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LOCAL AUTOMORPHISMS AND DERIVATIONS ON $\mathcal{B}(H)$

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ABSTRACT. Let $\mathcal{A}$ be an algebra. A mapping $\theta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a 2-local automorphism if for every $a, b \in \mathcal{A}$ there is an automorphism $\theta_{a,b} : \mathcal{A} \longrightarrow \mathcal{A}$, depending on a and b, such that $\theta_{a,b}(a) = \theta(a)$ and $\theta_{a,b}(b) = \theta(b)$ (no linearity, surjectivity or continuity of θ is assumed). Let H be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(H)$ be the algebra of all linear bounded operators on H. Then every 2-local automorphism $\theta : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ is an automorphism. An analogous result is obtained for derivations.

A mapping θ of an algebra $\mathcal{A}$ into itself is called a local automorphism (respectively, local derivation) if for every $a \in \mathcal{A}$ there exists an automorphism (respectively, derivation) θ_a of $\mathcal{A}$, depending on a, such that $\theta(a) = \theta_a(a)$. These two notions were introduced independently by Kadison [7] and Larson and Sourour [10]. In fact, their definitions were stronger. They have assumed that these mappings are also linear. For us, however, it will be more convenient to omit the linearity assumption in the definitions of local automorphisms and local derivations.

Larson and Sourour proved that every linear local derivation on $\mathcal{B}(X)$, the algebra of all bounded linear operators on a Banach space X, is a derivation [10, Theorem 1.2], and provided that X is infinite-dimensional, every surjective linear local automorphism of $\mathcal{B}(X)$ is an automorphism [10, Theorem 2.1]. Brešar and Šemrl [4] proved that the surjectivity assumption in the last result can be removed if X is a separable Hilbert space. For other results concerning linear local automorphisms and local derivations we refer to [2], [3], [7].

The famous Gleason-Kahane-Żelazko theorem [6], [8] asserts that every linear selection of the spectrum defined on a Banach algebra is multiplicative. As noted by Badea [1] this result can be equivalently reformulated as a result on linear local homomorphisms as follows: if θ is a linear functional on a unital Banach algebra $\mathcal{A}$ with $\theta(e) = 1$ such that for every $a \in \mathcal{A}$ there exists an algebra homomorphism $\theta_a : \mathcal{A} \longrightarrow \mathbb{C}$ with the property that $\theta(a) = \theta_a(a)$, then θ is multiplicative. Therefore, if a linear functional has a local behaviour like a character at every point, then the functional itself is a character. An interesting extension of this result was obtained by Kowalski and Slodkowski [9]. In particular, their result shows that at the cost of requiring the local behaviour like a character at every two points, the condition of linearity can be dropped [1, Corollary 3.7]. More precisely, if $\theta : \mathcal{A} \longrightarrow \mathbb{C}$ is a function having the property that for every a and b in $\mathcal{A}$ there exists a multiplicative

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linear functional $\theta_{a,b}$ on $\mathcal{A}$ such that $\theta(a) = \theta_{a,b}(a)$ and $\theta(b) = \theta_{a,b}(b)$, then θ itself is linear and multiplicative.

Motivated by the above considerations we introduce the following definition.

Definition. Let $\mathcal{A}$ be an algebra. A mapping $\theta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a 2-local automorphism (respectively, 2-local derivation) if for every $a, b \in \mathcal{A}$ there is an automorphism (respectively, derivation) $\theta_{a,b} : \mathcal{A} \longrightarrow \mathcal{A}$, depending on a and b, such that $\theta_{a,b}(a) = \theta(a)$ and $\theta_{a,b}(b) = \theta(b)$.

It is the aim of this note to show that if X is a separable Hilbert space, then the linearity and surjectivity assumptions can be removed in Larson and Sourour's results if we modify the definition of local automorphisms and local derivations as above. Before stating our results we fix some notation. Let H be a Hilbert space. For any $x, y \in H$ we denote the inner product of these two vectors by y^*x, while xy^* denotes the rank one operator given by $(xy^*)z = (y^*z)x$.

