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TOPOLOGIES ON THE IDEAL SPACE OF A BANACH ALGEBRA AND SPECTRAL SYNTHESIS

FERDINAND BECKHOFF

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ABSTRACT. Let the space $\mathrm{Id}(A)$ of closed two-sided ideals of a Banach algebra A carry the weak topology. We consider the following property called normality (of the family of finite subsets of A): if the net $(I_i)_i$ in $\mathrm{Id}(A)$ converges weakly to I, then for all $a \in A\backslash I$ we have $\liminf_i \|a + I_i\| > 0$ (e.g. C^*-algebras, $L^1(G)$ with compact $G, \dots$). For a commutative Banach algebra normality is implied by spectral synthesis of all closed subsets of the Gelfand space $\Delta(A)$, the converse does not always hold, but it does under the following additional geometrical assumption: $\inf\{\|\varphi_1 - \varphi_2\|; \varphi_1, \varphi_2 \in \Delta(A), \varphi_1 \neq \varphi_2\} > 0$.

1. Introduction

Let A be a Banach algebra. By $\mathrm{Id}(A)$ we denote the space of all closed two-sided ideals of A. A compact family is by definition a family $\mathcal{K}$ of compact subsets of A which is stable with respect to finite unions. Typical examples are

- the family $\mathcal{F}$ of all finite sets in A,
- the family $\mathcal{K}_s$ of all finite unions of norm convergent sequences together with their limits,
- the family of all compact subsets of A.

If $\mathcal{K}$ is a compact family, then the sets

$$U(K) := \{I \in \mathrm{Id}(A); I \cap K = \emptyset\}, \qquad K \in \mathcal{K},$$

together with $\mathrm{Id}(A)$ (this may be regarded as $U(\emptyset)$) are a base of a topology $\tau(\mathcal{K})$ on $\mathrm{Id}(A)$.

These topologies have been considered in [3]. $\tau(\mathcal{F})$ is well known as the hull-kernel or weak topology and is denoted by τ_w in [1] in the case of C^*-algebras or in [12] in the general Banach algebra case.

We say that a family is normal in $I \in \mathrm{Id}(A)$ iff the following holds:

If the net $(I_i)_i$ is $\tau(\mathcal{K})$-convergent to I, then for all $a \in A\backslash I$ we have $\liminf_i \|a + I_i\| > 0$.

$\mathcal{K}$ is normal iff $\mathcal{K}$ is normal in all ideals.

It is well known that $\mathcal{F}$ is normal in the case of a C^*-algebra, and other examples can be found in [3]. It turns out that normality is a strong property and that some results about C^*-algebras may be extended to the more abstract setting of a Banach algebra where $\mathcal{F}$ is normal.

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If A is a separable Banach algebra and $\mathcal{K}$ is a normal compact family, then $(\mathrm{Id}(A), \mathcal{K})$ is a second countable space by [3], Th. 5, and the converse holds if $\mathcal{K} \supset \mathcal{K}_s$ by [3], Th. 6. This converse does not hold for the family $\mathcal{K} = \mathcal{F}$, see the section "Examples" for this.

The main purpose of the present paper is to investigate the normality condition for $\mathcal{F}$ in the case of a *commutative* Banach algebra, and it turns out that it is closely related to spectral synthesis. If each closed subset of the Gelfand space $\Delta(A)$ admits spectral synthesis, then $\mathcal{F}$ is normal. The converse does not hold in general, but it does under an additional geometrical property of the Gelfand space, namely if

$$\inf\{\|\varphi_1 - \varphi_2\|; \varphi_1, \varphi_2 \in \Delta(A), \varphi_1 \neq \varphi_2\} > 0.$$

This distance property will be investigated in the following section. The main results about the relations between normality of $\mathcal{F}$ and spectral synthesis are treated in the third section. The last section presents some examples.

