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SEMI-FREE ACTIONS OF ZERO-DIMENSIONAL COMPACT GROUPS ON MENGER COMPACTA

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ABSTRACT. Let μ^n be the n-dimensional universal Menger compactum, X a Z-set in μ^n and G a metrizable zero-dimensional compact group with e the unit. It is proved that there exists a semi-free G-action on μ^n such that X is the fixed point set of every $g \in G \setminus \{e\}$. As a corollary, it follows that each compactum with dim $\leqslant n$ can be embedded in μ^n as the fixed point set of some semi-free G-action on μ^n.

In [Dr], Dranishnikov showed that every metrizable zero-dimensional compact group G acts freely on the n-dimensional universal Menger compactum[1] μ^n (cf. [Sa]). Here we consider the fixed point sets of semi-free actions[2] of G on μ^n. A closed set X in μ^n is called a *Z-set* if there are maps $f\colon \mu^n \to \mu^n \setminus X$ arbitrarily close to id. The following is our result:

Theorem. *Let G be a metrizable zero-dimensional compact group with e the unit and X a Z-set in μ^n. Then there exists a semi-free G-action on μ^n such that X is the fixed point set of every $g \in G \setminus \{e\}$.*

By [Be, 2.3.8], each compactum X with $\dim X \leqslant n$ can be embedded in μ^n as a Z-set. Then we have the following:

Corollary. *Let G be a metrizable zero-dimensional compact group. Each compactum X with $\dim X \leqslant n$ can be embedded in μ^n as the fixed point set of some semi-free G-action on μ^n.*

In the proof below, for two simplicial complexes K and L, $K \times L$ denotes the simplicial complex defined as the barycentric subdivision of the cell complex $\{\sigma \times \tau \mid \sigma \in K, \tau \in L\}$. For any simplicial map $f\colon K \to L$, the simplicial mapping cylinder of f is denoted by $M(f)$ (cf. [Wh, §6]). Notice that K and L are subcomplexes of $M(f)$. By $K^{(0)}$, we denote the set of vertices (0-skeleton) of K.

Proof of Theorem. We may only consider the case that G is non-trivial, i.e., $G \neq \{e\}$. By a well-known theorem of Pontryagin [Po, §46, C], G is the inverse limit of

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[1] A *compactum* is a compact metrizable space.

[2] An action of G on a space X is called *semi-free* if the isotropy subgroup G_x of G at each $x \in X$ is trivial or all of G, where $G_x = \{g \in G \mid gx = x\}$.

an inverse sequence of non-trivial finite groups

$$G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 \xleftarrow{\varphi_3} \cdots,$$

whence G is a subgroup of $\prod_{i\in\mathbb{N}} G_i$. We denote the unit of G_i by e_i. For each $i \in \mathbb{N}$, we denote

$$G'_i = \big\{(\varphi_1 \cdots \varphi_{i-1}(g), \cdots, \varphi_{i-1}(g), g) \bigm| g \in G_i\big\} \subset G_1 \times \cdots \times G_i$$

and $e'_i = (e_1, \cdots, e_i) \in G'_i \subset G_1 \times \cdots \times G_i$. Let L_i be the n-dimensional $(n-1)$-connected simplicial free G_i-complex. Such a complex is defined in [Sa]. Then the G_i-action on L_i extends naturally to the simplicial semi-free G_i-action on the cone $v_i * L_i$ over L_i such that the cone vertex v_i is the unique fixed point of every $g \in G_i \setminus \{e_i\}$. For each $i \in \mathbb{N}$, choose a vertex u_i of L_i.

By Freudenthal's Theorem (cf. [En, 1.13.2], [Ko]), we may assume that X is the inverse limit of the inverse sequence

$$|K_1| \xleftarrow{f_1} |K_2| \xleftarrow{f_2} |K_3| \xleftarrow{f_3} \cdots$$

such that $\operatorname{mesh} f_{i,\infty}^{-1}(K_i) \to 0$ $(i \to \infty)$, where each $f_{i,\infty} \colon X \to |K_i|$ is the projection, each K_i is a finite simplicial complex with $\dim K_i \leqslant \dim X \leqslant n$ and each f_i is PL (piece-wise linear).

