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EVERY LOCAL RING IS DOMINATED BY A ONE-DIMENSIONAL LOCAL RING

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ABSTRACT. Let $(R, \mathbf{m})$ be a local (Noetherian) ring. The main result of this paper asserts the existence of a local extension ring S of R such that (i) S dominates R, (ii) the residue field of S is a finite purely transcendental extension of $R/\mathbf{m}$, (iii) every associated prime of (0) in S contracts in R to an associated prime of (0), and (iv) $\dim(S) \leq 1$. In addition, it is shown that S can be obtained so that either $\mathbf{m}S$ is the maximal ideal of S or S is a localization of a finitely generated R-algebra.

1. INTRODUCTION

Let R be a commutative ring with identity having a unique maximal ideal $\mathbf{m}$. We write in this situation that $(R, \mathbf{m})$ is a quasilocal ring. A quasilocal extension ring $(S, \mathbf{n})$ of R is said to *dominate* R if $\mathbf{m} \subseteq \mathbf{n}$ or, equivalently, if $\mathbf{n} \cap R = \mathbf{m}$.

There are several well-known results concerning domination in the study of local rings and quasilocal rings. For example:

(1) Every quasilocal domain is dominated by a valuation domain [N, (11.9)].
(2) If $(R, \mathbf{m})$ is a quasilocal domain and F is a subfield of the field of fractions of R, then $R \cap F$ is a quasilocal domain and R dominates $R \cap F$.
(3) Every local ring is dominated by a complete local ring [N, (17.6)].
(4) (Chevalley) Every local domain is dominated by a rank-one discrete valuation domain (DVR) [Ch, page 26]. (For a generalization of (4), see [CHL].)

Concerning Result (4), the proof Chevalley gives for this also shows that a dominating DVR V of the local domain $(R, \mathbf{m})$ can be taken inside the field of fractions of R, and, in view of the fact that residue extensions are finite algebraic when passing to the integral closure of a Noetherian domain [N, (33.10)], V can be chosen so that the residue field of V as an extension field of $R/\mathbf{m}$ is a finitely generated field extension. It is often the case that the residue field of V is necessarily transcendental as an extension of $R/\mathbf{m}$, for if R is complete and the residue field of V is finite algebraic over $R/\mathbf{m}$, then by a result of Cohen [N, (30.6)], V is a finitely generated R-module and therefore $\dim(R) = 1$.

Since a DVR is a one-dimensional normal local domain (and conversely), a consequence of Result (4) is that a local domain is dominated by a one-dimensional local domain.

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The main result of this paper is a ring-theoretic version of this last fact. If $(R, \mathbf{m})$ is a local (Noetherian) ring, we establish in Theorems 3.1 and 3.7 the existence of a local extension ring S of R such that:

(1) S dominates R,
(2) the residue field of S is a finite purely transcendental extension field of $R/\mathbf{m}$,
(3) every associated prime of (0) in S contracts in R to an associated prime of (0), so every regular element of R remains regular in S, and
(4) $\dim(S) \leq 1$, and $\dim(S) = 1$ unless $\dim(R) = 0$.

In Theorem 3.1 we obtain an S satisfying the conditions above that also has the property that $\mathbf{m}S$ is the maximal ideal of S, while in Theorem 3.7 we obtain an S satisfying (1)–(4) that is a localization of a finitely generated R-algebra.[1]

We were originally motivated to consider the domination question of this paper by our work on the embedding of commutative rings in finite-dimensional rings [GH5]. For a local or quasilocal ring, a condition weaker than domination is that of having an extension ring of a certain form. It is well known that every Noetherian ring is a subring of an Artinian ring (see, for example, [GH2, (2.6)]). However, a local (Noetherian) ring having more than one associated prime is not a subring of a local Artinian ring. Our main result implies that every local ring is a subring of a one-dimensional local ring. There is no analogue to this result for arbitrary commutative rings in that [GH5, Example 1.6] establishes the existence, for each positive integer n, of a quasilocal ring R_n of dimension n that is not a subring of a ring of dimension less than n.