2. Distance properties of the Gelfand space

We will be in need of the following technical property $D(n)$ of a commutative Banach algebra A with Gelfand space $\Delta(A)$, where n is a positive integer:

$D(n)$: There is a constant $\alpha > 0$ such that whenever $\varphi_1, \dots, \varphi_n \in \Delta(A)$ and $\varphi_1 \notin \{\varphi_2, \dots, \varphi_n\}$, then

$$\mathrm{dist}(\varphi_1, \mathrm{span}\{\varphi_2, \dots, \varphi_n\}) > \alpha.$$

If $\varphi_1 \notin \{\varphi_2, \dots, \varphi_n\}$ it is clear that φ_1 is linearly independent of $\{\varphi_2, \dots, \varphi_n\}$, hence the norm distance of φ_1 to $\mathrm{span}\{\varphi_2, \dots, \varphi_n\}$ is greater than zero. The above condition requires that this does not depend on the characters chosen but only on n. With the usual convention that $\mathrm{span}(\emptyset) = \{0\}$ the condition $D(1)$ says nothing but $\inf\{\|\varphi\|; \varphi \in \Delta(A)\} > 0$.

In this section we will characterize this technical property by simpler properties that seem to be more natural and find some examples.

Proposition 1. *For a commutative Banach algebra A the following conditions are equivalent:*

(i) *There is a constant $K > 0$ such that whenever φ_1 and φ_2 are different characters there is an element $a \in A$ with $\|a\| \leq K$, $\hat{a}(\varphi_1) = 0$ and $\hat{a}(\varphi_2) = 1$.*
(ii) $\inf\{\|\varphi_1 - \varphi_2\|;\ \varphi_1,\ \varphi_2 \in \Delta(A),\ \varphi_1 \neq \varphi_2\} > 0$.
(iii) *A satisfies $D(1)$ and $D(2)$.*
(iv) *A satisfies $D(n)$ for all $n \in \mathbf{N}$.*

Proof. (i) $\Rightarrow$ (iv): We may assume that $\Delta(A)$ contains at least two elements since otherwise there is nothing to prove. If $\varphi \in \Delta(A)$, then there is another character different from φ and (i) implies that there is an $a \in A$ with $\|a\| \leq K$ such that $\varphi(a) = 1$. This means $\|\varphi\| \geq \frac{1}{K}$, hence $D(1)$ holds.

Now let $n \geq 2$ and let $\varphi_1, \dots, \varphi_n \in \Delta(A)$ such that $\varphi_1 \notin \{\varphi_2, \dots, \varphi_n\}$. Then for each $j = 2, \dots, n$ there is an element $a_j \in A$ with $\|a_j\| \leq K$ such that $\varphi_j(a_j) = 0$ and $\varphi_1(a_j) = 1$. Let $a := a_2 \cdot \ldots \cdot a_n$. Then $\|a\| \leq K^{n-1}$ and for $\lambda_2, \dots, \lambda_n \in \mathbf{C}$

we have

$$\begin{aligned}
&\left|\varphi_1(a) - \sum_{j=2}^{n} \lambda_j \varphi_j(a)\right| \\
&= \left|\underbrace{\varphi_1(a_2)\cdot \ldots \cdot \varphi_1(a_n)}_{=1} - \sum_{j=2}^{n} \lambda_j \underbrace{\varphi_j(a_2)\cdot \ldots \cdot \varphi_j(a_n)}_{=0}\right| = 1 \\
&\Rightarrow \left\|\varphi_1 - \sum_{j=2}^{n} \lambda_j \varphi_j\right\| \geq \frac{1}{K^{n-1}}.
\end{aligned}$$

This proves

$$\operatorname{dist}(\varphi_1, \operatorname{span}\{\varphi_2, \ldots, \varphi_n\}) \geq \frac{1}{K^{n-1}}.$$

(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Let A_1 be the Banach algebra obtained from A by adjunction of a unit. For $\varphi \in \Delta(A)$ let $\tilde{\varphi}\colon A_1 \to \mathbf{C}$, $a + \lambda 1 \mapsto \varphi(a) + \lambda$ and let $\omega\colon A_1 \to \mathbf{C}$, $a + \lambda 1 \mapsto \lambda$. Then it is well known that $\Delta(A_1) = \{\tilde{\varphi}; \varphi \in \Delta(A)\} \cup \{\omega\}$.

