PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 9, September 1997, Pages 2693–2697
S 0002-9939(97)03905-1

ISOMETRIC IMMERSIONS FROM THE HYPERBOLIC SPACE $H^2(-1)$ INTO $H^3(-1)$

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(Communicated by Christopher Croke)

ABSTRACT. In this paper, we transform the problem of determining isometric immersions from $H^2(-1)$ into $H^3(-1)$ into that of solving a degenerate Monge-Ampère equation on the unit disc. By presenting one family of special solutions to the equation, we obtain a great many noncongruent examples of such isometric immersions with or without umbilic set.

1. Introduction

Let $H^n(c)$ $(c < 0)$ be an n-dimensional hyperbolic space form with constant sectional curvature c; its Cayley model is the hypersurface $F : \langle X, X\rangle_L = \frac{1}{c}$, $x_{n+1} > 0$, in the Minkowski space R_1^{n+1}, where $\langle\cdot,\cdot\rangle_L$ denotes the inner product in R_1^{n+1}, i.e., $\langle X, Y\rangle_L = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}$, for $X = (x_1, \ldots, x_{n+1})$, $Y = (y_1, \ldots, y_{n+1}) \in R^{n+1}$.

Let us denote by $M^n(c)$ the n-dimensional space form of constant sectional curvature c, i.e., $M^n(0) = E^n$, $M^n(c) = S^n(\frac{1}{\sqrt{c}})$ for $c > 0$ and $M^n(c) = H^n(c)$ for $c < 0$. For the problem of isometric immersions from $M^n(c)$ into $M^{n+1}(c)$, the following global results are well-known:

(i) $c = 0$. Each isometrically immersed complete manifold M^n of E^n into E^{n+1} must be a cylinder over a plane curve, i.e., $M^n = E^{n-1} \times C$, where C is a curve over the plane orthogonal to E^{n-1}. This result is due to Hartman and Nirenberg [5] and Massey [8].

(ii) $c = 1$. An isometric immersion of $S^n(1)$ into $S^{n+1}(1)$ is rigid, i.e., it can only be a totally geodesic imbedding [1, 3, 11].

In the hyperbolic case, the situation looks quite different from the others. Isometric immersions seem much more abundant. Indeed, Nomizu [9] constructed explicitly a one-parameter family of examples of isometric immersion from $H^2(-1)$ into $H^3(-1)$ with three different kinds of properties. At the same time that [9] appeared, Ferus [4] showed that given a totally geodesic foliation of codimension 1 in $H^n(-1)$, there is a family of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ for which the relative nullity foliations coincide with the given foliation. Their result completely characterized the space of nowhere umbilic isometric immersions of

Received by the editors January 12, 1996 and, in revised form, April 12, 1996.
1991 *Mathematics Subject Classification.* Primary 53C42; Secondary 53C21.
Key words and phrases. Isometric immersion, hyperbolic space, Monge-Ampère equation.
This research was supported by the National Natural Science Foundation of China.

$H^n(-1)$ into $H^{n+1}(-1)$. By considering a broader class of isometric immersions, Abe and Haas [2] showed that given a differentiable lamination on $H^n(-1)$, there is a family of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ so that the induced relative nullity foliations are completely determined by the lamination. While including the results of [4] and [9], Abe and Haas' approach also provides examples of isometric immersions with umbilic sets.

In this paper, we are considering the following open problem posed by Nomizu in [9]: "To determine all isometric immersions from $H^2(-1)$ into $H^3(-1)$". By presenting a new approach we will transform this problem into a problem of pure analysis. Precisely, we shall transform the problem of determining isometric immersions from $H^2(-1)$ into $H^3(-1)$ into that of solving a degenerate Monge-Ampère equation on the unit disc D. Our main result is the following:

Theorem. *Every smooth isometric immersion of $H^2(-1)$ into $H^3(-1)$ corresponds to a solution of the Monge-Ampère equation*

$$\det\left(\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}\right) = 0, \qquad \xi = (\xi_1, \xi_2) \in D. \tag{1}$$