Theorem 1. *Let H be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(H)$ be the algebra of all linear bounded operators on H. Then every 2-local automorphism $\theta : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ (no linearity, surjectivity or continuity of θ is assumed) is an automorphism.*

Theorem 2. *Let H be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(H)$ be the algebra of all linear bounded operators on H. Then every 2-local derivation $\theta : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ (no linearity or continuity of θ is assumed) is a derivation.*

Remark. The same results hold also in the case that H is finite-dimensional. In this case, however, we were only able to get a long proof involving tedious computations, and so, we will not include these results. It would be interesting to get short proofs for finite-dimensional analogues of the above theorems.

Proof of Theorem 1. Let us fix an orthonormal basis $\{e_n : n = 1, 2, \dots\}$ in H. We define two operators $A, N \in \mathcal{B}(H)$ by

$$A = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n e_n^*$$

and

$$Ne_1 = 0 \quad \text{and} \quad Ne_n = e_{n-1}, \quad n = 2, 3, \dots .$$

It is easy to see that $T \in \mathcal{B}(H)$ commutes with A if and only if it is diagonal with respect to the basis $\{e_1, e_2, \dots\}$. More precisely, for any bounded sequence of complex numbers $\Lambda = (\lambda_n)$ we define $D(\Lambda) \in \mathcal{B}(H)$ by $D(\Lambda)e_n = \lambda_n e_n$, $n \in \mathbb{N}$. Then T commutes with A if and only if it is of the form $T = D(\Lambda)$ for some bounded sequence Λ. We claim that $T \in \mathcal{B}(H)$ commutes with N if and only if there exists a sequence of complex numbers $\Gamma = (\gamma_n)$ such that $T = U(\Gamma)$, where $U(\Gamma)$ is defined by

$$U(\Gamma)e_n = \gamma_n e_1 + \gamma_{n-1} e_2 + \dots + \gamma_1 e_n, \quad n = 1, 2, \dots . \tag{1}$$

If Γ is a complex sequence such that (1) defines a bounded linear operator on H, then it is easy to verify that $U(\Gamma)$ commutes with N. To prove the converse assume that T commutes with N. Then we have $0 = TNe_1 = NTe_1$, and therefore, there exists a complex number γ_1 such that $Te_1 = \gamma_1 e_1$. Applying an induction argument one can find a sequence $\Gamma = (\gamma_n)$ such that $T = U(\Gamma)$.

Every automorphism of $\mathcal{B}(H)$ is inner [5]. As θ is a 2-local automorphism we can find an invertible $S \in \mathcal{B}(H)$ such that $\theta(A) = SAS^{-1}$ and $\theta(N) = SNS^{-1}$. Replacing θ by the mapping $T \mapsto S^{-1}\theta(T)S$, if necessary, we can assume with no loss of generality that $\theta(A) = A$ and $\theta(N) = N$.

Let us denote by P_n the Hermitian projection on the linear span of the set $\{e_1, \dots, e_n\}$, $n \in \mathbb{N}$. Let $T \in \mathcal{B}(H)$ be any operator satisfying $P_nTP_n = T$ and $\mathrm{rank}(T) = n$. According to our assumption there exists an invertible operator $R \in \mathcal{B}(H)$ such that $\theta(A) = RAR^{-1}$ and $\theta(T) = RTR^{-1}$. But $\theta(A) = A$, and so, R commutes with A. Consequently, we have $R = D(\Lambda)$ for some complex sequence Λ, which further yields

$$P_n\theta(T)P_n = \theta(T). \tag{2}$$

If we consider operators N and T we get in a similar way that $\theta(T) = U(\Gamma)TU(\Gamma)^{-1}$ for some complex sequence $\Gamma = (\gamma_n)$. The operator $U(\Gamma)^{-1}$ also commutes with N, and therefore, it must be of the form $U(\Gamma)^{-1} = U(\Delta)$ for some complex sequence $\Delta = (\delta_n)$. Applying (2) we obtain