(ii) obviously implies $D(1)$, i.e. $\alpha := \inf\{\|\varphi\|;\ \varphi \in \Delta(A)\} > 0$. Let β be the infimum in (ii). If ψ_1, $\psi_2 \in \Delta(A_1)$ we see by the above description of $\Delta(A_1)$ that

$$\|\psi_1 - \psi_2\| \geq \|\psi_1|_A - \psi_2|_A\| \geq \min\{\alpha, \beta\} =: \gamma.$$

So there is an $x \in A_1$ such that $\|x\| \leq 1$, $|\psi_1(x) - \psi_2(x)| > \frac{\gamma}{2}$. Put $y := x - \psi_2(x)1$. Then $\|y\| \leq 2$, $\psi_2(y) = 0$ and $|\psi_1(y)| > \frac{\gamma}{2}$. So if $z := \frac{1}{\psi_1(y)} y$, then $\|z\| \leq \frac{4}{\gamma}$, $\psi_2(z) = 0$ and $\psi_1(z) = 1$.

Therefore A_1 satisfies the property in (i) with the constant $\frac{4}{\gamma}$. We now show that A has this property, too. Let φ_1, $\varphi_2 \in \Delta(A)$ be different characters. Then ω, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are pairwise different characters of A_1, and by what we proved above we can find z_1, $z_2 \in A_1$ such that $\|z_1\|$, $\|z_2\| \leq \frac{4}{\gamma}$ and $\omega(z_1) = 0$, $\tilde{\varphi}_1(z_1) = 1$, $\tilde{\varphi}_1(z_2) = 1$, $\tilde{\varphi}_2(z_2) = 0$. Then $a := z_1 z_2 \in \ker(\omega) = A$ and $\|a\| \leq (\frac{4}{\gamma})^2$, $\varphi_1(a) = 1$, $\varphi_2(a) = 0$. This finishes the proof. □

Definition 2. We say that A satisfies the distance property iff it satisfies the properties of the above proposition.

Examples. Each Banach algebra $\mathcal{C}_0(X)$, X a locally compact Hausdorff space, has the distance property, indeed, the distance between any two point evaluations equals 2.

Let G be a locally compact abelian group. If γ_1, γ_2 are different characters, then by [10], Th. 2.6.1, there is an element $f \in L^1(G)$ such that $\hat{f}(\gamma_1) = 1$, $\hat{f}(\gamma_2) = 0$ and $\|f\|_1 \leq 1$. So $L^1(G)$ satisfies the distance property.

The Lipschitz algebras $\operatorname{Lip}(X, d)$, (X, d) a compact metric space, do not have the distance property since for $s \neq t$ in X and $f \in \operatorname{Lip}(X, d)$

$$|f(s) - f(t)| = \frac{|f(s) - f(t)|}{d(s,t)} d(s,t) \leq \|f\| \cdot d(s,t).$$

The same consideration applies to the Banach algebras $\operatorname{lip}(X, d)$ and $\mathcal{C}^1[0, 1]$.

3. Spectral synthesis

We now relate the normality of the family $\mathcal{F}$ of finite subsets of a commutative Banach algebra A to the spectral synthesis of the closed sets of its Gelfand space $\Delta(A)$. We recall that a closed subset $E \subset \Delta(A)$ satisfies spectral synthesis iff there is exactly one closed ideal $I \in \mathrm{Id}(A)$ such that E is the hull of I. See [7], Ch. X, §39, for this concept. We start with the following easy proposition.

Proposition 3. *Let A be a commutative Banach algebra such that each closed set of the Gelfand space has spectral synthesis. Then $\mathcal{F}$ is normal.*

Proof. Each ideal $I \in \mathrm{Id}(A)$ has the form $I = \bigcap E$ where $E \subset \Delta(A)$ is the hull of I. Let $\bigcap E_i \to \bigcap E$ with respect to $\tau(\mathcal{F})$ and $x \in A \setminus \bigcap E$. We must show that $\liminf_i \|x + \bigcap E_i\| > 0$.

