix a large regular cardinal κ. The ideal-based forcing P derived from $A, \sqsubseteq, \mathfrak{I}$ has the following form:

$P = \{f : f$ is a finite function from ω_1 to H_κ, for $\alpha \in dom(f)$ $f(\alpha) = \langle M_\alpha, \xi_\alpha \rangle$ and

(E) the set $body(f) = \{\xi_\alpha : \alpha \in dom(f)\}$ is in A,
(F) every M_α is a countable elementary submodel of H_κ containing $A, \sqsubseteq, \mathfrak{I}, f \restriction \alpha$,
(G) $\xi_\alpha \notin \bigcup (M_\alpha \cap \mathfrak{I})\}$.

The order on P is defined by $f \leq g$ if $g \subset f$ and $body(g) \sqsubseteq body(f)$.

The forcing P adds a $\sqsubseteq$-filter $\{a \in A : a = body(p)$ for some $p \in G\}$ which meets all dense subsets of $\langle A, \sqsubseteq \rangle$ which are in some sense large as measured by $\mathfrak{I}$.

Frequently, for the sake of preservation of $\aleph_2$ one needs to consider an amended variation of P which has matrices of models as side conditions instead of just an $\in$-chain of models as above [T2]. We call such forcings *amended ideal-based*; since our proofs carry over to the class of amended ideal-based forcings with only more complicated notation, we concentrate on the class of ideal-based forcings proper. The point of course is that this class is reasonably wide; indeed, our scheme includes many of the side condition posets used in the literature. The following fact provides a by no means complete list.

Fact 8. The forcings for the following problems are (amended) ideal-based:

(1) (S) conjecture [T2],
(2) making a poset of uniform density $\aleph_1$ add $\aleph_1$ Cohen reals [SZ],
(3) classification of transitive relations on $\aleph_1$ [T1], [T3],
(4) shooting an uncountable set through a coherent sequence on ω_1 [T2].

Proof. We consider the case of (S) conjecture. As in [T2], it is only necessary to cope with the following problem. Let ω_1 be equipped with topology $\mathfrak{T}$ so that the space $(\omega_1, \mathfrak{T})$

(1) is hereditarily separable, that is, for every $X \subset \omega_1$ there is a countable subset $Y \subset X$ with the same closure,
(2) is not hereditarily Lindelöf, and it is even right separated, that is, for each $\alpha \in \omega_1$ there is an open set O_α such that $\alpha \in O_\alpha$ and the closure of O_α is a subset of $\alpha + 1$.

We wish to violate the hereditary separability of the space $(\omega_1, \mathfrak{T})$ by introducing an uncountable discrete subset to it. The forcing for doing that [T2] can be cast as an ideal-based forcing derived from $A = [\omega_1]^{<\aleph_0}$, $a \sqsubseteq b$ just in case $a \subset b$ and for every $\xi \in (b \setminus a)$ and every $\zeta \in a$ $\xi \notin O_\zeta$; furthermore, $\mathfrak{I} = \{X \subset \omega_1 :$ the closure of X is countable$\}$. It is not difficult to check the axioms (A) through (D) in the definition of ideal-based. (B) follows from hereditary separability and (D) from $\mathfrak{I}$-smallness of every O_α.

The intended uncountable discrete set will be $\bigcup\{body(f) : f \in G\}$, where $G \subset P$ is a generic filter. □

Theorem 9. *Any ideal-based forcing P is proper and friendly.*

Proof. Let $A, \sqsubseteq, \mathfrak{I}, \kappa$ be the parameters from which P is defined. To prove the properness, let $p_0 \in P, \lambda$ be a large regular cardinal, let $N \prec H_\lambda$ be a countable elementary submodel with $p_0, A \sqsubseteq, \mathfrak{I}, \kappa \in N$, and let $\delta = N \cap \omega_1$. By (C) there is a countable ordinal ξ such that $\xi \notin \bigcup(\mathfrak{I} \cap N)$ and $body(p_0) \sqsubseteq body(p_0) \cup \{\xi\} \in A$. Let $p_1 = p_0 \cup \{\langle \delta, \langle N \cap H_\kappa, \xi\rangle\rangle\}$. Obviously, $p_1 \leq p_0$ and we shall show that p_1 is a master condition for the model N. Thus, for any dense set D of P which happens to be in N, the set $D \cap N$ must be proved predense below p_1. Fix $p_2 \leq p_1$ and a dense set $D \in N$; we shall produce conditions $p_5 \leq p_2$ and $q \in D \cap N$ with $p_5 \leq q$, completing the proof of properness. By strengthening p_2 if necessary, it can be assumed that there is an element of D above p_2.