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity of S is contained in R, and hence is the unity of R. A nonzero element $r \in R$ is a *regular element of* R if r is not a zero divisor in R. All allusions to the dimension, $\dim(R)$, of a ring R refer to its Krull dimension. Thus $\dim(R) = n$ if there is a chain $P_0 < P_1 < \cdots < P_n$ of prime ideals of $\operatorname{Spec}(R)$ and no chain of longer length.

2. Ideals of R contracted from the extension ring $R(t)$

(2.1). Let $(R, \mathbf{m})$ be a local ring, let t be an indeterminate over R, and let $R(t) = R[t]_{\mathbf{m}[t]}$, the localization of the polynomial ring $R[t]$ at the multiplicative system of polynomials in $R[t]$ having a unit coefficient. It is clear that $R(t)$ is a local ring dominating R with maximal ideal $\mathbf{m}R(t)$ and residue field isomorphic to $(R/\mathbf{m})(t)$, a simple transcendental extension of the residue field $R/\mathbf{m}$ of R. It is known [N, pages 17-18], [G, (33.1)], [GH1] that each ideal of R is the contraction of its extension to $R(t)$. Since R is Noetherian, it follows from the altitude theorem of Krull [N, (9.3)] that $\dim(R) = \dim(R(t))$. More generally, if $n > 1$ is a positive integer and $t_1, \ldots, t_n$ are indeterminates over R, the ring $R(t_1, \ldots, t_n) = R[t_1, \ldots, t_n]_{\mathbf{m}[t_1,\ldots,t_n]}$ is a local ring isomorphic to $R(t_1)(t_2, \ldots, t_n)$ with residue field $R(t_1, \ldots, t_n)/\mathbf{m}R(t_1, \ldots, t_n)$, a pure transcendental extension of $R/\mathbf{m}$ in n indeterminates; each ideal of R is the contraction of its extension to $R(t_1, \ldots, t_n)$, and $\dim(R) = \dim(R(t_1, \ldots, t_n))$.

[1]As suggested by the referee, it would be interesting to know more about the structure of one-dimensional local rings S dominating a given R. One restriction on S is that it must have at least as many minimal prime ideals as does R (cf. [GH4, (2.1)]).

(2.2). With $(R, \mathbf{m})$ and t as in (2.1), suppose $a, b \in \mathbf{m}$, with b a regular element of R. Let $\phi : R[t] \to R[a/b]$ denote the R-algebra homomorphism into the total quotient ring of R such that $\phi(t) = a/b$. Assume that $\ker \phi \subseteq \mathbf{m}[t]$. Then $\mathbf{m}R[a/b]$ is a nonmaximal prime ideal of $R[a/b]$, and ϕ extends to a homomorphism of $R(t)$ onto $R[a/b]_{\mathbf{m}R[a/b]}$ such that $R(t)/(\ker \phi)R(t) \cong R[a/b]_{\mathbf{m}R[a/b]}$.

Lemma 2.3. *Suppose D is a Noetherian domain, $a, b \in D$ are such that the ideal $(a, b)D$ has height two, and P is a prime ideal of D with $(a, b)D \subseteq P$. Let t be an indeterminate over D and let $\phi : D[t] \to D[a/b]$ be the surjective D-algebra homomorphism such that $\phi(t) = a/b$. Then* $\ker \phi \subseteq P[t]$.

Proof. The result follows for every prime ideal containing $(a, b)D$ provided it is true for each minimal prime of the ideal, so we may assume that P is a minimal prime of $(a, b)D$, and is therefore of height two. Let $R = D_P$ and let $\mathbf{m} = PD_P$. Then R is a 2-dimensional local domain and $(a, b)R$ is $\mathbf{m}$-primary. Let R' denote the integral closure of R, let P' be a maximal ideal of R' of height two, and let $R^* = R'_{P'}$. Then R^* is a 2-dimensional normal local domain [N, (33.10) and (33.12)], and a, b form a regular sequence in R^*. Let $\phi^* : R^*[t] \to R^*[a/b]$ denote the R^*-algebra homomorphism such that $\phi^*(t) = a/b$. Then $\ker \phi^* = (bt - a)R^*[t] \subseteq \mathbf{m}R^*[t]$. Since ϕ^* restricts to ϕ on $D[t]$ and since $\mathbf{m}R^*[t] \cap D[t] = P[t]$, it follows that $\ker \phi \subseteq P[t]$. $\square$