Since $x \notin \bigcap E$ there is a $\varphi \in E$ such that $\varphi(x) \neq 0$. Let

$$U := \left\{ \psi \in \Delta(A); |\psi(x)| > \frac{1}{2}|\varphi(x)| \right\},$$

which is an open neighbourhood of φ. Because $\varphi \in E \cap U$ we have $E \not\subset U^c$, hence $\bigcap E \not\supset \bigcap U^c$. By the assumed $\tau(\mathcal{F})$-convergence we get $\bigcap E_i \not\supset \bigcap U^c$ for large i, hence $E_i \cap U \neq \emptyset$ for large i. Let $\psi_i \in E_i \cap U$. Then

$$\left\| x + \bigcap E_i \right\| \geq \|x + \ker(\psi_i)\| \geq |\psi_i(x)| \geq \frac{1}{2}|\varphi(x)|$$

for large i, and this finishes the proof. □

There are many examples of commutative Banach algebras having spectral synthesis for all closed sets of the Gelfand space:

- commutative C^*-algebras, see [14], Th. 85, for the unital case, [6], Th. 2.9.7 for the non-unital and even non-commutative case,
- $L^1(G)$, G a compact group, see [10], Th. 7.1.5,
- $\mathrm{lip}(X, d^\alpha)$, (X, d) a compact metric space, see [11], Cor. 4.3,
- the small Zygmund algebra λ, see [9], Cor. of Lemma 1,
- and others, see [4]; [5]; [7], Th. (38.7); [8].

The converse does not hold as we will see later, but it does under additional assumptions. For a closed subset $E \subset \Delta(A)$ the following ideals are important in the theory of spectral synthesis:

$$\begin{aligned} K(E) &:= \{a \in A; \hat{a} \text{ vanishes on } E\}, \\ J_0(E) &:= \left\{ a \in A; \begin{array}{l} \hat{a} \text{ vanishes in a neighbourhood of } E \\ \text{and } \hat{a} \text{ has compact support} \end{array} \right\}, \\ J(E) &:= \overline{J_0(E)}, \text{ the closure of } J_0(E). \end{aligned}$$

If A is semisimple and regular, then a closed subset $E \subset \Delta(A)$ has spectral synthesis iff $J(E) = K(E)$. So in order to prove a converse of the above proposition we want A to be regular:

Proposition 4. *Let A be a commutative Banach algebra such that $\mathcal{F}$ is normal and $\inf\{\|\varphi\|; \varphi \in \Delta(A)\} > 0$. Then A is regular.*

Proof. Let $E \subset \Delta(A)$ be closed and $\varphi \in \Delta(A)\backslash E$. We have to show that there is an $a \in A$ such that $\hat{a}|_E = 0$ and $\hat{a}(\varphi) \neq 0$. Assume that this were not the case. Then

$$\hat{a}|_E \neq 0 \quad \text{for all } a \in A\backslash \ker(\varphi). \tag{1}$$

Let $\mathcal{F}_0 := \{F \in \mathcal{F}; F \subset A\backslash \ker(\varphi)\}$. Then for all $F = \{a_1, \dots, a_n\}$ in $\mathcal{F}_0$ the product $a_1 \cdot \dots \cdot a_n$ is not in $\ker(\varphi)$ and so by (1) there is $\varphi_F \in E$ such that $(\hat{a}_1 \cdots \hat{a}_n)(\varphi_F) \neq 0$ and this means $\ker(\varphi_F) \in U(F)$. By construction we have a net $(\varphi_F)_{F\in\mathcal{F}_0}$ which $\tau(\mathcal{F})$-converges to $\ker(\varphi)$. Because $\Delta(A) \cup \{0\}$ is w^*-compact there is a subnet $(\varphi_{F_i})_i$ such that $\varphi_{F_i} \to \psi$ in $\Delta(A)\cup\{0\}$. Since E is closed we have $\psi \in E \cup \{0\}$, in particular $\varphi \neq \psi$. Then we can find an $a \in A$ such that $\varphi(a) \neq 0$ and $\psi(a) = 0$.