Conversely, for every smooth solution u of (1) *we define*

$$\begin{cases} g_{ij} = \lambda^{-4}(\lambda^2\delta_{ij} + \xi_i\xi_j), & \lambda = \sqrt{1-\xi_1^2-\xi_2^2}, \\ h_{ij} = \lambda^{-1}\partial^2 u/\partial\xi_i\partial\xi_j, & i,j = 1,2. \end{cases} \tag{2}$$

Then u determines a smooth isometric immersion of $H^2(-1)$ into $H^3(-1)$ with g, h as its first and second fundamental form, respectively.

Remark 1. We have studied in [6] the isometric immersions from $H^2(-1)$ into $H^3(c)$ $(c < -1)$. Based on results obtained by Li [7], we have partially classified those isometric immersions which are of bounded principal curvatures.

2. Proof of the theorem

Take the hyperplane $\Pi : x_3 = 1$ in R^3_1 and consider the central projection p of $F : x_1^2 + x_2^2 - x_3^2 = -1$, $x_3 > 0$, from the origin O of R^3 into Π. Then F is mapped in a one-to-one fashion onto an open unit disc $D : \xi_1^2 + \xi_2^2 < 1$. The mapping p is given by

$$p : F \to D, \qquad (x_1, x_2, x_3) \to (\xi_1, \xi_2)$$

where $x_3 = \sqrt{1 + x_1^2 + x_2^2}$ and $\xi_i = x_i/x_3,\ i = 1,2$.

The parametric representation of F with respect to $\{\xi_1, \xi_2\}$ is given by

$$F : v(\xi_1, \xi_2) = \left(\frac{\xi_1}{\lambda}, \frac{\xi_2}{\lambda}, \frac{1}{\lambda}\right), \qquad \lambda = \sqrt{1-\xi_1^2-\xi_2^2}.$$

The metric of $H^2(-1)$ with respect to $\{\xi_1, \xi_2\}$ is given by the tensor g with

$$g_{ij} = \lambda^{-4}(\lambda^2\delta_{ij} + \xi_i\xi_j), \qquad i,j = 1,2,$$

and $\det(g_{ij}) = \lambda^{-6}$. The Christoffel symbols are $\Gamma^k_{ij} = \lambda^{-2}(\xi_i\delta_{jk} + \xi_j\delta_{ik})$.

Let us denote by h_{ij} the second fundamental form of the immersion $x : H^2(-1) \to H^3(-1)$ w.r.t. the frame $\{\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}\}$ on $H^2(-1)$. From the fundamental theorem of hypersurface theory, an isometric immersion is uniquely determined, in the sense of congruence, by its first and second fundamental forms. The Gauss equation and the Codazzi equation are the integrability conditions. It is well known that, for an isometric immersion of $H^2(-1)$ into $H^3(-1)$, the Codazzi equation is equivalent to the second fundamental form h being a Codazzi tensor on $H^2(-1)$. We will need the following.

Lemma (cf. Proposition 1.3.3. of [10])**.** *Let (M, g) be a Riemannian manifold of constant sectional curvature c (possibly zero) and T a Codazzi tensor on M. Then for every point on M, there exist a neighborhood U and a smooth function $f : M \to R$ such that in U, $T = \text{Hess } f + cgf$. In addition, if M is simply connected then such a representation is available on all of M.*

Convrsely, on a manifold (M, g) of constant curvature c, any smooth function f generates a Codazzi tensor $T = \text{Hess } f + cgf$.

By applying the lemma to $H^2(-1)$, we see that h being a Codazzi tensor on $H^2(-1)$ is equivalent to the existence of a globally defined smooth function f on $H^2(-1)$ such that $h = \text{Hess } f - gf$.

