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ON THE EXISTENCE AND CONSTRUCTIONS OF ORTHONORMAL WAVELETS ON $L_2(\mathbb{R}^s)$

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Abstract. For a multiresolution analysis of $L_2(\mathbb{R}^s)$ associated with the scaling matrix T having determinant n we prove the existence of a wavelet basis with certain desirable properties if $2n-1>s$ and its real-valued counterpart if the scaling function is real-valued and $n-1>s$. That those results cannot be extended to $2n-1\leq s$ and $n-1\leq s$ respectively in general is demonstrated by Adams's theorem about vector fields on spheres. Moreover we present some new explicit constructions of wavelets, among which is a variation of Riemenschneider-Shen's method for $s\leq 3$.

1. Introduction

Let T be an $s\times s$ invertible matrix with integer entries. Then there exist $n:=|\det T|$ points $\{\delta\}_{k=0}^{n-1}\subseteq\mathbb{Z}^s$, called a full collection of representatives of $\mathbb{Z}^s/T\mathbb{Z}^s$, such that $\delta_0=0$ and the lattices $\delta_k+T\mathbb{Z}^s, k=0,\cdots,n-1$ partition $\mathbb{Z}^s$. In what following we always assume that the spectral of T is larger than one.

Given a function $\varphi\in L_2(\mathbb{R}^s)$ we define V_0 to be the $L_2(\mathbb{R}^s)$-closure of all finite linear combinations of $\{\varphi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^s}$ and $V_k=\{f(T^k\cdot)|f\in V_0\}$ for $k\in\mathbb{Z}$. We say $\{V_k\}$ form a multiresolution analysis (associated with the scaling matrix T) and φ is a scaling function if the following conditions are satisfied:

(i) $\overline{\bigcup_{k\in\mathbb{Z}} V_k}=L_2(\mathbb{R}^s), \quad \bigcap_{k\in\mathbb{Z}} V_k=\{0\}$,

(ii) $V_k\subset V_{k+1}, \quad k\in\mathbb{Z}$,

(iii) $\{\varphi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^s}$ is an orthonormal basis of V_0.

For a multiresolution analysis we denote by W the wavelet space, i.e., the orthogonal complement of V_0 in V_1. A set of $n-1$ (real-valued) functions $\{\psi_j\}_{j=1}^{n-1}$ is called a (real) wavelet set if $\{\psi_j(\cdot-\alpha)\}_{1\leq j\leq n-1,\alpha\in\mathbb{Z}^s}$ is an orthonormal basis for W. If $\{\psi_j\}_{j=1}^{n-1}$ is a wavelet set then $\{n^{\frac{k}{2}}\psi_j(T^k\cdot-\alpha)\}_{1\leq j\leq n-1,\alpha\in\mathbb{Z}^s,k\in\mathbb{Z}}$ forms an orthonormal basis of $L_2(\mathbb{R}^s)$.

There has been an extensive study of the existence and construction of wavelet sets (see [1], [4], [5] and references therein). For later use we describe here a standard method.

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From (ii) and (iii) we have a sequence $(a_\alpha)_{\alpha\in\mathbb{Z}^s} \in l_2(\mathbb{Z}^s)$ such that

$$\varphi(x) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \varphi(Tx-\alpha). \tag{1}$$

Let $A_{0k}(\omega) = n^{-\frac{1}{2}} \sum_{\alpha\in\mathbb{Z}^s} a_{\delta_k+T\alpha} e^{i\alpha\cdot\omega}, \quad k = 0,\cdots,n-1,\ \omega \in \mathbb{T}^s := [-\pi,\pi]^s$. Then (cf. [5])