Let $c := \frac{1}{2}\inf\{\|\varphi\|; \varphi \in \Delta(A)\} > 0$. For each i there is a b_i in A such that $\|b_i\| \leq \frac{1}{c}$ and $\varphi_{F_i}(b_i) = 1$. Then we have

$$\|a + \ker(\varphi_{F_i})\| \leq \|a - (a - \varphi_{F_i}(a)b_i)\| \leq \frac{1}{c}|\varphi_{F_i}(a)| \to \frac{1}{c}\psi(a) = 0.$$

Now the normality of $\mathcal{F}$ yields the contradiction $a \in \ker(\varphi)$. □

Proposition 5. *Let A satisfy the distance property and let $\mathcal{F}$ be normal in $J(E)$, $\emptyset \neq E \subset \Delta(A)$ closed. Then $J(E) = K(E)$.*

Proof. Let $\alpha(n)$ be the constant in the distance property $D(n)$; we clearly may assume $\alpha(1) \geq \alpha(2) \geq \dots$. There is nothing to prove if $A = J(E)$. Let $\mathcal{F}_0$ be the set of non-empty, finite subsets of $A\backslash J(E)$. Let $\mathcal{F}_1$ be the set of non-empty, finite subsets of $K(E)$. On $\mathcal{F}_0 \times \mathcal{F}_1 \times \mathbf{R}^+$ we define an order by

$$(F_0, F_1, \varepsilon) \leq (F_0', F_1', \varepsilon') :\Leftrightarrow F_0 \subset F_0', F_1 \subset F_1', \varepsilon \geq \varepsilon'.$$

This makes $\mathcal{F}_0 \times \mathcal{F}_1 \times \mathbf{R}^+$ a directed set and we will use it to construct a net which $\tau(\mathcal{F})$-converges to $J(E)$. To this end let (F_0, F_1, ε) be given and define

$$U(F_0, F_1, \varepsilon) := \left\{\varphi \in \Delta(A); |\hat{y}(\varphi)| < \frac{\varepsilon\alpha(|F_0|)}{2|F_0|} \text{ for all } y \in F_1\right\}.$$

$|F_0|$ denotes the number of elements in the set F_0. If $y \in F_1$, then $\hat{y}$ vanishes on E. Therefore $U(F_0, F_1, \varepsilon)$ is an open neighbourhood of E and its complement is compact because F_1 is finite and Gelfand transforms vanish at infinity. If $x \in F_0$, then $x \notin J(E)$ and so there must be a character $\varphi_{x,F_0,F_1,\varepsilon} \in U(F_0, F_1, \varepsilon)$ such that $\varphi_{x,F_0,F_1,\varepsilon}(x) \neq 0$. (This is the place where we need $E \neq \emptyset$ since this implies $U(F_0, F_1, \varepsilon) \neq \emptyset$.) Let

$$I_{(F_0,F_1,\varepsilon)} := \bigcap_{x\in F_0} \ker(\varphi_{x,F_0,F_1,\varepsilon}).$$

This finishes the construction of the net.

Let us show that the above net $\tau(\mathcal{F})$-converges to $J(E)$. If $x \in A\backslash J(E)$, then for $(F_0, F_1, \varepsilon) \geq (\{x\}, \{0\}, 1)$ we have $x \in F_0$, hence $\varphi_{x,F_0,F_1,\varepsilon}(x) \neq 0$, and this yields $x \notin I_{(F_0,F_1,\varepsilon)}$. But this is nothing but the stated $\tau(\mathcal{F})$-convergence.

