

FUNCTIONS ON THE MODULI SPACE OF FLAT G_2 -CONNECTIONS ON A RIEMANN SURFACE

JOSEF MATTES

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ABSTRACT. This paper contains $1\frac{1}{2}$ proofs that the construction of functions on the moduli space of flat G -connections from chord diagrams gives all functions in the case of the simple group G_2 .

1. INTRODUCTION

By now it is well known that there are deep connections between knot theory and quantum field theory ([Wit89],[RT90],[RT91] etc.). On the other hand, a major recent development in knot theory was the invention of Vassiliev invariants and the theory of chord diagrams ([Vas90]). These employed standard techniques of topology and singularity theory, but soon it was discovered that these invariants are related to Feynman diagrams and Lie algebras (see e.g. [BN95]).

Motivated by earlier work of Goldman [Gol86] and Turaev [Tur91], chord diagrams on an arbitrary Riemann surface Σ were introduced in [AMR]. Fix a Lie group G with invariant inner product on the Lie algebra. Functions on moduli space form a Poisson algebra whose quantization attracted a lot of attention (see for example [Ati90]). To each chord diagram D one can associate a function $f_{\rho,D}$ on the moduli space of flat G -connections over Σ (we have to colour D by representations ρ of G). It turns out that one can define a simple Poisson structure on the algebra of chord diagrams which trivially extends to coloured chord diagrams such that the map $D \mapsto f_{\rho,D}$ is a Poisson homomorphism. Since the Poisson structure on chord diagrams is independent of G this suggests the possibility of a universal quantization of moduli spaces.

In [AMR] it was shown for some groups that the map $D \mapsto f_{\rho,D}$ is surjective. This was done by reducing it to a problem about intertwiners for representations of the Lie algebra \mathfrak{g} of G ([AMR], Theorem 10). In this paper we show surjectivity for the simple group G_2 .

2. NOTATIONS AND CONVENTIONS

The Dynkin diagram will be taken to be $1 \Rightarrow 2$. The standard representation $V = V_{\omega_2} = V_{0,1}$ is also a representation of \mathfrak{b}_3 . A weight basis for V is denoted by

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v_1, v_2, v_3, \dots with v_1 being the highest weight vector. The adjoint representation is $V_{\omega_1} = V_{1,0}$.

Let $\{I_\mu\}$ be an orthonormal basis for \mathfrak{g}_2 with respect to the given invariant inner product. c_λ will denote the action of the quadratic Casimir $\sum I_\mu I_\mu$ on the irreducible representation V_λ .

c stands for $\sum I_\mu \otimes I_\mu$. Given a tensor product of representations $V_1 \otimes \dots \otimes V_n$ the action of $\sum 1 \otimes \dots \otimes 1 \otimes I_\mu^i \otimes 1 \otimes \dots \otimes I_\mu^j \otimes \dots \otimes 1$ is denoted by c_{ij} .

3. PROOF OF SURJECTIVITY

Since every irreducible representation of \mathfrak{g}_2 imbeds into some tensor power $V^{\otimes n}$ of the standard representation it follows from Theorem 10 of [AMR] that the map from chord diagrams to functions on moduli space is surjective provided

$$End^{\mathfrak{g}_2}(V^{\otimes n}) = \mathbb{C}(\{c_{ij}\}_{i < j}^n).$$

Lemma 3.1. $End^{\mathfrak{g}_2}(V^{\otimes 2}) = \mathbb{C}(c)$.

Proof. In the decomposition

$$V_{0,1} \otimes V_{0,1} = V_{0,0} \oplus V_{1,0} \oplus V_{0,1} \oplus V_{0,2}$$

the Casimir operator has a different eigenvalue on each summand. Therefore any intertwiner is a linear combination of c^0, \dots, c^3 . □

The following is an interesting consequence. Recall that $\mathfrak{g}_2 \subseteq \mathfrak{b}_3$ and hence $End^{\mathfrak{b}_3}(V^{\otimes n}) \subseteq End^{\mathfrak{g}_2}(V^{\otimes n})$.