$$\sum_{k=0}^{n-1} |A_{0k}(\omega)|^2 = 1 \quad \text{a.e.} \tag{2}$$

The vector $A(\omega) := (A_{00}(\omega),\cdots,A_{0n-1}(\omega))$ is called extensible if there exists an $n\times n$ matrix $M(\omega)$ with measurable functions entries and $A(\omega)$ as its first row such that $M(\omega)M^*(\omega) = I$ a.e., where $M^*(\omega)$ is the complex conjugate matrix of $M(\omega)$ and I the identity matrix. In this case $M(\omega)$ is known as a unitary extension of $A(\omega)$. For such $M(\omega) = (A_{jk}(\omega))_{j,k=0}^{n-1}$ we define functions ψ_j by the following equalities:

$$\widehat{\psi}_j(\omega) = n^{\frac{1}{2}} \sum_{k=0}^{n-1} A_{jk}(\omega)\widehat{\varphi}_k(\omega), \quad \omega \in \mathbb{R}^s, \tag{3}$$

$j = 1,\cdots,n-1$, where $\widehat{f}(\omega) = \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\omega}dx$ is the Fourier transform of $f \in L_2(\mathbb{R}^s)$ and $\varphi_k(x) = \varphi(Tx-\delta_k)$ for $k = 0,\cdots,n-1$. Then $\{\psi_j\}_{j=1}^{n-1}$ is a wavelet set. For a real-valued scaling function φ we have $A(\omega) = \overline{A(-\omega)}$ and $\{\psi_j\}_{j=1}^{n-1}$ is a real wavelet set provided $M(\omega) = \overline{M(-\omega)}$.

According to above recipe it is sufficient for the existence of a wavelet set to prove the extensibility of $A(\omega)$. Nevertheless we must in general require that the entries of the unitary extension are continuous in order to have some control of the decay of the wavelets. Under the assumption that $A(\omega)$ is a Hölder mapping Jia and Micchelli proved in [3] the existence of wavelet set when $2n-1 > s$, in which all the entries of associated unitary extension are continuous.

We establish the above mentioned result of [3] in Section 2 in a different case in which $A(\omega)$ is continuous with $[s/2]+1$ components satisfying a Lipschitz condition. Its analogy for real wavelet sets is also true for $n-1 > s$ provided that the scaling function is real-valued. As for the proof we are led to use the delicate Federer's theorem about the change of variables of integral. Our restriction on the number n cannot be loosened in general, which will be demonstrated by Adams's theorem about vector fields on spheres.

In order to construct (explicitly) a wavelet set a unitary extension of $A(\omega)$ should be constructed (explicitly). When the scaling function is skew-symmetric about some point $c_\varphi \in 2^{-1}\mathbb{Z}^s$ some unitary extensions of $A(\omega)$ are given (explicitly) in Section 3, among which is a variation of Riemenschneider-Shen's method [6] in low dimension. Moreover these give real wavelet sets if the scaling function φ is real-valued.

2. Existence of wavelet set

Let $S_{\mathbb{C}}^{n-1}$ and $S_{\mathbb{R}}^{n-1}$ be the unit spheres in $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively. For two $1\times n$ vectors $a, x \in S_{\mathbb{C}}^{n-1}$ with $a \neq x$ let $Q_a(x)$ be the Householder matrix

$$Q_a(x) = I - (a-x)^*(a-x)/(1-ax^*),$$

where a^* denotes, as before, the complex conjugate of a.

The following properties of $Q_a(x)$ may be easily verified.

Lemma 1 ([7]). *If $a \neq x$ then $Q_a(x)$ is unitary with $aQ_a(x) = x$.*

Now we prove the main auxiliary result, which is also of interest for its own right.