The assertion of the proposition is $K(E) \subset J(E)$. Let $y \in K(E)$. As $\mathcal{F}$ is normal at $J(E)$ it is sufficient to prove that $\liminf_{F_0,F_1,\varepsilon} \|y + I_{(F_0,F_1,\varepsilon)}\| = 0$. To this end let $\varepsilon > 0$ be given. Let x_1 be any element from $A\backslash J(E)$ and consider $(F_0, F_1, \eta) \geq (\{x_1\}, \{y\}, \varepsilon)$. Then we have $F_0 = \{x_1, \dots, x_n\}$. Let us abbreviate $\varphi_j := \varphi_{x_j,F_0,F_1,\eta} \in U(F_0, F_1, \eta)$. After renumbering we may assume that

$\varphi_1, \ldots, \varphi_m$ are the different elements of $\{\varphi_1, \ldots, \varphi_{|F_0|}\}$. By the property $D(m)$ we conclude

$$\operatorname{dist}(\varphi_j, \operatorname{span}\{\varphi_i; i \neq j\}) \geq \alpha(m), \qquad j = 1, \ldots, m.$$

The Hahn-Banach theorem provides us with functionals $\Psi_j \in A''$ such that $\|\Psi_j\| = 1$, $\Psi_j(\operatorname{span}\{\varphi_i; i \neq j\}) = 0$ and $\Psi_j(\varphi_j) = \operatorname{dist}(\ldots) \geq \alpha(m)$.

Let $H = \operatorname{span}(\Psi_1, \ldots, \Psi_m)$, $G = \operatorname{span}(\varphi_1, \ldots, \varphi_m)$. By the principle of local reflexivity [13] there is a linear isomorphism $T\colon H \to A_0 \subset A$ such that $\|T\|$, $\|T^{-1}\| \leq 2$ and $\Psi(\varphi) = \varphi(T\Psi)$ for all $\Psi \in H$, $\varphi \in G$.

Define $y_j = T\Psi_j$. Then $\|y_j\| \leq 2$, $\varphi_i(y_i) = 0$ for all $i \neq j$ and $\varphi_j(y_j) \geq \alpha(m)$. So if $x_j := \frac{1}{\varphi_j(y_j)} y_j$, then $\|x_j\| \leq \frac{2}{\alpha(m)}$ and $\varphi_i(x_j) = \delta_{i,j}$. Consider the projection

$$P\colon A \to \operatorname{span}\{x_1, \ldots, x_m\}, \qquad x \mapsto \sum_{i=1}^{m} \varphi_i(x) x_i.$$

Since the $x_1, \ldots, x_m$ are linearly independent elements, we have $\ker(P) = \bigcap_{i=1}^{m} \ker(\varphi_i) = I_{(F_0,F_1,\eta)}$. So for the given element y which is in F_1, we have

$$y - \sum_{i=1}^{m} \varphi_i(y) x_i = y - Py \in I_{(F_0,F_1,\eta)}, \ \Rightarrow$$

$$\begin{aligned}\|y + I_{(F_0,F_1,\eta)}\| &\leq \|Py\| \leq \max_{i=1,\ldots,m} |\varphi_i(y)| \cdot \sum_{i=1}^{m} \|x_i\| \\ &\leq \max_{i=1,\ldots,m} |\hat{y}(\varphi_i)| \cdot m \cdot \frac{2}{\alpha(m)} \leq \frac{\eta\alpha(|F_0|)}{2|F_0|} \cdot m \cdot \frac{2}{\alpha(m)} \leq \varepsilon.\end{aligned}$$

This finishes the proof. □

Theorem 6. *Let A be a commutative, semisimple Banach algebra satisfying the distance property such that $\tau(\mathcal{F})$ is normal. Then each closed subset of the Gelfand space has the property of spectral synthesis.*

Proof. The semisimplicity is equivalent to spectral synthesis of $\Delta(A)$. So let $E \subset \Delta(A)$ be a proper, non-empty subset. Because A is regular (by Prop. 4) and semisimple each ideal $I \in \operatorname{Id}(A)$ with hull E lies between $J(E)$ and $K(E)$, and so it is unique by the above Proposition 5 and the other assumptions of the theorem. The case $E = \emptyset$ causes some inconveniences:

