ON THE GANEA CONJECTURE FOR MANIFOLDS

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Abstract. Using a result of Singhof, we prove that $\text{cat}(M \times S^m) = \text{cat } M + 1$ provided $M$ is a connected closed PL manifold with $\dim M \leq 2\text{cat } M - 3$ and $S^m$ is the $m$-sphere, $m > 0$.

Let $\text{cat } X$ denote the Lusternik–Schnirelmann category of $X$ (normalized, i.e., $\text{cat } S^m = 1$). There is a long standing Ganea conjecture that $\text{cat}(X \times S^m) = \text{cat } X + 1$ for every connected finite CW-complex $X$ and every $m > 0$, see [1, Problem 2]. In [2, Corollary 6.7] Singhof proved the following

Singhof’s Theorem. (i) Let $M$ be a connected closed PL manifold such that

$$\text{cat } M \geq \frac{\dim M + m + 2}{2}.$$ Then $\text{cat}(M \times S^m) = \text{cat } M + 1$ provided $m > 0$.

(ii) In particular, if

$$\text{cat } M \geq \frac{\dim M + 3}{2}$$

then $\text{cat}(M \times S^1) = \text{cat } M + 1$.

However, it seems that nobody noted that Singhof’s Theorem implies a stronger result. Namely, the following theorem is valid:

Theorem. Let $M$ be a connected closed PL manifold such that

$$\text{cat } M \geq \frac{\dim M + 3}{2}.$$ Then $\text{cat}(M \times S^m) = \text{cat } M + 1$ for every $m > 0$.

Proof. First, we prove that $\text{cat}(M \times T^r) = \text{cat } M + r$ where $T^r$ is the $r$-torus. We prove this by induction. For $r = 1$ this follows from (ii). Now, suppose that $\text{cat}(M \times T^r) = \text{cat } M + r$. Then

$$\text{cat}(M \times T^r) = \text{cat } M + r \geq \frac{\dim M + 3}{2} + r \geq \frac{\dim M + r + 3}{2} = \frac{\dim(M \times T^r) + 3}{2}.$$
So, by (ii), $\text{cat}(M \times T^{r+1}) = \text{cat} M + r + 1$.

Now, given $m > 0$, we prove that $\text{cat}(M \times S^m) = \text{cat} M + 1$. Choose $r$ such that

$$\text{cat}(M \times T^r) = \text{cat} M + r \geq \frac{\dim(M \times T^r) + m + 2}{2}$$

(for example, $r \gg m$). Then, by (i),

$$\text{cat}(M \times T^r \times S^m) = \text{cat}(M \times T^r) + 1 = \text{cat} M + r + 1.$$ Now, if $\text{cat}(M \times S^m) \neq \text{cat} M + 1$ then $\text{cat}(M \times S^m) \leq \text{cat} M$. But then

$$\text{cat}(M \times S^m \times T^r) \leq \text{cat}(M \times S^m) + \text{cat} T^r \leq \text{cat} M + r.$$ This is a contradiction. \hfill \Box

**Corollary.** For every connected closed PL manifold $M$ there exists a natural number $k$ such that $\text{cat}(M \times T^k \times S^m) = \text{cat}(M \times T^k) + 1$ for every $m > 0$.

**Proof.** Indeed, you can find $k$ such that

$$\text{cat}(M \times T^k) \geq k \geq \frac{\dim(M \times T^k) + 3}{2}.$$

\hfill \Box

**References**