Lemma 2. *Let $n-1 > s$ and $F(\omega) = (f_1(\omega), \cdots, f_n(\omega))$ be a $2\pi\mathbb{Z}^s$-periodic continuous mapping from $\mathbb{T}^s$ to $S_{\mathbb{C}}^{n-1}$. Suppose that s components of $F(\omega)$ satisfy the Lipschitz condition. Then there exists a unitary extension $M(\omega)$ of $F(\omega)$ satisfying*

(i) *all entries of $M(\omega)$ are $2\pi\mathbb{Z}^s$-periodic continuous; moreover, they have the same order of differentiality as $F(\omega)$ has.*

(ii) *$M(\omega)$ is real whenever $F(\omega)$ is so.*

(iii) *$M(\omega) = \overline{M(-\omega)}$ whenever $F(\omega) = \overline{F(-\omega)}$.*

Proof. We first prove that $S_{\mathbb{R}}^{n-1}\backslash F(\mathbb{T}^s)$ is not empty, where $F(A) = \{F(\omega)|\omega \in A\}$. Without loss of generality we assume that $F(\mathbb{T}^s) \subseteq S_{\mathbb{R}}^{n-1}$, otherwise we consider $ReF(\omega)$ instead of $F(\omega)$ in what follows. Also we assume that $f_1, \cdots, f_s$ satisfy the Lipschitz condition, i.e., there exists a constant γ such that for $f(\omega) := (f_1(\omega), \cdots, f_s(\omega))$

$$|f(\omega_1) - f(\omega_2)| \leq \gamma|\omega_1 - \omega_2|, \quad \omega_1, \omega_2 \in \mathbb{R}^s, \tag{4}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^s$.

Denote by $E_1 = \{\omega \in \mathbb{T}^s | f \text{ is not differentiable (in common meaning) at } \omega\}$ and $E_2 = \{\omega \in \mathbb{T}^s\backslash E_1 | \det f'(\omega) = 0\}$, respectively, where $f'(\omega)$ is the Jacobian matrix of f at ω. It is well known that $E_1 \subseteq \mathbb{R}^s$ has (Lebesgue) measure zero (cf. [9], pp.48–49) and therefore so does $f(E_1)$ because of (4). To prove that $f(E_2)$ is also a null set in $\mathbb{R}^s$ we appeal to a special case of Federer's theorem (cf. [9], pp. 80–81)

$$\int_A |\det f'(\omega)| d\omega = \int_{\mathbb{R}^s} N(y, A) dy, \tag{5}$$

where $A \subset \mathbb{R}^s$ is a measurable set and $N(y, A)$ is the number of points (including ∞) in set $\{x \in A | f(x) = y\}$. By taking $A = E_2$ in (5) we conclude that $N(y, E_2) = 0$ a.e. $y \in \mathbb{R}^s$, or equivalently $f(E_2)$ has measure zero.

Having demonstrated that $f(E_1 \cup E_2)$ has measure zero we may choose a (in fact many) point $p = (p_1, \cdots, p_s) \in \mathbb{R}^s$ outside this set such that

$$\sum_{k=1}^{s} |p_k|^2 < 1.$$

Note f is differentiable at any point in $f^{-1}(p) := \{\omega \in \mathbb{T}^s | f(\omega) = p\}$ and $\det f'(\omega) \neq 0$ for any $\omega \in f^{-1}(p)$. Such p is called a regular value of f. The set $f^{-1}(p)$ is finite (possibly empty), otherwise there exists at least one accumulation point ω_0 of $f^{-1}(p)$, which certainly satisfies both $\omega_0 \in f^{-1}(p)$ and $\det f'(\omega_0) = 0$, a contradiction. Therefore $\{(f_{s+1}(\omega), \cdots, f_n(\omega)) | \omega \in f^{-1}(p)\}$ is a finite set and hence cannot contain the infinite set (note $n-1 > s$)

$$\{(x_{s+1}, \cdots, x_n) \in \mathbb{R}^{n-s} | \sum_{k=s+1}^{n} |x_k|^2 = 1 - \sum_{k=1}^{s} |p_k|^2\}.$$

This means that $F(\mathbb{T}^s)$ cannot contain $S_{\mathbb{R}}^{n-1}$.