Corollary 3.2. Any \mathfrak{b}_3 -intertwiner of $V^{\otimes n}$ is generated by Casimirs for \mathfrak{g}_2 , i.e. $End^{\mathfrak{b}_3}(V^{\otimes n}) \subseteq \mathbb{C}(\{c_{ij}\}_{i < j}^n)$.

Proof. Any \mathfrak{b}_3 -intertwiner of $V^{\otimes n}$ is a composition of intertwiners $c_{ij}^{\mathfrak{b}_3}$. These act nontrivially only in 2 factors in $V^{\otimes n}$ and the lemma implies they can be written as compositions of Casimirs. □

Lemma 3.3. $End^{\mathfrak{g}_2}(V^{\otimes 3}) = \mathbb{C}(c_{12}, c_{13}, c_{23})$

Proof. We compare

$$(3.1) \quad V^{\otimes 3} = V_{0,0} \oplus 2V_{1,0} \oplus 4V_{0,1} \oplus 2V_{1,1} \oplus 3V_{0,2} \oplus V_{0,3}$$

with

$$V^{\otimes 3} =_{\mathfrak{b}_3} 3V_{1,0,0} \oplus 2V_{1,1,0} \oplus V_{0,0,2} \oplus V_{3,0,0}.$$

The \mathfrak{b}_3 modules branch as follows: $V_{1,0,0} = V_{0,1}, V_{1,1,0} = V_{1,0} \oplus V_{1,1} \oplus V_{0,2}, \Lambda^3 V = V_{0,0,2} = V_{0,0} \oplus V_{0,1} \oplus V_{0,2}, V_{3,0,0} = V_{0,3}$.

Using the fact that $c_{V^{\otimes 3}} = 3c_{\omega_1} + 2(c_{12} + c_{13} + c_{23}) \in \mathbb{C}(c_{12}, c_{13}, c_{23})$ has distinct eigenvalues on the summands in (3.1) we can project onto any summand in

$$\begin{aligned}
 End^{\mathfrak{g}_2}(V^{\otimes 3}) &= End(V_{0,0}) \oplus End(2V_{1,0}) \oplus End(4V_{0,1}) \\
 &\quad \oplus End(2V_{1,1}) \oplus End(3V_{0,2}) \oplus End(V_{0,3}).
 \end{aligned}$$

The summands $V_{0,0}$ and $V_{0,3}$ pose no problem. Looking at the branching over \mathfrak{b}_3 we see that $2V_{1,0} \subseteq 2V_{1,1,0}$ and $2V_{1,1} \subseteq V_{1,1,0}$. Using Corollary 3.2 we get any intertwiner of $2V_{1,0}$ and therefore also all intertwiners in $End(2V_{1,0}) \oplus End(2V_{1,1})$.

Next consider the summand $End(3V_{0,2})$. We have

$$3V_{0,2} \subseteq 2V_{1,1,0} \oplus ((3V_{0,2}) \cap \Lambda^3 V).$$

So we can get any intertwiner in $(End(2V_{0,2}) \oplus (0)) \subseteq End(3V_{0,2})$. Now we show that the 3rd copy $V_{0,2} = (3V_{0,2}) \cap \Lambda^3 V$ is not invariant under the c_{ij} : This holds since the highest weight vector is $v_1 \wedge v_2 \wedge v_3$. If $c_{ij}(v_1 \wedge v_2 \wedge v_3) \in \Lambda^3 V$ this would imply $c_{12}(v_1 \wedge v_2 \wedge v_3) = c_{23}(v_1 \wedge v_2 \wedge v_3)$, but c acts differently on $v_1 \wedge v_2$ than on $v_2 \wedge v_3$. Simple linear algebra shows now that $End(3V_{0,2}) \subseteq \mathbb{C}(c_{12}, c_{13}, c_{23})$.

$End(4V_{0,1})$ can be dealt with similarly. □

Theorem 3.4. $End^{\mathfrak{g}_2}(V^{\otimes n}) = \mathbb{C}\{c_{ij}\}_{i < j}^n$.