$$P_nU(\Gamma)TU(\Delta)(I - P_n) = P_nU(\Gamma)P_nTP_nU(\Delta)(I - P_n) = 0.$$

But obviously, $\mathrm{rank}(P_nU(\Gamma)P_nT) = n$, and so, we have $P_nU(\Delta)(I - P_n) = 0$. This yields $\delta_k = 0$ for all $k > 1$. Therefore we have $U(\Gamma) = \gamma_1 I$. As a consequence, we get

$$\theta(T) = T \tag{3}$$

for every T satisfying $P_nTP_n = T$ and $\mathrm{rank}(T) = n$.

In the next step we will prove that $\theta(P) = P$ for every idempotent P of rank one satisfying $P_nPP_n = P$ for some positive integer n. Such an operator P is of the form $P = xy^*$ with $P_nx = x$, $P_ny = y$, and $y^*x = 1$. Clearly, $\theta(P)$ is an idempotent of rank one, and so, there exist $u, v \in H$ such that $\theta(P) = uv^*$ with $v^*u = 1$. We can find an operator $B \in \mathcal{B}(H)$ such that $P_nBP_n = B$, $\mathrm{rank}(B) = n$, $Bx = x$, and $\dim \mathcal{N}(B - I) = 1$. Here, $\mathcal{N}(B - I)$ denotes the null space of $B - I$. Since $BP = P$, we have

$$\theta(P) = \theta_{B,P}(P) = \theta_{B,P}(BP) = \theta_{B,P}(B)\theta_{B,P}(P) = B\theta(P),$$

or equivalently, $Bu = u$. This further yields $u = \lambda x$ for some complex number λ. Similarly, we get $v = \mu y$ for some $\mu \in \mathbb{C}$. It follows from $y^*x = v^*u$ that $\theta(P) = P$.

Now, let T be any operator in $\mathcal{B}(H)$. We choose any two vectors x, y from the linear span of the set $\{e_1, \dots, e_n\}$ satisfying $y^*x = 1$. We have already proved that $\theta(xy^*) = xy^*$. As a consequence we get

$$(y^*\theta(T)x)xy^* = xy^*\theta(T)xy^* =$$

$$\theta_{T,xy^*}(xy^*Txy^*) = (y^*Tx)\theta(xy^*).$$

It follows that $P_n\theta(T)P_n = P_nTP_n$ for every positive integer n. Hence, $\theta(T) = T$ for every $T \in \mathcal{B}(H)$. This completes the proof. □

Proof of Theorem 2. The proof of Theorem 1 was based on an observation that an automorphism of $\mathcal{B}(H)$, which maps the operators A and N into themselves, must be the identical automorphism. The proof of Theorem 2 uses a similar idea, so we will give here just its outline.

Replacing θ by $\theta - \theta_{A,N}$, if necessary, we can assume that $\theta(A) = \theta(N) = 0$. Every derivation on $\mathcal{B}(H)$ is inner. It follows that for every $T \in \mathcal{B}(H)$ there exist complex sequences Λ and Γ such that $\theta(T) = TD(\Lambda) - D(\Lambda)T = TU(\Gamma) - U(\Gamma)T$. A similar approach as above gives us that $\theta(T) = 0$ for every T satisfying $P_nTP_n = T$ and $\operatorname{rank}(T) = n$, and consequently, $\theta(P) = 0$ for every idempotent of rank one such that $P_nPP_n = P = xy^*$. This further yields that

$$P\theta(T)P = \theta_{P,T}(P)TP + P\theta_{P,T}(T)P + PT\theta_{P,T}(P) =$$

$$\theta_{P,T}(PTP) = (y^*Tx)\theta_{P,T}(P) = 0$$

for every $T \in \mathcal{B}(H)$. It is now easy to complete the proof. □

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