Theorem 2.4. *Let $(R, \mathbf{m})$ be a local domain and let P be a prime ideal of R such that $\dim(R/P) = d \geq 2$. There exists a prime ideal Q of $R(t)$ such that*

1. $\dim(R(t)/Q) < d$,
2. $Q \cap R = P$, *and*
3. *each P-primary ideal of R is the contraction of a Q-primary ideal of $R(t)$.*

Proof. Since $\dim(R/P) \geq 2$, we can choose $a \in \mathbf{m} - P$, and $b \in \mathbf{m}$ so that b is not in any minimal prime of $(P, a)R$ nor in any minimal prime of aR. It follows that $\mathrm{ht}((a, b)R) = 2$. Let $R' = R/P$ and let a', b' denote the images of a, b in R'. Our choice of a and b implies that $\mathrm{ht}((a', b')R') = 2$. Let t be an indeterminate and consider the R-algebra homomorphism $\phi : R[t] \to R[a/b]$ such that $\phi(t) = a/b$, and the R'-algebra homomorphism $\phi' : R'[t] \to R'[a'/b']$ such that $\phi'(t) = a'/b'$. Since $b \notin P$, $R[a/b]$ is a subring of R_P and $R_P = R[a/b]_{\mathbf{q}}$, where $\mathbf{q} = PR_P \cap R[a/b]$. By Lemma (2.3) and Remark (2.2), ϕ extends to a surjective R-algebra homomorphism $\psi : R(t) \to R[a/b]_{\mathbf{m}R[a/b]}$. We claim that

$$\mathbf{q} = PR_P \cap R[a/b] \subseteq \mathbf{m}R[a/b]. \tag{2.5}$$

Assuming (2.5), we have $R[a/b]_{\mathbf{m}R[a/b]} \subseteq R[a/b]_{\mathbf{q}} = R_P$. Let

$$Q = \psi^{-1}(\mathbf{q}R[a/b]_{\mathbf{m}R[a/b]}).$$

Since $\psi(R(t)) \subseteq R_P$ and every P-primary ideal of R is the contraction of a PR_P-primary ideal of R_P, it follows that every P-primary ideal of R is the contraction of a Q-primary ideal of $R(t)$. Since $PR(t) < Q$ and $R(t)/PR(t) \cong (R/P)(t)$ is of dimension d, $\dim(R(t)/Q) < d$.

It remains to prove (2.5). Let $g : R_P \to R_P/PR_P$ denote the canonical surjection. Consider the following commutative diagram, where f is the restriction of g, the map of $R[t]$ onto $R'[t]$ is obtained by reducing coefficients modulo P, and α and

β are inclusion maps.

$$\begin{tikzcd} R[t] \arrow[r, "\phi"] \arrow[d] & R[a/b] \arrow[r, "\alpha"] \arrow[d, "f"'] & R_P \arrow[d, "g"'] \\ R'[t] \arrow[r, "\phi'"] & R'[a'/b'] \arrow[r, "\beta"] & R_P/PR_P \end{tikzcd} \tag{2.6}$$

Let $\mathbf{m}' = \mathbf{m}/P$ denote the maximal ideal of R'. By Lemma (2.3), $\ker \phi' \subseteq \mathbf{m}'[t]$. By commutativity of (2.6), $\ker f = \mathbf{q} \subseteq \mathbf{m}R[a/b]$. This establishes (2.5) and completes the proof of Theorem 2.4. $\square$

Corollary 2.7. *Let $(R, \mathbf{m})$ be a local domain and let I be an ideal of R such that $\dim(R/I) = d \geq 2$. There exists an ideal J of $R(t)$ such that $J \cap R = I$, every associated prime of J in $R(t)$ contracts in R to an associated prime of I, and $\dim(R(t)/J) < d$.*