Proof. It follows from [AMR], Theorem 10, and [Kup94], Theorem 3.2, that it is sufficient to show that we can obtain any \mathfrak{g}_2 -intertwiner for $V^{\otimes 3} = V_{0,1}^{\otimes 3}$. □

4. SOME EXPLICIT CALCULATIONS

A few hours of computer time give the following explicit forms for the intertwiners c_{12}, c_{13}, c_{23} in a basis of eigenvectors for c_{12} , acting on $4V_{0,1}$:

$$\begin{aligned} c_{12} &= \begin{pmatrix} -12 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \\ c_{13} &= \begin{pmatrix} 0 & 0 & \frac{36}{7} & 0 \\ 0 & -3 & 0 & 21 \\ 2 & 0 & -6 & 24 \\ 0 & 1 & \frac{7}{2} & -7 \end{pmatrix}, \\ c_{23} &= \begin{pmatrix} 0 & 0 & -\frac{36}{7} & 0 \\ 0 & -3 & 0 & -21 \\ -2 & 0 & -6 & -24 \\ 0 & -1 & -\frac{7}{2} & -7 \end{pmatrix}. \end{aligned}$$

Again we see that they algebraically generate all of $End(4V)$: Since all eigenvalues of c_{12} are different there are polynomials p_i such that $p_i(c_{12})(e_j) = \delta_{i,j}e_i$. Therefore for any nonzero entry $a_{i,j}$ of a matrix A that we can construct from we can construct $E_{i,j} = \frac{1}{a_{i,j}}p_j(c_{12}) \circ A \circ p_i(c_{12})$. So all we have to show is that for every i, j there is an A with $(A)_{i,j} \neq 0$. We can take for example $c_{13}, (c_{13})^2, (c_{13})(c_{23})^2$.

5. SKETCH OF AN ALTERNATIVE PROOF

Define the following ordering on the weight lattice: $(p, q) \geq (m, n)$ iff $p+q > m+n$ or $p+q = m+n$ and $p \geq m$.

Computations suggest the following:

$$V_{0,p} \otimes V_{n,m} = V_{n,m+p} \oplus \sum_{(k,l) < (n,m+p)} V_{l,k}$$

(this might be known, but I could not find a reference).

Assuming this and the following assertion about symmetric powers,

$$(5.1) \quad S^p V_{0,1} = \sum_k V_{0,p-2k},$$

$$(5.2) \quad S^p V_{1,0} = V_{p,0} \oplus \sum_{(k,l) < (p,0)} V_{k,l},$$

we can give an alternative proof of Theorem 3.4, avoiding the use of [Kup94], Theorem 3.2. We will do this by induction using the above ordering on weights and employing Corollary 3.2.

The induction will start as follows: Every \mathfrak{g}_2 intertwiner of $nV = \oplus_n V$ is determined by its action on the highest weight vectors, thus is in fact a \mathfrak{b}_3 -intertwiner. Consider now $nV_{1,0} : n\Lambda^2 V_{0,1} = nV_{1,0} \oplus nV_{0,1}$. So every \mathfrak{g}_2 intertwiner of $nV_{1,0}$ is a \mathfrak{b}_3 intertwiner of $n\Lambda^2 V_{0,1}$ minus an intertwiner of $nV_{0,1}$.

Note that $V_{n,m} \subseteq S^n(V_{1,0} + V_{0,1}) \otimes S^m(V_{0,1})$. According to the above decompositions the right-hand side is of the form

$$\begin{aligned} S^n(V_{1,0} + V_{0,1}) \otimes S^m(V_{0,1}) &= \left(V_{n,0} \oplus \sum_{(k,l) < (n,0)} V_{k,l} \right) \otimes \left(\sum_k V_{0,m-2k} \right) \\ &= V_{n,m} \oplus \sum_{(k,l) < (n,m)} V_{l,k}. \end{aligned}$$

Therefore any \mathfrak{g}_2 -intertwiner of $\oplus_k V_{n,m}$ is the difference of a \mathfrak{b}_3 -intertwiner of $kS^n(\Lambda^2 V_{0,1}) \otimes S^m(V_{0,1})$ and intertwiners of $kV_{\alpha,\beta}$ that we already constructed by induction. This finishes the sketch of the alternative proof.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT DAVIS, DAVIS, CALIFORNIA 95616

E-mail address: mattes@math.math.ucdavis.edu