We have to show that $\{a \in A; \operatorname{supp}(\hat{a}) \subset \Delta(A) \text{ is compact}\}$ is dense in A (i.e. that A is Tauberian). Let J be the closure of this ideal and let us assume that $J \neq A$. Then $\Delta(A)$ is not compact. Define

$$\begin{aligned}\mathcal{F}_0 :=& \{\text{non-empty finite subset of } A \backslash J\},\\ \mathcal{F}_1 :=& \{\text{non-empty finite subsets of } A\}.\end{aligned}$$

Consider the set $\mathcal{F}_0 \times \mathcal{F}_1 \times \mathbf{R}^+$ with the order as defined in the proof of Proposition 5 and define $U(F_0, F_1, \varepsilon)$ as in that proof. Since $\Delta(A) \backslash U(F_0, F_1, \varepsilon)$ is compact it cannot contain the support of any element in F_0. So for each $x \in F_0$ there must be a character $\varphi_{x,F_0,F_1,\varepsilon} \in U(F_0, F_1, \varepsilon)$ which does not vanish in x. Again define

$$I_{(F_0,F_1,\varepsilon)} := \bigcap_{x \in F_0} \ker(\varphi_{x,F_0,F_1,\varepsilon}).$$

Repeat the corresponding part of the proof of Proposition 5 to see that $I_{(F_0,F_1,\varepsilon)} \to J$ and $\|y + I_{(F_0,F_1,\varepsilon)}\| \to 0$ for all $y \in A$. The normality then implies $J = A$ which is the desired contradiction. □

4. Examples

Example 1. The algebra

$$A := \{f \in \mathcal{C}[0,1]; f \text{ is differentiable in } 0\}$$

is a Banach algebra under the norm

$$\|f\| := \sup_{t\in[0,1]} |f(t)| + \sup_{t\in(0,1]} \left|\frac{f(t)-f(0)}{t}\right|.$$

The following facts can be shown, the proofs are not always easy:

- The Gelfand space is homeomorphic to the unit interval identifying a point with the corresponding point evaluation.
- Let $M := \{f \in A; f(0) = f'(0) = 0\}$. The closed ideals of A have the form $K(E)$ or $K(E) \cap M$ for a unique closed subset $E \subset [0,1]$. A closed subset $E \subset [0,1]$ has spectral synthesis iff 0 is not an isolated point of E (i.e. $0 \notin E$ or 0 is an accumulation point of $E\backslash\{0\}$).
- The space $(\mathrm{Id}(A), \tau(\mathcal{F}))$ is second countable.
- $\mathcal{F}$ is not normal. Even more is true: the family of all compact sets is not normal.

This example shows that second countability of $\mathcal{F}$ in general does not imply normality of $\mathcal{F}$.

Example 2. Consider the commutative C^*-algebra $A = \mathcal{C}[0,1]$ with the supremum norm. Let $z \in A$ be the identity map, i.e. $z(t) = t$, $t \in [0,1]$. Define a new multiplication $\diamond$ on A by $f \diamond g := fzg$, and denote the resulting Banach algebra by B.

Then it is easy to see that $\mathrm{Id}(A) = \mathrm{Id}(B)$ and that the topologies $\tau(\mathcal{K})$ mean the same thing on $\mathrm{Id}(A)$ and on $\mathrm{Id}(B)$. Since A is a C^*-algebra it follows that $\mathcal{F}$ is normal (for A and for B). B does not have spectral synthesis for the following reason:

The Gelfand space is easily seen to be $\Delta(B) = \{t\cdot\omega_t; t \in (0,1]\}$ where ω_t denotes the point evaluation in t. $\ker(\omega_0)$ is a maximal ideal in B, but it is not regular, hence the hull of $\ker(\omega_0)$ is empty. Therefore the empty set $\emptyset \subset \Delta(B)$ does not have spectral synthesis. This example shows that the converse of Proposition 3 does not hold in general.

Example 3. As another example let us consider group algebras.