Let $K_0 = \{v_0\}$ be the simplicial complex consisting of only one vertex. Then the constant map $f_0 \colon K_1 \to K_0$ is simplicial. Let $K'_1 = K_1$ and inductively choose simplicial subdivisions K'_i and K^*_i of K_i so that K^*_i is a subdivision of the barycentric subdivision of K'_i and $f_i \colon K'_{i+1} \to K^*_i$ is simplicial.

Let M_1 be the n-skeleton of

$$M(f_0) \times L_1 \cup_{K_1 \times L_1} K_1 \times (v_1 * L_1),$$

which is $(n-1)$-connected. We regard K^*_1 as a subdivision of $K_1 \times \{v_1\}$. Then M_1 has the simplicial subdivision M^*_1 with $(M^*_1)^{(0)} = (M_1)^{(0)} \cup (K^*_1)^{(0)}$ which contains $M(f_0) \times L_1$ and K^*_1 as subcomplexes. Using the G_1-actions on L_1 and $v_1 * L_1$, we define a simplicial G_1-action on M^*_1 by $g(x, y) = (x, gy)$ on $|M(f_0)| \times |L_1|$ and $|K_1| \times |v_1 * L_1|$. Observe that $|K_1| = |K_1| \times \{v_1\}$ is the fixed point set of every $g \in G_1 \setminus \{e_1\}$. Let $N_{1,1} = M^*_1$ and define $N_{1,i+1}$ inductively as the n-skeleton of $N_{1,i} \times L_{i+1}$ and let $p_{1,i} \colon |N_{1,i+1}| \to |N_{1,i}|$ be the projection. Then we have the inverse sequence

$$|N_{1,1}| \xleftarrow{p_{1,1}} |N_{1,2}| \xleftarrow{p_{1,2}} |N_{1,3}| \xleftarrow{p_{1,3}} \cdots$$

such that the inverse limit N_1 is a compact μ^n-manifold as is shown in [Sa] by using [GHW, Theorem 1]. Since $|N_{1,1}| = |M_1|$ is $(n-1)$-connected, N_1 is indeed homeomorphic to μ^n.

Note that $f_1 \colon K'_2 \to K^*_1$ is simplicial. Let $i_1 \colon K^*_1 \subset M^*_1 = N_{1,1}$ be the inclusion. Then $i_1 f_1 \colon K'_2 \to N_{1,1}$ is also simplicial and $M(i_1 f_1) = N_{1,1} \cup_{K^*_1} M(f_1)$. The G_1-action on $N_{1,1}$ extends simplicially to $M(i_1 f_1)$ so that $|M(f_1)|$ is the fixed point set for every $g \in G_1 \setminus \{e_1\}$.

Let M_2 be the n-skeleton of

$$M(i_1 f_1) \times L_2 \cup_{K'_2 \times L_2} K'_2 \times (v_2 * L_2),$$

which is $(n-1)$-connected. We regard K^*_2 as a subdivision of $K'_2 \times \{v_2\} \subset M_2$. Then M_2 has the simplicial subdivision M^*_2 with $(M^*_2)^{(0)} = (M_2)^{(0)} \cup (K^*_2)^{(0)}$ which contains $M(i_1 f_1) \times L_2$ and K^*_2 as subcomplexes. Observe that $N_{1,2} = M^*_1 \times L_2 \subset$