Now we follow the idea of [3]. For $a \in S_{\mathbb{R}}^{n-1} \backslash F(\mathbb{T}^s)$ we define $M(\omega) = Q_e(F(\omega))$ if $a = e := (1, 0, \cdots, 0) \in \mathbb{R}^n$ and $M(\omega) = Q_e(G(\omega)) Q_a^*(e)$ otherwise, where $G(\omega) = F(\omega) Q_a(e)$.

Obviously $M(\omega)$ is unitary and satisfies (i), (ii) and (iii). It remains only to verify that $M(\omega)$ has $F(\omega)$ as its first row, or, equivalently, $eM(\omega) = F(\omega)$. This is obvious when $a = e$ by virtue of Lemma 1. For $a \neq e$ we have

$$\begin{aligned} eM(\omega) &= eQ_e(G(\omega))Q_a^*(e) = G(\omega)Q_a^*(e) \\ &= G(\omega)Q_a^{-1}(e) = F(\omega). \end{aligned}$$

The proof is complete.

Remarks. 1. In general Lemma 2 does not hold true for $n \leq s+1$ except possibly $n \in \{2, 4, 8\}$. For example, if $n = s+1$, let $F(\omega)$ be the spherical coordinates. Then $F(\mathbb{T}^{n-1}) = S_{\mathbb{R}}^{n-1}$ and Adams's theorem tells us $F(\omega)$ has a real unitary extension iff $n \in \{2, 4, 8\}$.

2. In the proof of Lemma 2 if we only want to derive that $S_{\mathbb{C}}^{n-1} \backslash F(\mathbb{T}^s)$ (instead of $S_{\mathbb{R}}^{n-1} \backslash F(\mathbb{T}^s)$) is nonempty, it suffices to assume that $2n > s+1$ and $[s/2]+1$ components of the $2\pi\mathbb{Z}^s$-periodic continuous mapping $F(\omega)$ satisfy a Lipschitz condition. In this case, however, (ii) and (iii) are not true.

Invoking the description of the relation between wavelet set and unitary extension given in Section 1, together with Lemma 2 and Remark 2 we have immediately the main result in this section.

Theorem 1. *Let T be a scaling matrix with $|\det T| = n$ and φ a scaling function (real-valued scaling function, respectively) such that $A(\omega)$ defined as in Section 1 is continuous and $\left[\frac{s}{2}\right] + 1$ (s, respectively) components of $A(\omega)$ satisfy a Lipschitz condition. Then there exists wavelet set (real wavelet set, respectively) $\{\psi_j\}_{j=1}^{n-1}$ with all $A_{jk}(\omega)$ in (3) being continuous provided $2n-1 > s$ ($n-1 > s$, respectively).*

3. Constructions of wavelet set

We present some explicit constructions of wavelet sets from a multiresolution analysis having scaling function φ which is skew-symmetric about some point $c_\varphi \in 2^{-1}\mathbb{Z}^s$, i.e.,

$$\varphi(c_\varphi + x) = \overline{\varphi(c_\varphi - x)}, \quad x \in \mathbb{R}^s. \tag{6}$$

To this end the following algorithm is useful.

Algorithm. (i) *Let $A_{0k}(\omega) = n^{-\frac{1}{2}} \sum_{\alpha \in \mathbb{Z}^s} a_{\delta_k + T\alpha} e^{i\omega\cdot\alpha}, k = 1, \cdots, n.$*

(ii) *Find $a \in S_{\mathbb{C}}^{n-1} \backslash A(\mathbb{T}^s)$ and construct $M(\omega)$ as in Lemma 2.*

(iii) *Write $M(\omega) = (A_{jk}(\omega))_{j,k=0}^{n-1}$.*

(iv) *Define ψ_j as in (3), $j = 1, \cdots, n-1$.*

We remark that if φ is real-valued and the point a in (ii) belongs to $S_{\mathbb{R}}^{n-1}\backslash A(\mathbb{T}^s)$ then $\{\psi_j\}_{j=1}^{n-1}$ is a real wavelet set. For this reason we are more interested in finding $a \in S_{\mathbb{R}}^{n-1}\backslash A(\mathbb{T}^s)$ in the following.