Proposition 7. *Let G be a locally compact, non-compact, abelian group. Then the Banach algebra $L^1(G)$ has at least one of the following two properties:*

(i) $\tau(\mathcal{F}) \neq \tau(\mathcal{K}_s)$.
(ii) *$\mathcal{F}$ is not first countable.*

Proof. Assume that both properties do not hold. Then $\tau(\mathcal{F}) = \tau(\mathcal{K}_s)$ is first countable, hence we have normality by [3], Th. 6. Since $L^1(G)$ satisfies the distance property we can conclude by Theorem 6 that $L^1(G)$ admits spectral synthesis. But this is not the case by Malliavin's theorem ([7], Th. 42.19, or [10], Th. 7.6.1). □

Example 4. Let (X, d) be a complete metric space. It is well-known that $\mathrm{lip}(X, d^\alpha)$ is a regular semisimple commutative Banach algebra satisfying spectral synthesis for all closed sets. In particular $\tau(\mathcal{F})$ is normal by Proposition 3. In general Theorem 6 is not applicable in this situation because $\mathrm{lip}(X, d^\alpha)$ lacks the distance property. If (X, d) is the unit interval with the Euclidean distance and if δ_s denotes the point evaluation in $s \in [0,1]$, then we have for each $f \in \mathrm{lip}([0,1], |\cdot|^\alpha)$

$$|\delta_s(f) - \delta_t(f)| = \frac{|f(s) - f(t)|}{|s-t|^\alpha}|s-t|^\alpha \leq \|f\||s-t|^\alpha,$$

i.e. $\|\delta_s - \delta_t\| \leq |s-t|^\alpha$, hence the distance property is not satisfied.

Let us finish with some questions:

- The last proposition prompts us to ask which of the two properties hold, maybe both of them?
- If $\mathcal{K}$ is a compact family containing $\mathcal{K}_s$, then first countability and second countability of $(\mathrm{Id}(A), \tau(\mathcal{K}))$ are equivalent by [3], Th. 6. Does this equivalence also hold for $\mathcal{F}$?
- Is it possible to weaken the distance property to some property (P) such that spectral synthesis for all closed sets of the Gelfand space is equivalent to normality of $\mathcal{F}$ and (P)?
- How can the results of Section 3 be generalized to the non-commutative case?

References

[1] R. J. Archbold, *Topologies for primal ideals*, J. London Math. Soc. (2) 36 (1987) 524–542. MR **89h:**46076

[2] F. Beckhoff, *Topologies on the space of ideals of a Banach algebra*, Studia Mathematica 115 (2) (1995), 189–205. CMP 95:17

[3] F. Beckhoff, *Topologies of compact families on the ideal space of a Banach algebra*, Studia Mathematica 118 (1) (1996), 63–75. MR **96m:**46087

[4] A. Beurling, *Construction and analysis of some convolution algebras*, Ann. Inst. Fourier, Grenoble 14 (1964), 1–32. MR **32:**321

[5] T. Ceauşu, D. Gaşpar, *Generalized Lipschitz spaces as Banach algebras with spectral synthesis*, Analele Universităţii din Timişoara XXX (1992), 173–182, Seria Ştiinţe Matematice. MR **96b:**46070

[6] J. Dixmier, *C^*-algebras*, North-Holland-Publishing Company 1977. MR **56:**16388

[7] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis* II, Springer (1970). MR **41:**7378

[8] L. G. Khanin, *Spectral synthesis of ideals in algebras of functions having generalized derivatives*, English transl. in Russian Math. Surveys 39 (1984), 167–168. MR **85d:**46072

[9] L. G. Khanin, *The Structure of Closed Ideals in Some Algebras of Smooth Functions*, Amer. Math. Soc. Transl. 49 (1991), 97–113. MR **92f:**00031

[10] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers (1962). MR **27:**2808

[11] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), 240–272. MR **28:**4385

[12] D. W. B. Somerset, *Minimal primal ideals in Banach algebras*, Math. Proc. Camb. Philos. Soc. 115 (1994), 39–52. MR **94k:**46090

[13] C. Stegall, *A proof of the principle of local reflexivity*, Proc. Amer. Math. Soc. 78, (1980), 154–156. MR **81e:**46012

[14] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), 375–481.

Mathematisches Institut der Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: `beckhof@math.uni-muenster.de`