ON REDUCIBILITY OF SEMIGROUPS OF COMPACT QUASINILPOTENT OPERATORS

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Abstract. The following generalization of Lomonosov’s invariant subspace theorem is proved. Let \( S \) be a multiplicative semigroup of compact operators on a Banach space such that \( \hat{r}(S_1, \ldots, S_n) = 0 \) for every finite subset \( \{S_1, \ldots, S_n\} \) of \( S \), where \( \hat{r} \) denotes the Rota-Strang spectral radius. Then \( S \) is reducible.

This result implies that the following assertions are equivalent:

(A) For each infinite-dimensional complex Hilbert space \( H \), every semigroup of compact quasinilpotent operators on \( H \) is reducible.

(B) For every complex Hilbert space \( H \), for every semigroup of compact quasinilpotent operators on \( H \), and for every finite subset \( \{S_1, \ldots, S_n\} \) of \( S \) it holds that \( \hat{r}(S_1, \ldots, S_n) = 0 \).

The question whether the assertion (A) is true was considered by Nordgren, Radjavi and Rosenthal in 1984, and it seems to be still open.

Let \( X \) be a (real or complex) Banach space and \( B(X) \) the Banach algebra of all bounded linear operators on \( X \). Let \( (S_1, \ldots, S_n) \) be an \( n \)-tuple of operators of \( B(X) \). The Rota-Strang spectral radius \( \hat{r}(S_1, \ldots, S_n) \) is defined by

\[
\hat{r}(S_1, \ldots, S_n) = \lim_{m \to \infty} \max \left\{ \|S_{k_1}S_{k_2}\cdots S_{k_m}\|^{1/m} : 1 \leq k_i \leq n, 1 \leq i \leq m \right\}.
\]

In [7] it is shown that this limit always exists. Similarly, we introduce the local spectral radius of \( n \)-tuple \( (S_1, \ldots, S_n) \) at a vector \( x \in X \) by the formula

\[
\hat{r}(S_1, \ldots, S_n; x) = \limsup_{m \to \infty} \max \left\{ \|S_{k_1}S_{k_2}\cdots S_{k_m}x\|^{1/m} : 1 \leq k_i \leq n, 1 \leq i \leq m \right\}.
\]

Let \( C \) be a collection of operators of \( B(X) \). We say that \( C \) is reducible if there exists a closed subspace of \( X \), other than \( \{0\} \) and \( X \), which is invariant under every member of \( C \). The collection \( C \) is irreducible if it is not reducible. If there exists even a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of \( C \), then \( C \) is said to be simultaneously triangularizable.

Let us define the number \( \hat{r}(C) \in [0, \infty] \) by

\[
\hat{r}(C) = \sup \{\hat{r}(S_1, \ldots, S_n) : n \in \mathbb{N}, S_1, \ldots, S_n \in C\}.
\]

Similarly, we set, for any \( x \in X \),

\[
\hat{r}(C; x) = \sup \{\hat{r}(S_1, \ldots, S_n; x) : n \in \mathbb{N}, S_1, \ldots, S_n \in C\}.
\]
The proof of our first result is based on the famous Lomonosov-Hilden technique; see e.g. [4].

**Theorem 1.** Let $X$ be a Banach space of dimension greater than one, and let $S$ be a multiplicative semigroup in $\mathcal{B}(X)$ containing a compact operator $K$. If $\hat{r}(SK;x_0) = 0$ and $Kx_0 \neq 0$ for some vector $x_0 \in X$, then $S$ is reducible.

**Proof.** There is no loss of generality in assuming that $\|K\| = 1$, since we can assume that $S = \mathbb{R}^+ S$. Replacing $x_0$ by $\lambda x_0$ for an appropriate scalar $\lambda > 0$, we can also assume that $\|x_0\| > 1$ and $\|Kx_0\| > 1$. Let $U = \{x \in X : \|x - x_0\| \leq 1\}$ be the closed unit ball centered at $x_0$. Obviously, 0 is not in $U$ nor in the closure $\overline{K(U)}$ of $K(U)$.