Proof. Let $I = \bigcap_{i=1}^n \mathbf{w}_i$ be a minimal primary decomposition (cf. [AM, page 52]) of I in R, where $\mathbf{w}_i$ is P_i-primary for $1 \leq i \leq n$. For each i such that $\dim(R/P_i) = d$, Theorem 2.4 guarantees existence of a prime ideal Q_i of $R(t)$ such that $\dim(R(t)/Q_i) < d$ and such that there exists a Q_i-primary ideal $\mathbf{q}_i$ of $R(t)$ such that $\mathbf{q}_i \cap R = \mathbf{w}_i$. On the other hand, if $\dim(R/P_i) < d$, let $\mathbf{q}_i = \mathbf{w}_i R(t)$. In this latter case we have $R(t)/\mathbf{q}_i \cong (R/\mathbf{w}_i)(t)$, so $\dim(R(t)/\mathbf{q}_i) < d$. Let $J = \bigcap_{i=1}^n \mathbf{q}_i$. Then $\dim(R(t)/J) < d$ and $J \cap R = \bigcap_{i=1}^n (\mathbf{q}_i \cap R) = \bigcap_{i=1}^n \mathbf{w}_i = I$, so every associated prime of J in $R(t)$ contracts in R to an associated prime of I. $\square$

Theorem 2.8. *Let $(R, \mathbf{m})$ be a local domain and let I be an ideal of R such that $\dim(R/I) = d \geq 1$. Let $t_1, \ldots, t_{d-1}$ be indeterminates over R and let $S = R(t_1, \ldots, t_{d-1})$.*[2] *There exists an ideal J of S such that $J \cap R = I$, every associated prime of J in S contracts in R to an associated prime of I, and $\dim(S/J) = 1$.*

Proof. We proceed by induction on $d = \dim(R/I)$. If $d = 1$, the assertion is clear. Assume $\dim(R/I) = d \geq 2$ and that the assertion holds for all local domains T and ideals L of T such that $1 \leq \dim(T/L) < d$. By (2.7), there exists an ideal J of $R(t_1)$ such that $J \cap R = I$, every associated prime of J in $R(t_1)$ contracts in R to an associated prime of I, and $\dim(R(t_1)/J) < d$. Since a zero-dimensional local ring has nilpotent maximal ideal and therefore cannot dominate a local ring of positive dimension, we have $\dim(R(t_1)/J) \geq 1$. Applying the induction hypothesis to the ideal J of the ring $R(t_1)$ now yields the result. $\square$

We summarize our results of this section in the following corollary.

Corollary 2.9. *Let $(R, \mathbf{m})$ be a local domain and let I be an ideal of R such that $\dim(R/I) = d \geq 1$. Let $t_1, \ldots, t_{d-1}$ be indeterminates over R and let $S = R(t_1, \ldots, t_{d-1})$.*[3] *Then S is a local extension domain of R such that:*

1. *$\mathbf{m}S$ is the maximal ideal of S,*
2. *$S/\mathbf{m}S$ is a purely transcendental extension field of $R/\mathbf{m}$ of transcendence degree $d-1$, and*
3. *there exists an ideal J of S such that $J \cap R = I$, every associated prime of J in S contracts in R to an associated prime of I, and $\dim(S/J) = 1$.*

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[2] If $d = 1$, we take $S = R$.

[3] Again, if $d = 1$, we take $S = R$.

Proof. The first two assertions about S are noted in (2.1), and the third condition is established in (2.8). $\square$

3. Domination by a one-dimensional local ring

As a first application of Theorem 2.8, we have:

Theorem 3.1. *Let $(R, \mathbf{m})$ be a local ring with $\dim(R) = d \geq 1$. There exists a local extension ring S of R such that:*

1. *$\mathbf{m}S$ is the maximal ideal of S,*
2. *$S/\mathbf{m}S$ is a pure transcendental extension field of $R/\mathbf{m}$ in $d-1$ indeterminates,*
3. *every associated prime of (0) in S contracts in R to an associated prime of (0) in R, so regular elements of R remain regular in S, and*
4. *$\dim(S) = 1$.*

