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THE K-THEORY OF GROMOV'S TRANSLATION ALGEBRAS AND THE AMENABILITY OF DISCRETE GROUPS

GÁBOR ELEK

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ABSTRACT. We prove the following theorem. A finitely generated group Γ is amenable if and only if $\mathbf{1} \neq \mathbf{0}$ in $K_0(T(\Gamma))$, the algebraic K-theory group of its translation algebra.

1. INTRODUCTION

The translation algebras of finitely generated groups were introduced by Gromov ([3], page 262). Let Γ be a finitely generated group and let C_Γ be a Cayley-graph of Γ. Then Γ can be equipped with the shortest distance metric, $\mathrm{dist} : \Gamma \times \Gamma \to \mathbf{R}$. A *matrix of finite propagation* is a function $A : \Gamma \times \Gamma \to \mathbf{R}$ with the following properties. There exist positive constants k_A, m_A such that for any $(\gamma, \delta) \in \Gamma \times \Gamma$:
1. $|A(\gamma, \delta)| \leq m_A$,
2. $A(\gamma, \delta) = 0$, if $\mathrm{dist}(\gamma, \delta) > k_A$.

The matrices of finite propagation form the *translation algebra* of Γ, which we denote by $T(\Gamma)$. (Note that $T(\Gamma)$ does not depend on the choice of the Cayley-graph C_Γ.) The goal of this paper is to prove the following theorem.

Theorem 1. *Let Γ be a finitely generated group. Then Γ is amenable if and only if $\mathbf{1} \neq \mathbf{0}$ in $K_0(T(\Gamma))$, the algebraic K-theory group of $T(\Gamma)$.*

The reader should note that our theorem can be regarded as an analog of Theorem 3.1 in [1].

2. AMENABLE GROUPS

Let Γ be a finitely generated amenable group. Then the following proposition holds.

Proposition 2.1. *There exists a finite trace Tr_ω on $T(\Gamma)$, such that $\mathrm{Tr}_\omega(\mathbf{1}) = 1$ and $\mathrm{Tr}_\omega(\mathbf{0}) = 0$.*

Note that the proposition is a combinatorial analogue of a result of John Roe [4]. Finite traces can be extended to the algebraic K-theory group; hence we have the following corollary.

Proposition 2.2. *Let Γ be as above. Then $\mathbf{1}$ and $\mathbf{0}$ are not representing the same element in $K_0(T(\Gamma))$.*

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Proof of Proposition 2.1. Let ω be a Banach-mean (ultralimit) on the bounded, infinite real sequences $\mathbf{a} = (a_1, a_2, \dots)$, such that $\omega(\mathbf{a}) = \lim_{n\to\infty} a_n$ if $\lim_{n\to\infty} a_n$ exists. Again, let us consider a Cayley-graph of Γ and the induced metric structure on the group. The amenability of Γ is characterized by the following Følner-property [5].

Let $B_n = \{y \in \Gamma |\ \mathrm{dist}(y,e) \le n\}$, the n-ball centered at the identity element of Γ. Then, for any $k > 0$,

$$\lim_{n\to\infty} \frac{\#(B_{n+k} \setminus B_n)}{\#(B_n)} = 0 \quad .$$

We define Tr_ω as follows. Let $A \in T(\Gamma)$; then $\mathrm{Tr}_\omega(A) = \omega(\mathbf{a})$, where

$$a_n = \frac{1}{\#(B_n)} \sum_{x\in B_n} A(x,x) \quad .$$

Obviously, $\mathrm{Tr}_\omega(\mathbf{1}) = 1$ and $\mathrm{Tr}_\omega(\mathbf{0}) = 0$. Hence the only thing that needs to be shown is that, for any $R, S \in T(\Gamma)$, $\mathrm{Tr}_\omega(RS) = \mathrm{Tr}_\omega(SR)$.

For $A \in T(\Gamma), n > 0$, let A_n be a $\#(B_n)\times\#(B_n)$-matrix with entries parametrized by pairs of vertices of B_n, such that $(A_n)_{i,j} = A(i,j)$, if $i, j \in B_n$.

