A CONVERSE OF THE GELFAND THEOREM

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Abstract. In this short note we obtain a converse to the Gelfand theorem: a Riemannian manifold is homogeneous if the isometrically invariant operators on the manifold form a commutative algebra.

1. Introduction

A remarkable property of a symmetric space is that the isometrically invariant differential operators on it form a commutative algebra. This is known as the Gelfand theorem ([4]). The converse of the theorem is not true in general. There are many non-symmetric homogeneous Riemannian manifolds on which the algebra of isometrically invariant differential operators is commutative ([6], [11]). However, there still are many results relating to the converse of the Gelfand theorem (See [1], [7], [8], etc.). Among them is Kowalski and Vanhecke’s theorem which asserts that a homogeneous Kähler manifold is locally Hermitian symmetric if all isometrically invariant differential operators on the manifold commute each other.

In this short note, we prove a converse of the Gelfand theorem:

Theorem. If the algebra of isometrically invariant differential operators on a complete connected Riemannian manifold $M$ is commutative, $M$ is a Riemannian homogeneous manifold.

A Riemannian homogeneous manifold with commutative invariant differential operators is called a commutative space ([10]). Our result says that the “homogeneous” assumption can be removed from the above definition. Commutative spaces are contained in the class of Riemannian manifolds with volume-preserving local geodesic symmetries ([9]). It is known that a naturally reductive homogeneous space of dimension $\leq 5$ is necessarily a commutative space ([10]). This is no longer true for dimension $\geq 6$ ([5]). On the other hand, a commutative space is not necessarily a naturally reductive homogeneous space due to the results of Kaplan and Ricci’s on Heisenberg type Lie groups ([6], [11]). It is an interesting problem to characterize commutative spaces.

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2. Proof of the Theorem

In this section, $M$ denotes a complete connected Riemannian manifold. Denote by $C^\infty(M)$ the Fréchet space of smooth functions on $M$ with the topology under which a sequence of functions is convergent if for any integer $k \geq 0$, the sequence of the $k$-th derivatives of the functions is uniformly convergent in any compact subset of $M$. As usual, a differential operator on $M$ is a continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ which has local form $A = \sum |\alpha| \leq k f_\alpha D^\alpha$. A differential operator $A$ is called isometrically invariant if $g^*A = A$ holds for any isometry $g$ of $M$, where $g^*A$ is defined by $g^*A(u)(x) = A(u \circ g)(g^{-1}x)$. It is easy to see that the Laplacian $\triangle$ on $M$ is an isometrically invariant differential operator. To list other examples of the invariant operators, we need the following definition of the spherical mean operator.

The spherical mean operator $L_r : C^\infty(M) \to C^\infty(M)$ is defined by

$$L_r f(x) = \int_{S_x(M)} f(\exp_x r\xi) d\mu_x(\xi)$$

where $d\mu_x$ stands for the normalized canonical measure on the unit tangent sphere $S_x(M) = \{ \xi \in T_xM \mid \| \xi \| = 1 \}$. We refer to [3] as a reference for the operator $L_r$. It is easy to see that the spherical mean operator $L_r$ is an isometrically invariant operator. We notice that $L_r$ is smooth in $r$. Let $Z$ be the vector field on $S(M)$ generating the geodesic flow $G^t$. A direct calculation shows that for $k = 1, 2, \cdots$,

$$\frac{d^k}{dr^k} |_{r=0} L_r f(x) = \int_{S_x(M)} (Z^k \pi^* f)(\xi) d\mu_x(\xi),$$

$$Z^k (\pi^* f)(\xi) = \nabla_{\xi^1} \cdots \nabla_{\xi^k} f = \sum_{\alpha_1 \cdots \alpha_k} \xi^1_{\alpha_1} \cdots \xi^k_{\alpha_k} \nabla_{\alpha_1} \cdots \nabla_{\alpha_k} f,$$

where $\pi : S(M) \to M$ is the natural projection, $\xi = \sum \xi^i \frac{\partial}{\partial \xi^i} \in S_x(M)$, and $\nabla$ is the covariant differential on $M$. Applying the above identity to the Taylor expansion of $L_r$, we get that, for each integer $n \geq 0$,

$$L_r = \sum_{k \leq n} \frac{1}{(2k)!} P_{2k} r^{2k} + \frac{d^{2n+1}}{dr^{2n+1}} |_{r=0} (L_r) \frac{r^{2n+1}}{(2n+1)!}$$

where

$$P_{2k} f(x) = \int_{S_x(M)} \sum \xi^1_{\alpha_1} \cdots \xi^k_{\alpha_k} \nabla_{\alpha_2} \cdots \nabla_{\alpha_1} f(x) d\mu_x(\xi).$$

Since $L_r$ is an isometrically invariant operator, so is each coefficient $P_{2k}$.