Proof. Let $\widehat{R}$ denote the completion of R with respect to the $\mathbf{m}$-adic topology. Then $\widehat{R}$ is a local extension ring of R with maximal ideal $\mathbf{m}\widehat{R}$, every associated prime of (0) in $\widehat{R}$ contracts in R to an associated prime of (0) in R, (cf. [N, (18.11)]), and $R/\mathbf{m} = \widehat{R}/\mathbf{m}\widehat{R}$ with respect to the canonical inclusion of $R/\mathbf{m}$ into $\widehat{R}/\mathbf{m}\widehat{R}$. A structure theorem of Cohen [C, Theorem 12],[N, (31.1)] implies the existence of a complete local domain $(T, \mathbf{n})$ having an ideal I such that $T/I \cong \widehat{R}$. By Corollary 2.9, there exists a local extension domain $W = T(t_1, \dots, t_{d-1})$ of T such that: (i) $\mathbf{n}W$ is the maximal ideal of W, (ii) $W/\mathbf{n}W$ is a pure transcendental extension of $T/\mathbf{n}$ in $d-1$ indeterminates, (iii) there exists an ideal J of W such that $J \cap T = I$, every associated prime of J in W contracts in T to an associated prime of I, and (iv) $\dim(W/J) = 1$. We have $R \hookrightarrow \widehat{R} \cong T/I \hookrightarrow W/J$. Therefore $S = W/J$ is a local extension of R of the desired form. $\square$

An alternate proof of parts of Theorem 3.1 can be obtained by using the concept of gluing of maximal ideals. This gluing process has the merit of yielding a one-dimensional local extension ring S dominating R which is a localization of a finitely generated R-algebra.

The following lemma extends (2.3) to a ring context.

Lemma 3.2. *Let E be a Noetherian ring such that (0) in E is $\mathbf{p}$-primary. Suppose $a, b \in E$ are such that $\mathrm{ht}((a,b)E) = 2$ and let P be a prime ideal of E such that $(a,b)E \subseteq P$. Since $\mathrm{ht}((a,b)E) = 2$, $b \notin \mathbf{p}$, so $E[a/b] \subseteq E_{\mathbf{p}}$, the total quotient ring of E. Let t be an indeterminate over E and let $\phi : E[t] \to E[a/b]$ be the E-algebra homomorphism such that $\phi(t) = a/b$. Then $\ker \phi \subseteq P[t]$.*

Proof. Since $E[a/b] \subseteq E_{\mathbf{p}}$, the ideal (0) of $E[a/b]$ is primary and $\mathbf{p}E_{\mathbf{p}} \cap E[a/b] = \mathbf{q}$ is the nilradical of $E[a/b]$. Let a', b' denote the images of a, b in $D = E/\mathbf{p}$. Consider the following commutative diagram:

$$\begin{array}{ccccc} E[t] & \xrightarrow{\phi} & E[a/b] & \xrightarrow{\alpha} & E_{\mathbf{p}} \\ \downarrow & & f\downarrow & & g\downarrow \\ D[t] & \xrightarrow{\phi'} & D[a'/b'] & \xrightarrow{\beta} & E_{\mathbf{p}}/\mathbf{p}E_{\mathbf{p}} \end{array} \tag{3.3}$$

where $\phi' : D[t] \to D[a'/b']$ is the D-algebra homomorphism such that $\phi'(t) = a'/b'$, the map $E[t] \to D[t]$ is reduction of coefficients modulo $\mathbf{p}$, $g : E_{\mathbf{p}} \to E_{\mathbf{p}}/\mathbf{p}E_{\mathbf{p}}$ is the canonical surjection, f is the restriction of g, and α and β are inclusion maps. By Lemma 2.3, $\ker\phi' \subseteq (P/\mathbf{p})D[t]$. Hence $\ker\phi \subseteq P[t]$. $\square$

Theorem 3.4. *Let $(R, \mathbf{m})$ be a local ring such that (0) in R is $\mathbf{p}$-primary. Assume that $\dim(R) = d \geq 2$. There exists a local extension ring T of R such that*

(1) *T is a localization of a finitely generated R-algebra,*
(2) *$\mathbf{m}T$ is the maximal ideal of T so that, in particular, T dominates R,*
(3) *T is a subring of the total quotient ring of R, so the ideal (0) of T is primary,*
(4) *$\dim(T) = 1$, and*
(5) *there exists a positive integer $n \leq d-1$ such that the residue field of T is canonically isomorphic to $(R/\mathbf{m})(t_1, \ldots, t_n)$, a pure transcendental extension of $R/\mathbf{m}$ in n indeterminates.*