Let $A$ be the linear manifold in $\mathcal{B}(X)$ generated by $S$. Since $S$ is a semigroup, $A$ is equal to the subalgebra of $\mathcal{B}(X)$ generated by $S$. For each fixed $y \in X$ the set

$$
L_y = \{Ay : A \in A\}
$$

is a linear manifold in $X$ which is invariant under every member of $A$. If $L_y = \{0\}$ for some $y \neq 0$, then the one-dimensional subspace generated by $y$ is invariant under every member of $S$. So, we may assume that $L_y \neq \{0\}$ for all $y \neq 0$. We shall show that there exists $y \neq 0$ such that $L_y$ is not dense in $X$. This will prove the theorem. Assume on the contrary that $L_y$ is dense in $X$ for all $y \neq 0$. Then for each $y \neq 0$ there is an $A \in A$ such that $Ay \in \text{int} U$, where $\text{int} U$ denotes the interior of $U$. We therefore have

$$
\overline{K(U)} \subseteq X \setminus \{0\} \subseteq \bigcup_{A \in A} A^{-1}(\text{int} U).
$$

Since $\overline{K(U)}$ is a compact set in $X$, there exists a finite set $\{A_1, \ldots, A_n\} \subseteq A$ such that

$$
\overline{K(U)} \subseteq \bigcup_{i=1}^{n} A_i^{-1}(\text{int} U).
$$

Therefore, it follows from $Kx_0 \in K(U)$ that $Kx_0 \in A_i^{-1}(U)$ for some $i \in \{1, \ldots, n\}$, and hence $A_i, Kx_0 \in U$. Then $KA_i Kx_0 \in K(U)$ implies that $KA_i Kx_0 \in A_i^{-1}(U)$ for some $i_2$, and so $A_i K A_i Kx_0 \in U$. Proceeding with this "ping-pong" of Hilden’s, after $m$ steps we obtain an integer $i_m \in \{1, 2, \ldots, n\}$ such that

$$
A_{i_m} K A_{i_{m-1}} \cdots A_i K x_0 \in U.
$$

Clearly there exist $S_1, S_2, \ldots, S_N \in S$ and scalars $c_j^{(i)}$, $i = 1, \ldots, n$, $j = 1, \ldots, N$, such that

$$
A_i = \sum_{j=1}^{N} c_j^{(i)} S_j
$$

for all $i = 1, \ldots, n$. Set $c = \max\{\|c_j^{(i)}\| : i = 1, \ldots, n, j = 1, \ldots, N\}$. Putting $B_i = A_i K$ for $i = 1, \ldots, n$, we then have

$$
\|B_{i_m} B_{i_{m-1}} \cdots B_{i_1} x_0\| \leq c^m N^m \max\{\|(S_{k_m} K) \cdots (S_{k_1} K)x_0\| : 1 \leq k_i \leq N\}.
$$

Since $\hat{r}(S_1 K, \ldots, S_N K; x_0) = 0$, we conclude that

$$
\lim_{m \to \infty} \|B_{i_m} B_{i_{m-1}} \cdots B_{i_1} x_0\|^{1/m} = 0,
$$

as desired. 

\[\square\]
which implies that

$$\lim_{m \to \infty} \|B_{m}B_{m-1}\cdots B_{1}x_{0}\| = 0 ,$$

and so $0 \in U$. This contradiction completes the proof of the theorem. \,

In [2] an irreducible semigroup of nilpotent operators of index two is constructed. Let us use that semigroup to show that the compactness hypothesis of Theorem 1 cannot be omitted.

**Example 2.** If $A$ is a $k \times k$ matrix, then let $N_{A}$ be the $2k \times 2k$ matrix

$$\begin{bmatrix} A & -A \\ A & -A \end{bmatrix},$$

and let $T_{A}$ be the direct sum of denumerably many copies of $N_{A}$. We regard $T_{A}$ as an operator on a separable Hilbert space $l^{2}$.

For each $i \in \mathbb{N}$ let $S_{i}$ denote the linear space of all operators $T_{A}$ as $A$ ranges over all $2^{i-1} \times 2^{i-1}$ matrices, and let $S$ be the union of all $S_{i}$. It is shown in [2] that $S$ is an irreducible semigroup of nilpotent operators of index two. In order to show that $\hat{\tau}(S) = 0$, let $(S_{1}, \ldots, S_{n})$ be an $n$-tuple of non-zero operators of $S$. Let $i(S_{p})$ $(1 \leq p \leq n)$ denote the uniquely determined number $i$ such that $S_{p}$ belongs to $S_{i}$. We claim that

$$S_{k_{1}}S_{k_{2}}\cdots S_{k_{m}} = 0$$

for all $m \geq 2^{n}$ and for all $1 \leq k_{i} \leq n$. Indeed, there exist $1 \leq p < q \leq m$ such that $k_{p} = k_{q}$ and $i(S_{k_{p}}) < i(S_{k_{q}})$ for each $p < j < q$. Using the facts that $S_{i}S_{j} \subseteq S_{i}$ and $S_{j}S_{i} \subseteq S_{i}$ for all $i > j$ and that $S_{i}S_{i} = \{0\}$ for all $i$, we now conclude that $S_{k_{1}}\cdots S_{k_{q}} = 0$, which proves the claim. It follows that $\hat{\tau}(S_{1}, \ldots, S_{n}) = 0$, so that $\hat{\tau}(S) = 0$.