Another feature of above Algorithm is that when $\{V_j\}$ is a r-regular multiresolution analysis (cf. [5], Def. 2, Chapter II) then for any $\alpha = (\alpha_1, \cdots, \alpha_s)$ with $|\alpha|_1 := \sum_{k=1}^{s} \alpha_k \leq r$, any m and $j = 1, \cdots, n-1$

$$\left|\frac{\partial^\alpha \psi_j}{\partial x^\alpha}\right| \leq C_m(1+|x|)^{-m}, m = 1, 2, \cdots .$$

The verification of this fact can proceed as in ([5], Theorem 2, Chapter II).

For any δ_k we can find a unique k' for which $\delta_k + \delta_{k'} \in T\mathbb{Z}^s$. There is thus a unique $\beta_k \in \mathbb{Z}^s$ such that

$$\delta_k + \delta_{k'} = T\beta_k, \quad k = 0, \cdots, n-1. \tag{7}$$

For later use we want to establish a result which ensures that $\{\beta_k\}_{k=0}^{n-1}$ contains a basis of $\mathbb{Z}^s$, i.e.,

$$\mathbb{Z}^s = \left\{\sum_{k=0}^{n-1} \alpha_k \beta_k | \alpha_k \in \mathbb{Z}, k = 0, \cdots, n-1\right\}.$$

We are able to do so for a special but important type of T.

Lemma 3. *Assume that the greatest common factor of all entries of T is larger than one in modulus. Then we can find explicitly a full collection of representatives $\{\delta_k\}_{k=0}^{n-1}$ for which $\{\beta_k\}_{k=0}^{n-1}$ given as in (7) contains a basis of $\mathbb{Z}^s$.*

Proof. Invoking the invariant factor theory (cf. [2]) we can construct two $s \times s$ matrices A and B with integer entries and $|\det A| = |\det B| = 1$ such that $T = A\text{diag}(\sigma_1, \cdots, \sigma_s)B$, where $\sigma_1 = \triangle_1, \sigma_2 = \triangle_2/\triangle_1, \cdots, \sigma_s = \triangle_s/\triangle_{s-1}$ and $\triangle_k$ is the greatest common factor of all $k \times k$ minors of $T, k = 1, \cdots, s$. The $\sigma_1, \cdots, \sigma_s$ are called the invariant factors of T and they satisfy $\sigma_k | \sigma_{k-1}, k = 2, \cdots, s$. Our assumption yields that $|\sigma_k| \geq 2, k = 1, \cdots, s$. Without loss of generality we assume that $\sigma_k \geq 2, k = 1, \cdots, s$.

Let $E_0 = \{(\nu_1, \cdots, \nu_s)^T \in \mathbb{Z}^s | 0 \leq \nu_k \leq \sigma_k - 1, k = 1, \cdots, s\}$, where x^T stands for the transpose of the vector x. We conclude that $\{\delta_k\}_{k=0}^{n-1} := AE_0$ satisfies our demand.

First we verify that $\{\delta_k\}_{k=0}^{n-1}$ is a full collection of representatives of $\mathbb{Z}^s/T\mathbb{Z}^s$. Obviously E_0 is a full collection of representatives of $\mathbb{Z}^s/T_0\mathbb{Z}^s$ with $T_0 = \text{diag}(\sigma_1, \cdots, \sigma_s)$. Note $A\mathbb{Z}^s = A^{-1}\mathbb{Z}^s = B\mathbb{Z}^s = B^{-1}\mathbb{Z}^s = \mathbb{Z}^s$. We have for any $\alpha \in \mathbb{Z}^s$

$$A^{-1}\alpha = T_0\beta + \mu$$

for some $\beta \in \mathbb{Z}^s$ and $\mu \in E_0$ and hence

$$\alpha = T\gamma + A\mu$$

for $\gamma = B^{-1}\beta$ and $A\mu \in \{\delta_k\}_{k=0}^{n-1}$.