$M(i_1f_1) \times L_2 \subset M_2^*$. Using the G_1-action on $M(i_1f_1)$ and the G_2-actions on L_2 and $v_2 * L_2$, we define a simplicial semi-free action of G_2' on M_2^* by $(g_1, g_2)(x_1, x_2) = (g_1x_1, g_2x_2)$ on $|M(i_1f_1)| \times |L_2|$ and $|K_2| \times |v_2 * L_2|$. Then $|K_2| = |K_2| \times \{v_2\}$ is the fixed point set of every $(g_1, g_2) \in G_2' \setminus \{e_2'\}$. In other words, this action induces the free G_2'-action on $|M_2| \setminus |K_2|$. We have a retraction $r_1 = c_1p_1 \colon |M_2| \to |M_1|$, where $p_1 \colon |M_2| \to |M(i_1f_1)|$ is the projection and $c_1 \colon |M(i_1f_1)| \to |M_1|$ is the collapsing. Then $r_1((g_1, g_2)x) = g_1r_1(x)$ for each $x \in |M_2|$ and $(g_1, g_2) \in G_2'$, r_1 induces an isomorphism of homotopy groups of dimension $\leqslant n-1$ and $r_1||K_2| = f_1$. Let $N_{2,1} = M_2^*$ and define $N_{2,i+1}$ inductively as the n-skeleton of $N_{2,i} \times L_{i+2}$ and let $p_{2,i} \colon |N_{2,i+1}| \to |N_{2,i}|$ be the projection. Then $N_{1,i+1} \subset N_{2,i}$ for each $i \in \mathbb{N}$. Similarly as above, we have the inverse sequence

$$|N_{2,1}| \overset{p_{2,1}}{\longleftarrow} |N_{2,2}| \overset{p_{2,2}}{\longleftarrow} |N_{2,3}| \overset{p_{2,3}}{\longleftarrow} \cdots,$$

such that the inverse limit N_2 is homeomorphic to μ^n.

Note that $f_2 \colon K_3' \to K_2^*$ is simplicial. Let $i_2 \colon K_2^* \subset M_2^* = N_{2,1}$ be the inclusion. Then $i_2f_2 \colon K_3' \to N_{2,1}$ is also simplicial and $M(i_2f_2) = N_{2,1} \cup_{K_2^*} M(f_2)$. The G_2'-action on $N_{2,1}$ extends simplicially to $M(i_2f_2)$ so that $|M(f_2)|$ is the fixed point set for every $g \in G_2' \setminus \{e_2'\}$.

By induction, we have the following diagram of the inverse sequences:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle p_{1,2}}\big\downarrow & & {\scriptstyle p_{2,2}}\big\downarrow & & {\scriptstyle p_{3,2}}\big\downarrow & & \\
|N_{1,2}| & & |N_{2,2}| & & |N_{3,2}| & & \\
{\scriptstyle p_{1,1}}\big\downarrow & & {\scriptstyle p_{2,1}}\big\downarrow & & {\scriptstyle p_{3,1}}\big\downarrow & & \\
|N_{1,1}| & & |N_{2,1}| & & |N_{3,1}| & & \\
\| & & \| & & \| & & \\
|M_1| & \underset{\supset}{\overset{r_1}{\longleftarrow}} & |M_2| & \underset{\supset}{\overset{r_2}{\longleftarrow}} & |M_3| & \underset{\supset}{\overset{r_3}{\longleftarrow}} & \cdots \\
\cup & & \cup & & \cup & & \\
|K_1| & \underset{f_1}{\longleftarrow} & |K_2| & \underset{f_2}{\longleftarrow} & |K_3| & \underset{f_3}{\longleftarrow} & \cdots,
\end{array}$$

where the following conditions are satisfied:

(1) each $|M_i|$ is $(n-1)$-connected,
(2) each $N_{i,1}$ is a subdivision of M_i which has a simplicial G_i'-action,
(3) each $|K_i|$ is the fixed point set of every $g \in G_i' \setminus \{e_i'\}$,
(4) r_i is a retraction such that $r_i||K_{i+1}| = f_i$ and

$$r_i((g_1, \cdots, g_{i+1})x) = (g_1, \cdots, g_i)r_i(x)$$

for each $x \in |M_i|$ and $(g_1, \cdots, g_{i+1}) \in G_i'$,
(5) r_i induces an isomorphism of homotopy groups of dimension $\leqslant n-1$,
(6) $N_{i,j} \subset N_{i+1,j-1} \subset \cdots \subset N_{i+j-1,1}$, $p_{i,j} = r_{i+j-1}||N_{i,j+1}|$ and
(7) the inverse limit N_i of the inverse sequence

$$|N_{i,1}| \overset{p_{i,1}}{\longleftarrow} |N_{i,2}| \overset{p_{i,2}}{\longleftarrow} |N_{i,3}| \overset{p_{i,3}}{\longleftarrow} \cdots$$

is homeomorphic to μ^n.