Proof. Since $d \geq 2$ there exist $a, b \in \mathbf{m}$ such that $\mathrm{ht}((a,b)R) = 2$. It follows that $b \notin \mathbf{p}$ and hence that b is a regular element of R, so $R[a/b]$ is a subring of the total quotient ring $R_{\mathbf{p}}$ of R. Let $\phi : R[t] \to R[a/b] \subseteq R_{\mathbf{p}}$ be the R-algebra homomorphism such that $\phi(t) = a/b$. Lemma 3.2 implies that $\ker\phi \subseteq \mathbf{m}[t]$. By (2.2), ϕ extends to a surjective R-algebra homomorphism of $R(t)$ onto $R[a/b]_{\mathbf{m}R[a/b]}$ and $R(t)/(\ker\phi)R(t) \cong R[a/b]_{\mathbf{m}R[a/b]}$. It is clear that $S = R[a/b]_{\mathbf{m}R[a/b]}$ is a local ring with maximal ideal $\mathbf{m}S$. The ideal (0) of S is primary since the zero ideal of $R_{\mathbf{p}}$, the total quotient ring of S, is primary. Each of the local rings $R(t)$ and S has residue field isomorphic to $(R/\mathbf{m})(t)$. The polynomial $bt - a$ is an element of $(\ker\phi)R(t)$ that is not in $\mathbf{p}R(t)$, the unique minimal prime of $R(t)$. Hence $S \cong R(t)/(\ker\phi)R(t)$ has dimension less than $\dim(R(t)/\mathbf{p}R(t)) = d$. Since $\mathbf{m}$ is not nilpotent, $\mathbf{m}S$ is not nilpotent, so $\dim(S) > 0$. The ring $(S, \mathbf{m}S)$ satisfies the hypothesis of Theorem 3.4, so by repetition of the process above if necessary (that is, if $\dim(S) > 1$), we obtain after at most $\dim(S) - 1$ repetitions a one-dimensional local ring T which satisfies the conditions of Theorem 3.4 $\square$

(3.5). Suppose $(S, \mathbf{n})$ is a one-dimensional local ring in which (0) is primary and $t_1, \ldots, t_k$ are indeterminates over S. Then $S(t_1, \ldots, t_k)$ is a one-dimensional local ring with maximal ideal $\mathbf{n}S(t_1, \ldots, t_k)$ and residue field $(S/\mathbf{n})(t_1, \ldots, t_k)$, a pure transcendental extension of $S/\mathbf{n}$ in k indeterminates; moreover, the zero ideal of $S(t_1, \ldots, t_k)$ is primary. Therefore with notation as in Theorem 3.4, by passing from T to $W = T(t_1, \ldots, t_k)$, one obtains a local ring W satisfying all the assertions in (3.4) except that W is not a subring of the total quotient ring of R. Thus for each positive integer $s \geq d-1$, there exists a one-dimensional local extension ring W_s of R such that the residue field of W_s is canonically isomorphic to $(R/\mathbf{m})(t_1, \ldots, t_s)$, a pure transcendental extension of $R/\mathbf{m}$ in s indeterminates, and such that W_s satisfies all the conditions of (3.4) except that of being a subring of the total quotient ring of R.

(3.6). The process of gluing maximal ideals proceeds as follows; the reader is referred to [DL] for details. Assume that $M_1, \ldots, M_k$ are maximal ideals of a ring T such that each T/M_i is isomorphic to a field F. Suppose $\phi_i : T \to F$ is a surjective ring homomorphism such that $\ker\phi_i = M_i$ for $1 \leq i \leq k$. Let

$$S = \{t \in T : \phi_1(t) = \phi_2(t) = \cdots = \phi_k(t)\}.$$

Then S is a subring of T containing $M = M_1 \cap \cdots \cap M_k$ as a maximal ideal, $S/M \cong F$, T is a finitely generated integral extension of S, and each of the maximal ideals M_i lies over M in S. We say that S is *a gluing of the maximal ideals*

$M_1, \ldots, M_k$. If the ring T is Noetherian, then by Eakin's theorem [M, page 18], S is Noetherian. Since S is a subring of T, each associated prime of (0) in S is the contraction to S of an associated prime of (0) in T.