A recent extension of Rota’s theorem [3, Corollary 1] gives the following result that will be needed in the sequel.

**Proposition 3.** Let $H$ be a complex Hilbert space, and let $C$ be a collection of compact quasinilpotent operators on $H$ which is simultaneously triangularizable. Then $\hat{\tau}(C) = 0$.

**Proof.** Let $(S_{1}, \ldots, S_{n})$ be an $n$-tuple of operators of $C$, and let $\epsilon > 0$. By [3, Corollary 1] there exists a positive invertible operator $T$ such that $\|T^{-1}S_{k}T\| < \epsilon$ for all $k = 1, 2, \ldots, n$. We then have

$$\|S_{k_{1}}\cdots S_{k_{m}}\| \leq \|T\|\|T^{-1}S_{k_{1}}T\|\cdots\|T^{-1}S_{k_{m}}T\|\|T^{-1}\| \leq \|T\|\|T^{-1}\|\epsilon^{m}$$

for every $m \in \mathbb{N}$ and every $k_{1}, \ldots, k_{m} \in \{1, \ldots, n\}$. It follows that $\hat{\tau}(S_{1}, \ldots, S_{n}) \leq \epsilon$. Therefore, $\hat{\tau}(S_{1}, \ldots, S_{n}) = 0$, and so $\hat{\tau}(C) = 0$. \,

Let $H$ denote an arbitrary complex Hilbert space, and let $S(H)$ be the collection of all multiplicative semigroups of compact quasinilpotent operators on $H$.

**Theorem 4.** The following assertions are equivalent:

(a) For each $H$ of dimension greater than one, every semigroup $S \in S(H)$ is reducible.

(b) For each $H$, every semigroup $S \in S(H)$ is simultaneously triangularizable.

(c) $\hat{\tau}(S) = 0$ for every $H$ and for every $S \in S(H)$.

(d) $\hat{\tau}(S;x_{0}) = 0$ for every $H$, for every $S \in S(H)$, and for every $x_{0} \in H$. 

(e) For each $\mathcal{H}$ and for each $S \in \mathcal{S}(\mathcal{H})$ there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $\hat{r}(S; x_0) = 0$.

Proof. The idea of the proof that (a) implies (b) is well-known. Since the proof is (for example) a slight modification of the proof of Corollary 1 in [5], we will omit it. By Proposition 3, (b) implies (c). It is easy to show that (c) implies (d) and that (d) implies (e).

Assume that (e) holds. Given $\mathcal{H}$ of dimension greater than one and $S \in \mathcal{S}(\mathcal{H})$, there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $\hat{r}(S; x_0) = 0$. If $Sx_0 = 0$ for all $S \in \mathcal{S}$, then the one-dimensional subspace generated by $x_0$ is invariant under every member of $\mathcal{S}$, so that $\mathcal{S}$ is reducible. Otherwise, there is a $K \in \mathcal{S}$ such that $Kx_0 \neq 0$. Since $\hat{r}(SK; x_0) = 0$, $\mathcal{S}$ is reducible by Theorem 1, and so (a) holds.

Note that the problem (a) of the last result was studied in [5], where it is shown that every semigroup of trace class quasinilpotent operators is reducible and consequently simultaneously triangularizable.

We conclude by mentioning that the assertion (c) of Theorem 4 is not true if the compactness hypothesis is dropped. Namely, Guinand [1] constructed two weighted shifts $T_1$ and $T_2$ on $l^2$ of norm 1 such that every word in $\{T_1, T_2\}$ is nilpotent of index three. In other words, the semigroup $\mathcal{S}$ generated by $\{T_1, T_2\}$ consists of nilpotent operators on $l^2$. It was shown in [6, example] that $\hat{r}(T_1, T_2) = 1$, which implies that $\hat{r}(\mathcal{S}) = 1$.

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References


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