Second, we show

$$(1, 0, \cdots, 0)^T, \cdots, (0, \cdots 0, 1)^T \in \{B\beta_k\}_{k=0}^{n-1}.$$

Let $\delta_{k_1} = A(1,0,\cdots,0)^T$. Then it is easy to see that $\delta_{k_1'} = A(\sigma_1 - 1, 0, \cdots, 0)^T$ and therefore

$$\delta_{k_1} + \delta_{k_1'} = AT_0(1,0,\cdots,0)^T = TB^{-1}(1,0,\cdots,0)^T,$$

which implies that $\beta_{k_1} = B^{-1}(1,0,\cdots,0)^T$. This proves that $(1,0,\cdots,0)^T \in \{B\beta_k\}_{k=0}^{n-1}$. Similarly we have

$$(0,1,0,\cdots,0)^T, \cdots, (0,\cdots,0,1)^T \in \{B\beta_k\}_{k=0}^{n-1}.$$

Noting $B^{-1}\mathbb{Z}^s = \mathbb{Z}^s$ once again we get the desired conclusion. The proof is complete.

Now we assume that the scaling function φ is skew-symmetric about $c_\varphi \in \mathbb{Z}^s$, which we may assume to be zero, and that $A_{0k}(\omega)$ is continuous for all $0 \le k \le n-1$. Therefore $\{a_\alpha\}_{\alpha\in\mathbb{Z}^s}$ given as in (1) satisfies

$$a_\alpha = \overline{a}_{-\alpha}, \quad \alpha \in \mathbb{Z}^s,$$

from which it follows that

(8)
$$\begin{aligned} A_{0k}(\omega) &= \sum_{\alpha\in\mathbb{Z}^s} \overline{a}_{-\delta_k + T\alpha} e^{-i\omega\cdot\alpha} = e^{-i\beta_k\cdot\omega} \sum_{\alpha\in\mathbb{Z}^s} a_{\delta_{k'} + T(\alpha-\beta_k)} e^{-i\omega\cdot(\alpha-\beta_k)} \\ &= e^{-i\omega\cdot\beta_k}\overline{A_{0k'}(\omega)}, \quad k = 0, \cdots, n-1. \end{aligned}$$

Theorem 2. *Assume that T satisfies the conditions of Lemma 3 and $\{\delta_k\}_{k=0}^{n-1}$ is chosen so that $\{\beta_k\}_{k=0}^{n-1}$ contains a basis of $\mathbb{Z}^s$. Suppose that $\widehat{\varphi}(\omega)$, the Fourier transform of the scaling function φ satisfying* (6) *with $c_\varphi = 0$, is continuous at $\omega = 0$ with $\widehat{\varphi}(0) \neq 0$ and $A(\omega)$ is a continuous mapping. Then $-n^{-\frac{1}{2}}(1,\cdots,1)\overline{\in} A(\mathbb{T}^s)$.*

Moreover if $T = 2I$ we can take $\{\delta_k\}_{k=0}^{n-1} = E$, the extremal points of $[0,1]^s$, for which $\{\beta_k\}_{k=0}^{n-1}$ certainly contains a basis of $\mathbb{Z}^s$. Then any point $a = (x_0, \cdots, x_{n-1}) \in (\mathbb{R}\backslash\{0\})^n$ satisfies $a \overline{\in} A(\mathbb{T}^s)$ with the possible exceptions in case $\sum_{k=0}^{n-1} x_k = 2^{s/2} (= n^{\frac{1}{2}})$ or, more precisely, only with the possible exception that $a = A(0)$.

Proof. We first claim that $A(\omega) \neq -n^{-\frac{1}{2}}(1,\cdots,1)$ for any $\omega \in \mathbb{T}^s$ with $\omega \neq 0$.