For an isometric immersion of $H^2(-1)$ into $H^3(-1)$, the Gauss equation is equivalent to $\det(h) = 0$, i.e.,

$$\det(\text{Hess } f - gf) = 0. \tag{3}$$

Define by $u = \lambda f$ a function on D, the components of Hess $f - gf$ are

$$\nabla_{ij} f - g_{ij} f = \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} - \Gamma^k_{ij} \frac{\partial f}{\partial \xi_k} - g_{ij} f = \lambda^{-1} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j},$$

where ∇_{ij} are the operators of second covariant differentiation in the metric g. Then, equation (3) is changed into the Monge-Ampère equation (1) in D.

Hence, every smooth isometric immersion x of $H^2(-1)$ into $H^3(-1)$ corresponds to a smooth solution u of (1) such that the first and second fundamental forms g and h of x are given by (2).

On the other hand, if u is a smooth solution of (1), then h defined by (2) is a Codazzi tensor on $H^2(-1)$ and g, h determine a smooth isometric immersion x of $H^2(-1)$ into $H^3(-1)$ with (1) as its Gauss equation.

This completes the proof of our theorem.

3. Further discussions

Assume that u is a smooth solution of (1), and that the first and second fundamental forms g and h of the immersion x which corresponds to u are given by (2). If we denote by σ the principal curvature of the immersion x, then

$$\det(h_{ij} - \sigma g_{ij}) = 0. \tag{4}$$

Now, we mention the following two obvious facts:

1°. Two solutions u and $\overline{u}$ of (1) determine two congruent immersions of $H^2(-1)$ into $H^3(-1)$ if $u - \overline{u}$ is a linear function in ξ_1, ξ_2.

2°. For any $a, b \in R$, $a^2 + b^2 \neq 0$, we denote $(-\sqrt{a^2+b^2}, \sqrt{a^2+b^2})$ by $I_{a,b}$. Then for each $G(t) \in C^\infty(I_{a,b})$, $u = G(a\xi_1 + b\xi_2)$ is a smooth solution of (1).

Then for an arbitrary $G(t) \in C^\infty(I_{a,b})$, $u_G = G(a\xi_1 + b\xi_2)$ is a smooth solution of (1). A direct computation shows that the generally non-zero principal curvature of the immersion corresponding to u_G satisfies

$$\sigma_{G,a,b} = \lambda G''(a\xi_1 + b\xi_2)[a^2 + b^2 - (a\xi_1 + b\xi_2)^2].$$

Thus by choosing the function G properly, we can obtain a great many non-congruent isometric immersions of $H^2(-1)$ into $H^3(-1)$ possessing one of the following properties, respectively:

(1) The immersion possesses bounded principal curvature with no umbilics.

(2) The immersion possesses one-side unbounded principal curvature with no umbilics.

(3) The immersion possesses bounded principal curvature with the umbilic set consisting of finitely many arbitrary curves.

(4) The immersion possesses one- (or two-) sided unbounded principal curvatures with umbilic set consisting of finitely many arbitrary curves.

To study isometric immersions from $H^2(-1)$ into $H^3(-1)$, one may consider the Dirichlet boundary problem of the degenerate Monge-Ampère equation:

$$\begin{cases} \det\left(\partial^2 u/\partial\xi_i\partial\xi_j\right) = 0 & \text{in } D, \\ u = \varphi & \text{on } \partial D. \end{cases} \tag{5}$$

However, the regularity properties of the solution to (5) are quite complicated. This is clearly shown by the example $u = \sqrt{\xi_1^2 + \xi_1^2}$; in this case, u satisfies (1) in $D\backslash\{0\}$ and $u|_{\partial D} \equiv 1$, whereas $\xi = 0$ is a singular point of u.

On the other hand, if $\varphi \in C^0(\partial D)$, (5) does have a unique convex generalized solution. One can see Wachsmuth [12] for more detailed discussions and references on the problem (5).

Acknowledgement

The first author would like to thank Professors K. Abe and H. Mori for helpful information. Both authors would like to express their gratitude to Professor A. M. Li for his constant encouragement.

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