Theorem 3.7. *Let $(R, \mathbf{m})$ be a local ring. There exists a local extension ring S of R such that*

1. *S dominates R,*
2. *the residue field of S is of the form $(R/\mathbf{m})(t_1, \ldots, t_k)$, a pure transcendental extension of $R/\mathbf{m}$,*
3. *S is a localization of a finitely generated R-algebra,*
4. *every associated prime of (0) in S contracts in R to an associated prime of (0), so every regular element of R remains regular in S, and*
5. *$\dim(S) \le 1$, and $\dim(S) = 1$ unless $\dim(R) = 0$.*

Proof. Let $d = \dim(R)$. If $d \le 1$, then for any positive integer k, $S = R(t_1, \ldots, t_k)$ satisfies the required conditions.

If $d \ge 2$, let $(0) = \bigcap_{i=1}^n \mathbf{q}_i$ be a minimal primary decomposition of (0) in R and let $R_i = R/\mathbf{q}_i$. Then R_i is a local ring with maximal ideal $\mathbf{m}_i = \mathbf{m}/\mathbf{q}_i$, $\dim(R_j) \le d$ for each j, and $\dim(R_j) = d$ for at least one integer $j, 1 \le j \le n$. By (3.4) and (3.5), there exist local rings $(T_i, M_i), 1 \le i \le n$, such that T_i is the localization of a finitely generated R_i-algebra, the ideal (0) of T_i is primary, $\mathbf{m}_i T_i$ is the maximal ideal of T_i, $\dim(T_i) \le 1$, and each $T_i/\mathbf{m}_i T_i$ is canonically isomorphic to $(R/\mathbf{m})(t_1, \ldots, t_k)$ for some positive integer k; moreover, $\dim(T_j) = 1$ for some j, and hence $\dim(T) = 1$. The ring $T = T_1 \oplus \cdots \oplus T_n$ is Noetherian with n maximal ideals $N_1, \ldots, N_n$ where, for $1 \le i \le n$, $T/N_i \cong T_i/\mathbf{m}_i T_i \cong (R/\mathbf{m})(t_1, \ldots, t_k)$. Moreover, T has n minimal primes and every associated prime of (0) in T is a minimal prime. We identify R as a subring of T via the diagonal embedding $r \mapsto (r + \mathbf{q}_1, \ldots, r + \mathbf{q}_n)$. Let ϕ_i denote the canonical surjection of T onto $(R/\mathbf{m})(t_1, \ldots, t_k)$ such that $\ker \phi_i = N_i$, and let S denote the gluing of $N_1, \ldots, N_n$ with respect to $\phi_1, \ldots, \phi_n$. Then R is a subring of S. Each associated prime of (0) in S is the contraction to S of an associated prime of (0) in T, and therefore contracts in R to an associated prime of (0). Moreover, $\mathbf{n} = N_1 \cap \cdots \cap N_n$ is the unique maximal ideal of S. The ring S is Noetherian since T is a Noetherian ring and T is a finitely generated S-module. Because T_i is a localization of a finitely generated R_i-algebra for each i, it follows easily that the ring T is a localization of a finitely generated extension ring of R in T. Consequently, S is a localization of a finitely generated R-algebra [HL, Prop. 1.1, page 2867]. We have $\dim(S) = \dim(T) = 1$, so S is a one-dimensional local ring. □

Remark 3.8. The gluing construction in the proof of (3.7) provides an S so that $\mathbf{m}S$ is primary for the maximal $\mathbf{n}$ of S, but, in general, not with $\mathbf{m}S = \mathbf{n}$. With notation as in the proof of (3.7), we have

$$\mathbf{m}T = \mathbf{m}_1 T_1 \oplus \cdots \oplus \mathbf{m}_n T_n = M_1 \oplus \cdots \oplus M_n,$$

and the projection of $\mathbf{m}S$ onto T_i is $M_i = \mathbf{m}_i T_i$ for each i. But the kernels of these projection maps on T are not contained in S, and one may have $\mathbf{m}S < \mathbf{n} = \mathbf{m}T$. In the case where R has only one associated prime of (0), Theorem 3.4 implies the existence of an extension S of R satisfying both the condition in (3.1) that $\mathbf{m}S$ is the maximal ideal of S and the condition of (3.7) that S is a localization of a finitely generated R-algebra. It would be interesting to know more generally whether there exists an S simultaneously satisfying the conditions of (3.1) and (3.7).

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