Let $p_3 = p_2 \cap N$. Obviously $p_3 \in P \cap N$ and $p_2 \leq p_3$, by (A). The whole point of the proof is to find a way of carefully extending p_3 within N while preserving compatibility with p_2. Let $k = |p_2 \setminus p_3|$ and $\xi_0 \ldots \xi_{k-1}$ enumerate $body(p_2) \setminus body(p_3)$ in the increasing order. By induction on $l < k$ define sets $S(t)(l) \subset \omega_1$ for all $t \in {}^{<\omega}\omega_1$ simultaneously by

(1) $S(t)(0) = \{\zeta \in \omega_1 \setminus max(body(p_3) \cup rng(t)) : \exists p_4 \leq p_3$ such that p_4 has an element of D above it and $body(p_3) \cup rng(t) \sqsubseteq body(p_4) = body(p_3) \cup rng(t) \cup \{\zeta\} \in A\}$.
(2) $S(t)(l+1) = \{\zeta \in \omega_1 \setminus max(body(p_3) \cup rng(t)) : body(p_3) \cup rng(t) \sqsubseteq body(p_3) \cup rng(t) \cup \{\zeta\} \in A$ and the set $S(t^\frown\langle\zeta\rangle)(l)$ is $\mathfrak{I}$-positive$\}$.

Claim 10. The set $S(\langle\rangle)(k-1)$ is $\mathfrak{I}$-positive.

Proof. Note that the system $\{S(t)(l) : t \in {}^{<\omega}\omega_1, l < k\}$ belongs to all the models mentioned in p_2 above δ since it is in $N \cap H_{\aleph_2}$. The claim will be proved by contradiction. If $S(\langle\rangle)(k-1)$ were an element of $\mathfrak{I}$, by induction on $l < k$ one could show that $\xi_l \notin S(\langle \xi_{l'} : l' < l\rangle)(k-1-l)$. But this is a contradiction to the case (1) of the definition of the system $\{S(t)(l) : t \in {}^{<\omega}\omega_1, l < k\}$, since $\xi_{k-1} \in S(\langle \xi_l : l < k-1\rangle)(0)$ as witnessed by the condition p_2. □

Now by induction on $l < k$ build ζ_l, T_l, X_l so that

(1) for $l \leq k$, $a_l = \langle \zeta_{l'} : l' < l\rangle$ is an increasing sequence of countable ordinals larger than $max(body(p_3))$ in N,
(2) $T_l \in N$ is an $\mathfrak{I}$-positive countable subset of the $\mathfrak{I}$-positive set $S(a_l)(k-l-1)$, by (B),
(3) $body(p_2)$ and $body(p_3) \cup rng(a_l)$ are $\sqsubseteq$-compatible and an $\mathfrak{I}$-large set $X_l \subset \omega_1$ is a witness to (D) for $body(p_2) \cup rng(a_l)$,
(4) $\zeta_l \in T_l \cap X_l$.

By the construction, $a_k \in N$, $body(p_2)$ and $body(p_3) \cup rng(a_k)$ are $\sqsubseteq$-compatible and moreover, there is a condition $p_4 \leq p_3$ such that there is an element of D above it and $body(p_4) = body(p_3) \cup rng(a_k)$. By the elementarity of the model N, there are such p_4 and $q \in D$ above it already in N. By the definition of the forcing P, $p_5 = p_4 \cup p_2$ is a lower bound of p_4 and p_2 and has $q \in D \cap N$ above it as desired.

The friendliness of P is proved by a trick similar to the one in Section 2. Let us adopt the framework from the proof of properness of P, in particular, choose

$p_0, N \dots$ and the master condition $p_1 \leq p_0$ for the model N as constructed above. Let $f \in {}^\omega\omega$ be an N-big function. We shall show that $p_1 \Vdash$ "$\check{f}$ is $N[G]$-big".

For contradiction, let $p_2 \leq p_1, \dot{h} \in N$ and $n \in \omega$ be such that

(1) $p_2 \Vdash$ "for all integers $m, |\dot{h}(m)| \leq m$",
(2) $p_2 \Vdash$ "for all integers $m > n, \check{f}(m) \in \dot{h}(m)$".