Let M be the inverse limit of the sequence

$$|M_1| \xleftarrow{r_1} |M_2| \xleftarrow{r_2} |M_3| \xleftarrow{r_3} \cdots .$$

We can regard X and each N_i as subspaces of M. Observe that

$$M \setminus X = \bigcup_{i\in\mathbb{N}} N_i = \bigcup_{i\in\mathbb{N}} \mathrm{int}_M N_i \quad \text{and}$$
$$\mathrm{int}_M N_i = r_{i\infty}^{-1}(|M_i| \setminus |K_i|),$$

where $r_{i\infty}\colon M \to |M_i|$ is the projection. Hence $M \setminus X$ is a $(n-1)$-connected μ^n-manifold. It is easy to see that M is LC^{n-1} and X is a Z-set in M. Then M is $(n-1)$-connected. By Bestvina's characterization of μ^n [Be, 5.2.3], M is homeomorphic to μ^n.

We define an action of $G \subset \prod_{i\in\mathbb{N}} G_i$ on $M \subset \prod_{i\in\mathbb{N}} |M_i|$ as follows:

$$(g_1, g_2, \cdots)(x_1, x_2, \cdots) = (g_1x_1, (g_1, g_2)x_2, (g_1, g_2, g_3)x_3, \cdots).$$

Each $x = (x_1, x_2, \cdots) \in X$ is a fixed point of every $g = (g_1, g_2, \cdots) \in G$ since $x_i \in |K_i|$ is a fixed point of $(g_1, \cdots, g_i) \in G'_i$ for each $i \in \mathbb{N}$. On the other hand, $gx \neq x$ for each $x = (x_1, x_2, \cdots) \in M \setminus X$ and $g = (g_1, g_2, \cdots) \in G \setminus \{e\}$. In fact, $x \in \mathrm{int}_M N_i$ for some $i \in \mathbb{N}$. Let

$$x_{i+1} = (x_i, x'_{i+1}) \in |M_i| \times |L_{i+1}|,$$
$$x_{i+2} = (x_i, x'_{i+1}, x'_{i+2}) \in |M_i| \times |L_{i+1}| \times |L_{i+2}|,$$
$$\vdots$$

Identifying $x = (x_i, x'_{i+1}, x'_{i+2}, \cdots) \in (|M_i| \setminus |K_i|) \times \prod_{j>i} |L_j|$,

$$\begin{aligned} gx &= ((g_1, \cdots, g_i)x_i, g_{i+1}x'_{i+1}, g_{i+2}x'_{i+2}, \cdots) \\ &\neq (x_i, x'_{i+1}, x'_{i+2}, \cdots) = x \end{aligned}$$

because the G'_i-action on $|M_i| \setminus |K_i|$ and the G_j-action on $|L_j|$ $(j > i)$ are free. Therefore X is the fixed point set for every $g \in G \setminus \{e\}$. Since X is a Z-set in M, we have the result by the Z-set unknotting theorem [Be, 3.1.5]. □

Concerning our result, the following question arises:

Question. *If X is a closed set in μ^n but not a Z-set, is the theorem still true?*

Remark. This question has been solved affirmatively by Iwamoto. In his paper "*Fixed point sets of transformation groups of Menger manifolds, their pseudo-interior and their pseudo-boundaries*" [Topology Appl. **68** (1996), 267–283], by extending the method of this paper, he proved that if M is a μ^n-manifold and X is a closed set in M then there exists a semi-free G-action on M such that X is the fixed point set of every $g \in G \setminus \{e\}$. Moreover it is also proved that M has a G-invariant pseudo-interior $\nu(M)$. Then we have the same result for any pseudo-interior $\nu(M)$ of a μ^n-manifold M.

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