Lemma 2.3.

$$\lim_{n\to\infty} \frac{\mathrm{Tr}(RS)_n - \mathrm{Tr}(R_nS_n)}{\#(B_n)} = 0 \quad .$$

Proof.

$$\mathrm{Tr}(RS)_n = \sum_{i\in B_n}\sum_{j\in\Gamma} R(i,j)S(j,i)$$

$$= \mathrm{Tr}(R_nS_n) + \sum_{i\in B_n}\ \sum_{j\in B_{n+k_S}\setminus B_n} R(i,j)S(j,i)$$

by the finite propagation property of S. However, $R(i,j)S(j,i) = 0$, if $\mathrm{dist}(i,j) > k_R$. Therefore, we have the following estimate.

$$\mathrm{Tr}(R_nS_n) + \sum_{i\in B_n}\ \sum_{j\in B_{n+k_S}\setminus B_n} R(i,j)S(j,i) \le m_R m_S \#(B_{k_R})\#(B_{n+k_S} \setminus B_n) \quad .$$

That is,

$$\left| \frac{\mathrm{Tr}(RS)_n - \mathrm{Tr}(R_nS_n)}{\#(B_n)} \right| \le \frac{m_R m_S \#(B_{k_R})\#(B_{n+k_S} \setminus B_n)}{\#(B_n)}.$$

The right-hand side of the inequality above tends to zero by the Følner-property. This proves our lemma. □

Now we return to the proof of Proposition 2.1. By our definition, $\mathrm{Tr}_\omega(RS) = \omega(\mathbf{x})$, where $x_n = \frac{1}{\#(B_n)}\mathrm{Tr}(RS)_n$. Hence by our previous lemma,

$$\mathrm{Tr}_\omega(RS) = \omega(\mathbf{y})\,,$$

where $y_n = \frac{1}{\#(B_n)}\mathrm{Tr}(R_nS_n)$. On the other hand, $\mathrm{Tr}_\omega(SR) = \omega(\mathbf{z})$, where $z_n = \frac{1}{\#(B_n)}\mathrm{Tr}(SR)_n$. Thus,

$$\mathrm{Tr}_\omega(SR) = \omega(\mathbf{w})\,,$$

where $w_n = \frac{1}{\#(B_n)}\mathrm{Tr}(S_nR_n)$. However, $y_n = w_n$, so $\mathrm{Tr}_\omega(RS) = \mathrm{Tr}_\omega(SR)$. □

3. Non-amenable groups

First let us review a result of Deuber, Simonovits and Sós on paradoxical metric spaces [2].

Let X be a metric space. Then $Y \subset X$ is *wobbling equivalent* to X, if there exists a bijection $\phi : X \to Y$, such that

$$\sup_{x\in X} \operatorname{dist}(\phi(x), x) < \infty \quad .$$

According to Theorem 3.1 [2], if X is a vertex space of a graph with exponential growth, then X can be written as the disjoint union of Y_1 and Y_2, where both Y_1 and Y_2 are wobbling-equivalent to X. Now let Γ be a finitely generated non-amenable group. Then Γ has exponential growth [5]. Hence we have the partition of Γ into $\Gamma_1 \cup \Gamma_2$, where $\phi_1 : \Gamma \to \Gamma_1$ and $\phi_2 : \Gamma \to \Gamma_2$ are bijections such that

$$\sup_{x\in X} \operatorname{dist}(\phi_1(x), x) < \infty \quad \text{and} \quad \sup_{x\in X} \operatorname{dist}(\phi_2(x), x) < \infty \quad .$$

Let U be a *co-isometry* defined the following way. For each $\gamma \in \Gamma$, $U(e_\gamma) = e_{\phi_1(\gamma)}$, where e_γ is the characteristic function of γ. Then $U^*U = \mathbf{1}, UU^* = P$, where the idempotent P is the multiplication by the characteristic function of Γ_1. Note that both U and U^* are elements of the translation algebra; consequently $\mathbf{1} = P$ in $K_0(T(\Gamma))$. The same way, $\mathbf{1} = Q$ in $K_0(T(\Gamma))$, where $Q = 1 - P$, the multiplication by the characteristic function of Γ_2. Therefore, $\mathbf{1} + \mathbf{1} = \mathbf{1}$ in $K_0(T(\Gamma))$, which shows the remaining part of Theorem 1. □

References

1. J. Block, S. Weinberger, *Aperiodic tilings, positive scalar curvature and amenability of spaces*, Journal of the Amer. Math. Soc. **5** (1992), 907–918 MR **93d:**53054
2. W.A.Deuber,M.Simonovits and V.T.Sós, *A note on paradoxical metric spaces*, Studia Sci.Hung.Math. **30** (1995), 17–23 MR **96i:**54025
3. M.Gromov, *Asymptotic Invariants of Infinite Groups*, London Math. Society, Lecture Note Series **182** (1993) MR **95m:**20041
4. J. Roe, *An index theorem on open manifolds I-II*, Journal of Differential Geometry **27** (1988), 87–136 MR **89a:**58102
5. P.M.Soardi, *Potential Theory on Infinite Networks*, Lecture Notes in Mathematics **1590** MR **96i:**31005

Department of Mathematics, Purdue University, West Lafayette, Indiana 47906

E-mail address: elekgab@math.purdue.edu

Current address: Mathematical Institute, Hungarian Academy of Science, P. O. Box 127, H-1364 Budapest, Hungary

E-mail address: elek@math-inst.hu