Let $p_3 = p_2 \cap N$. Then $p_3 \geq p_2$ is in N and by strengthening the condition p_2 if necessary we may assume that $p_3 \Vdash$ "for all integers $m, |\dot{h}(m)| \leq m$". Let $k = |p_2 \setminus p_3| \geq 1$.

Work in N. Fix an integer $m > n$. By a tree induction construct a tree $X \subset {}^{<\omega}\omega$, its subset X_0 and functions $F : X \to P, T : X_0 \to V$ so that:

(1) the empty sequence 0 is in X and $F(0) = p_3$,
(2) for every $s \subset t$ both in X, $F(t) \leq F(s)$ holds in P.

Moreover, at each $s \in X$, exactly one of the two following cases will hold.

Case 1. There is a tree $T(s)$ on ${}^{\leq k}\omega_1$ such that

(a) the empty sequence is in $T(s)$ and T consists of increasing sequences of ordinals above $max(body(F(s)))$,
(b) for every $t \in T(s)$ of length $< k$ the set $\{\zeta \in \omega_1 : t^\frown\langle\zeta\rangle \in T(s)\}$ is countable and $\mathfrak{I}$-positive; moreover for each ζ in this set, $body(F(s)) \cup rng(t) \sqsubseteq body(F(s)) \cup rng(t) \cup \{\zeta\} \in A$,
(c) for every $t \in T(s)$ of length k (i.e. a terminal node) there is a condition $q_t \leq F(s)$ in P such that $body(q_t) = body(F(s)) \cup rng(t)$ and $q_t \Vdash$ "length$(s) \in \dot{h}(m)$".

In this case, let $s \in X_0$, $T(s)$ will be the value of T at s and $s^\frown\langle l\rangle \in X$ for all $l \in \omega$. Also, $F(s^\frown\langle l\rangle) : n \in \omega$ enumerates the set $\{q_t : t \in T(s)$ is a terminal node$\}$.

Case 2. No such tree exists. Then $s \notin X_0$, the only successor of the sequence s in X is $s^\frown\langle 0\rangle$ and $F(s) = F(s^\frown\langle 0\rangle)$.

This completes the inductive definition of the tree X in N. Define a set $o(X) \subset \omega$ by $i \in o(X)$ if there exists a collection $\{A(s,t) : s \in X_0, t \in T(s)$ is not a terminal node$\}$ of $\mathfrak{I}$-large sets so that for each sequence $u \in X$ of length i if (*) below holds for all $j < i$ then $u \in X_0$.

(*) If $u \restriction j \in X_0$ then $F(u \restriction (j+1)) = q_t$ for some terminal node $t \in T(u \restriction j)$ such that $\forall l < k\ t(l+1) \in A(u \restriction j, t \restriction l)$.

Claim 11. $|o(X)| \leq m$.

Proof. For contradiction, suppose that there are $m+1$ elements of $o(X)$ enumerated in the increasing order as i_0 through i_m. Pick witnesses $\{A_l(s,t) : s \in X_0, t \in T(s)$ is not a terminal node$\}$ for $i_l \in o(X)$ and define $\{A(s,t) : s \in X_0, t \in T(s)$ is not a terminal node$\}$ by $A(s,t) = \bigcap_{l \leq m} A_l(s,t)$. Now each $A(s,t)$ is still an $\mathfrak{I}$-large set and so it is possible to find a sequence $u \in X$ of length $i_m + 1$ such that (*) holds for each $j < lth(u)$. But then, from the construction of X and F it follows that $F(u) \Vdash$ "$\{i_0 \dots i_m\} \subset \dot{h}(m)$", which is absurd since $F(u) \leq p_3$ and $p_3 \Vdash$ "$|\dot{h}(m)| \leq m$". ☐

Claim 12. $f(m) \in o(X)$.

Proof. It is necessary to define the witness collection; our candidate lies outside of N. For $s \in X_0$ and $t \in T(s)$ there will be two cases:

(1) Either $body(p_2)$ and $body(F(s)) \cup rng(t)$ are $\sqsubseteq$-compatible. In such a case let $A(s,t)$ be an $\mathfrak{I}$-large witness to (D) for $body(p_2) \cup body(F(s)) \cup rng(t)$.
(2) Otherwise let $A(s,t) = \omega_1$.