If the algebra of isometrically invariant differential operators on $M$ is commutative, $M$ has cyclically parallel Ricci tensor by a theorem of Z. Szabo’s ([12]). Therefore, $M$ is real analytic (see [2], [12]). In this case, for a local analytic function $f$,

$$L_r f(x) = \sum_k \frac{1}{(2k)!} P_{2k}(f)(x) r^{2k}$$

holds for small enough $r > 0$ such that $\exp_x(r \xi)$ is contained in the analytic domain of $f$. Since analytic functions are locally dense in $C^\infty(M)$, we get the following lemma:
Lemma 1. If $M$ is a Riemannian manifold with commutative algebra of isometrically invariant differential operators, then the spherical mean operator $L_r$ commutes with all isometrically invariant differential operators.

In the rest of the paper, we denote by $G$ the isometry group of $M$. To prove the theorem, we need the following lemma:

Lemma 2. Let $x \in M$ and $U \subseteq M$ be an open set with compact closure $\bar{U}$. The set $K_x = \{ g \in G \mid gx \in U \}$ is pre-compact in $G$ with regard to the compact-open topology.

Proof. Let $\{ g_n \}_{i=1}^\infty \subseteq K_x$ be a sequence. According to the definition of $K_x$, we see that $\{ g_n x \} \subseteq \bar{U}$. There is a subsequence $\{ g_{n_j} \}$ of $\{ g_n \}$, due to the compactness of $\bar{U}$, such that $\{ g_{n_j} (x) \}$ converges to a point $x_1 \in M$. Let $\{ e_1, \ldots, e_n \}$ be an orthonormal frame of $M$ at $x$. Since $g_{n_j} (e_j)$ are contained in a compact subset of the sphere tangent bundle $S(M)$, by passing to a subsequence if necessary, we may assume that $\lim_{j \to \infty} (g_{n_j})_*(e_j)$ exists for all $j = 1, \ldots, n$. Therefore, $\{ (g_{n_j})_*(e) \}$ is uniformly convergent in $S_x M$.

We will show that $\{ g_{n_j}(p) \}$ is uniformly convergent for $p$ in any compact set of $M$. To this end, choose a geodesic $\gamma : [0, l] \to M$ connecting $x$ with $p$. Since $g_{n_j}$ are isometries, $\varphi_{n_j}(t) = g_{n_j}(\gamma(t))$ are also geodesics with initial data $\varphi_{n_j}(0) = g_{n_j}(x), \varphi_{n_j}'(0) = (g_{n_j})_* \gamma'(0)$. Hence, by the convergence of $g_{n_j}(x)$ and $(g_{n_j})_* \gamma'(0)$ and the continuous dependence of ordinary differential equations on the initial data, we see that $g_{n_j}(p) = \varphi_{n_j}(l)$ is convergent; more precisely, $g_{n_j}(p)$ is uniformly convergent about $p$ in any compact set.

Define $g(p) = \lim_{j \to \infty} g_{n_j}(p)$. It is easy to see that $g$ is an isometry of $M$ and $\lim g_{n_j} = g$ under the compact-open topology of $G$. This proves our lemma. $\square$

Proof of Theorem. For each $y \in M$, the orbit $G(y)$ is a smooth submanifold of $M$. Since $G(y)$ is a closed subset of $M$ (see [13]), it suffices to prove $\dim(G(y)) = \dim(M)$ for each $y \in M$.

Assume that $\dim(G(y)) < \dim(M)$. Take an open neighborhood $U$ of $y$ such that $\text{dist}^2(x, G(y))$ is smooth in $U$ and $U$ has compact closure $\bar{U}$. Let $\lambda(x)$ be a smooth cut-off function of $U$ and define

$$ h(x) = \lambda(x) \text{dist}^2(x, G(y)). $$

It is easy to see that $h(x)$ is a global smooth function satisfying $h(x) > 0$ for each $x \in U \setminus G(y)$ and $\text{supp}(h) \subseteq \bar{U}$. Furthermore, we define

$$ h_1(x) = \int_G h(gx) d\mu(g). $$

Here $d\mu$ is the Harr measure on $G$. Since $h(gx) \neq 0$ implies that $g \in K_x = \{ g \in G \mid gx \in \bar{U} \}$, according to Lemma 2, we see that the integrand $g \to h(gx)$ has compact support in $G$. Therefore, the function $h_1$ is well-defined. Moreover, $h_1$ is a non-negative $G$-invariant function and satisfies $h_1 > 0$ in $U \setminus G(y)$.

Obviously, $h_1 \triangle$ is a $G$-invariant differential operator. Here $\triangle$ is the Laplacian on $M$. According to Lemma 1, we get $[h_1 \triangle, L_r] = 0$. On the other hand, we still have $[\triangle, L_r] = 0$. Therefore, it holds that $h_1 L_r \triangle = L_r (h_1 \triangle)$. Take $f \in C^\infty(M)$ such that $\triangle f = 1$ in $U$. Then for $r > 0$ small enough such that $\exp_y(r\xi) \in U$ for
\[ \xi \in S_y(M), \text{ we have } h_1(y)L_r(1) = L_r(h_1), \text{ i.e.,} \]
\[ h_1(y) = \int_{S_y(M)} h_1(\exp_y(r\xi)) d\mu(\xi). \]

However \( h_1(y) = 0 \) and \( \int_{S_y(M)} h_1(\exp_y(r\xi)) d\mu(\xi) > 0 \). This is a contradiction. \( \square \)

REFERENCES


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