In fact for $\omega \in \mathbb{T}^s$ with $\omega \neq 0$ we can find some $\alpha \in \mathbb{Z}^s$ such that $e^{i\omega\cdot\alpha} \neq 1$. Since $\{\beta_k\}_{k=0}^{n-1}$ contains a basis of $\mathbb{Z}^s$ we have some β_k for which $e^{i\beta_k\cdot\omega} \neq 1$, which together with (8) gives that $A(\omega) \neq -n^{-\frac{1}{2}}(1,\cdots,1)$ for $\omega \in \mathbb{T}^s\backslash\{0\}$.

Second we prove $A(0) \neq -n^{-\frac{1}{2}}(1,\cdots,1)$. To this end we define function

$$A_0(\omega) := \sum_{k=0}^{n-1} e^{i\omega\cdot\delta_k} A_{0k}(T^*\omega), \tag{9}$$

which satisfies $\widehat{\varphi}(\omega) = n^{-\frac{1}{2}} A_0(T^{*-1}\omega)\widehat{\varphi}(T^{*-1}\omega)$. Thus $A_0(0) = n^{\frac{1}{2}}$. Now the desired conclusion follows from (9) by setting $\omega = 0$.

As for the case $T = 2I$ and $\{\delta_k\}_{k=0}^{n-1} = E$ it holds naturally that $k = k'$ and $\beta_k = \delta_k$ for all $k = 0, \cdots, n-1$, where k' and β_k are determined by k as in (7). Therefore

$$A_{0k}(\omega) = e^{-i\omega\cdot\delta_k}\overline{A_{0k}(\omega)}, k = 0, \cdots, n-1,$$

from which we conclude as before that $a \neq A(\omega)$ for any $a \in (\mathbb{R}\backslash\{0\})^n$ and $\omega \in \mathbb{T}^s\backslash\{0\}$. Moreover $a = A(0)$ only if $\sum_{k=0}^{n-1} x_k = 2^{\frac{s}{2}}$ by setting $\omega = 0$ in (9). We complete the proof.

Xiao [7] and Zhou [8] have proved respectively that $k(\sigma_1, \cdots, \sigma_n)\overline{\in} A(\mathbb{T}^s)$ for some k and some $\sigma_k = 0, 1$, and $-n^{-\frac{1}{2}}(1, \cdots, 1)\overline{\in} A(\mathbb{T}^s)$ for $T = 2I$ and $\{\delta_k\}_{k=0}^{n-1} = E$.

At the end of the paper we give another construction of unitary extensions in low dimension $s = 1, 2, 3$. We recall that $\eta : E \to E$ is called an admissible mapping if $\eta(0) = 0$ and $(\eta(\mu) + \eta(\mu)) \cdot (\mu + \nu)$ is odd for $\mu, \nu \in E$ with $\mu \neq \nu$. The following result, as a variation of [6], gives a unitary extension of a vector in a way different from that described in above Algorithm.

Theorem 3. *Suppose that a sequence $(x_\alpha)_{\alpha \in \mathbb{Z}^s}$ satisfies $x_\alpha = x_{\alpha+2\beta}$ for all $\alpha, \beta \in \mathbb{Z}^s$, $\sum_{\alpha \in E} |x_\alpha|^2 = 1$ and for some $c \in E$*

$$x_\mu = \overline{x}_{c-\mu}, \quad \mu \in E.$$

If η is a admissible mapping, then the matrix $(\omega_{\mu\nu})_{\mu\nu \in E}$ defined by

$$\omega_{\mu\nu} = \begin{cases} (-1)^{\mu\cdot\nu} x_{\nu-\eta(\mu)}, & \text{if } c\cdot\mu \text{ is even,} \\ \\ (-1)^{\mu\cdot\nu} \overline{x}_{\nu-\eta(\mu)}, & \text{if } c\cdot\mu \text{ is odd,} \end{cases}$$

is unitary.

The proof is similar to that in [6] and is omitted.

During the revision of the paper Prof. Long Ruilin told me that he had also gotten such variations.

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