Now suppose that a sequence $u \in X$ of length $f(m)$ satisfies (*) for all $j < f(m)$. By the definition of $A(s,t)$ and the tree X, necessarily $body(F(u))$ and $body(p_2)$ are $\sqsubseteq$-compatible and therefore p_2 and $F(u)$ are compatible conditions in P. The proof of $u \in X_0$ is essentially a repetition of the proof of the properness of P with the condition p_2 replaced with $F(u) \cup p_2$ and the phrase "element of D above it" replaced with "forces $f(m)$ into $\dot{h}(m)$". $\square$

Now the definition of $X, o(X)$ was uniform for integers $m > n$. Thus within the model N there is a sequence $X_m, o(X_m) : m > n$ such that $|o(X_m)| \leq m$ and $f(m) \in o(X_m)$ for every $m > n$. But then the sequence $o(X_m) : m > n$ in the model N, understood as a function of m, contradicts N-bigness of the function f. $\square$

It should be remarked that while ideal-based forcings preserve $add(\mathcal{N})$, they can add dominating functions. If $\langle f_\alpha : \alpha \in \omega_1 \rangle \subset {}^\omega\omega$ is a modulo finite increasing unbounded sequence of increasing functions then it is possible to derive an S-space from it [T2]. Then the ideal-based forcing killing that space adds a function which modulo finite dominates all $f_\alpha : \alpha \in \omega_1$.

4. Conclusion

At last, we are in a position to construct some interesting models of set theory with the additivity of the null ideal equal to $\aleph_1$. The classical iteration vehicle gives

Lemma 13. *Let $\langle P_\alpha : \alpha \leq \theta, \dot{Q}_\alpha : \alpha < \theta \rangle$ be a countable support iteration of forcings such that $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ is friendly" for each $\alpha < \theta$. Then $P_\theta \Vdash$ "the union of the null sets coded in the ground model is not null".*

It seems likely that in fact P_θ is a friendly forcing, but we have no argument for that.

Proof. By induction on $\beta \leq \theta$ we shall demonstrate that $P_\beta \Vdash$ "the union of the null sets coded in the ground model is not null". The limit step is handled by [BJ, Theorem 6.3.41]. For the successor step, assume that $P_\beta \Vdash$ "the union of the null sets coded in the ground model is not null"; we shall prove the same statement for $P_{\beta+1}$. Choose a generic filter $G \subset P_\beta$ and work in $V[G]$. It is enough to show that $Q_\beta \Vdash$ "$V \cap {}^\omega\omega$ is not localized". For contradiction, suppose $q \in Q_\beta, \dot{h}$ are such that $q \Vdash_{Q_\beta}$ "for all $n \in \omega$, $|\dot{h}(n)| \leq n$ and $\dot{h}$ localizes $V \cap {}^\omega\omega$". Choose a large regular cardinal κ and a countable elementary submodel $N \prec H_\kappa$ with $q, Q_\beta, \dot{h} \in N$.

Claim 14. There is an N-big function $f \in V \cap {}^\omega\omega$.

Proof. By an easy bookkeping argument, there is a function $k : \omega \to [\omega]^{<\aleph_0}$ such that $\forall n \in \omega\ |k(n)| \leq n^2$ and for every function $l \in N$ with $l : \omega \to [\omega]^{<\aleph_0}$, $\forall n \in \omega\ |l(n)| \leq n$ there is an integer m_0 such that for every $m > m_0, l(m) \subset k(m)$. By the induction hypothesis, $V \cap {}^\omega\omega$ is not 2-localized and therefore there is a function $f \in V \cap {}^\omega\omega$ such that the set $\{m \in \omega : f(m) \notin k(m)\}$ is infinite. Obviously, the function f is N-big. $\square$

The postulated friendly master condition $r \leq q$ for N, f contradicts the assumption $q \Vdash$ "$\exists n\ \forall m > n\ f(m) \in \dot{h}(m)$". $\square$

Therefore, starting from a model of the Continuum Hypothesis, any sufficiently generic iteration of length ω_2 of proper, $\aleph_2$-p.i.c. [S, pg. 262] and friendly forcings will provide for $\Vdash$"$add(\mathcal{N}) = \aleph_1, \mathfrak{c} = \aleph_2$, all branchless trees of size $\aleph_1$ are special, (S) conjecture holds, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals etc." In the construction, it is necessary to use amended ideal-based forcings in order to ensure $\aleph_2$-c.c. of the resulting iteration. The standard bookkeeping arguments are left to the reader.

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