PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 2867–2873
S 0002-9939(97)03882-3

A WEIGHTED L^2 ESTIMATE FOR THE COMMUTATOR OF THE BOCHNER-RIESZ OPERATOR

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(Communicated by Christopher D. Sogge)

ABSTRACT. A weighted L^2 estimate with power weights is established for the maximal operator associated with the commutator of the Bochner-Riesz operator. An application of this weighted estimate is also given.

1. INTRODUCTION

We will work on $\mathbb{R}^n$, $n \geq 2$. Let λ and r be two positive numbers. The Bochner-Riesz operator T_λ^r is defined in terms of the Fourier transforms by

$$(T_\lambda^r f)^\wedge(\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)_+^\lambda \hat{f}(\xi), \tag{1}$$

where $\hat{f}$ denotes the Fourier transform of f. Let k be a positive integer and b a BMO function, the $k-$th order commutator of T_λ^r is defined by

$$T_{\lambda;\,b,k}^r f(x) = T_\lambda^r\left((b(x) - b(\cdot))^k f\right)(x). \tag{2}$$

If $\lambda \geq (n-1)/2$, a result of Shi and Sun [8] states that T_λ^r is bounded on $L^p(\mathbb{R}^n,\, w(x)dx)$ provided that $1 < p < \infty$ and $w \in A_p$, where A_p denotes the weight function class of Muckenhoupt (see [9, Chapter V] for definition and properties of A_p). Therefore, by the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Babgy, Kurtz and Pérez (see [1, Theorem 2.13]), we see that in this case $T_{\lambda;\,b,k}^r$ is also bounded on $L^p(\mathbb{R}^n,\, w(x)dx)$ for all $1 < p < \infty$ and $w \in A_p$. On the other hand, as was pointed by Hu and Lu [6], if $0 < \lambda < (n-1)/2$, the boundedness criterion of Alvarez-Babgy-Kurtz-Pérez does not apply to $T_{\lambda;\,b,k}^r$. Hu and Lu [6] showed that for the special case $n = 2$ or the case $\lambda > (n-1)/2(n+1)$, $T_{\lambda;\,b,k}^r$ enjoys some L^p mapping properties which are parallel to that of the operator T_λ^r. Furthermore, Hu and Lu [7] also considered the maximal operator associated with the commutator of the Bochner-Riesz operator, defined by

$$T_{\lambda;\,b,k}^* f(x) = \sup_{r>0} |T_{\lambda;\,b,k}^r f(x)|, \tag{3}$$

and obtained the following result.

Received by the editors December 1, 1995 and, in revised form, April 1, 1996.
1991 *Mathematics Subject Classification.* Primary 42B15.
Key words and phrases. Weighted L^2 estimate, commutator, Bochner-Riesz operator, maximal operator.
This work was supported by the NNSF of China.

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Theorem A. *Let k be a positive integer and b belong to* $\mathrm{BMO}(\mathbb{R}^n)$. *If $\lambda > 0$, then $T^*_{\lambda;\,b,k}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n,k,\lambda)\|b\|^k_{\mathrm{BMO}}$.*

The purpose of this paper is to establish a weighted L^2 estimate with power weights for the operator $T^*_{\lambda;\,b,k}$. Our main result is

Theorem 1. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $\lambda > 0$ and $0 < \alpha < 1 + 2\lambda < n$, then*

$$\int_{\mathbb{R}^n} |T^*_{\lambda;\,b,k} f(x)|^2 |x|^{-\alpha} dx \le C(n,k,\lambda,\alpha)\|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

This estimate is of interest since it implies

Theorem 2. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$ for some p with $2 \le p < 2n/(n-1-2\lambda)$, then $\lim_{r\to\infty} T^r_{\lambda;\,b,k} f(x) = 0$ almost everywhere.*

2. Proof of theorems

We begin with some lemmas which will be used in the proof of Theorem 1.

Lemma 1 (see [5]). *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Denote by $M_{b,k}$ the $k-$th order commutator of the Hardy-Littlewood maximal operator defined by*

$$M_{b,k} f(x) = \sup_{r>0} r^{-n} \int_{|x-y|<r} |b(x) - b(y)|^k |f(y)| dy.$$

If $1 < p < \infty$ and $w \in A_p$, then $M_{b,k}$ is bounded on $L^p(\mathbb{R}^n, w(x)dx)$ with bound $C(n,k,p)\|b\|^k_{\mathrm{BMO}}$.

Lemma 2 (see [7]). *Let $\varphi \in C^\infty_0(\mathbb{R}^n)$ be a radial function such that $\mathrm{supp}\,\varphi \subset \{1/4 \le |x| \le 4\}$, φ is identically one on $\{1/2 \le |x| \le 2\}$ and*

$$\sum_{l\in\mathbb{Z}} \varphi(2^{-l}x) = 1, \quad |x| > 0.$$

Denote by g_l the multiplier operator

$$(g_l f)^\wedge(\xi) = \varphi(2^{-l}\xi)\hat{f}(\xi).$$

Then for any positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, the $k-$th order commutator defined by

$$g_{l;\,b,k} f(x) = g_l\Big((b(x) - b(\cdot))^k f\Big)(x)$$

satisfies

$$\|(\sum_{l\in\mathbb{Z}} |g_{l;\,b,k} f|^2)^{1/2}\|_{p,w} \le C\|b\|^k_{\mathrm{BMO}}\|f\|_{p,w},$$

provided that $1 < p < \infty$ and $w \in A_p$.

Lemma 3. *Let $0 < \delta < 1/2$, m_δ be a radial C^∞ function supported in $\{1 - \delta \le |\xi| \le 1\}$ and such that*

$$0 \le m_\delta(\xi) \le 1, \;\; |D^\nu m_\delta(\xi)| \le C\delta^{-|\nu|} \;\; \textit{for all multi-index}\ \nu.$$

Let S^t_δ be the multiplier operator defined by

$$(S^t_\delta f)^\wedge(\xi) = m_\delta(t\xi)\hat{f}(\xi).$$

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For k a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, denote by $S^t_{\delta;\,b,k}$ the $k-th$ order commutator of S^t_δ. Then for any $1<\alpha<n$ and $\beta>0$, there exists a positive constant $C=C(n,k,\alpha,\beta)$ such that

$$\int_1^2\int_{\mathbb{R}^n}|S^t_{\delta;\,b,k}f(x)|^2|x|^{-\alpha}dx\frac{dt}{t}\le C\|b\|^{2k}_{\mathrm{BMO}}\delta^{2-\alpha-\beta}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}dx.$$

Proof. Let $\psi_0,\,\psi\in C_0^\infty(\mathbb{R}^n)$ be radial functions such that

$$\mathrm{supp}\psi\subset\{1/4\le|x|\le 4\}$$

and that

$$\psi_0(x)+\sum_{l=1}^\infty\psi(2^{-l}x)=1,\quad \text{if } |x|>0.$$

Set $\psi^l(x)=\psi(2^{-l}x)$ for $l\ge 1$ and $K^t_\delta(x)=(m_\delta(t\cdot))^\wedge(x)$. Write

$$K^t_\delta(x)=K^t_\delta(x)\psi_0(\delta x)+\sum_{l=1}^\infty K^t_\delta(x)\psi^l(\delta x)=\sum_{l=0}^\infty K^{t,l}_\delta(x).$$

It is easy to see that m_δ is integrable on $L^1(\mathbb{R}^n)$ with integral less than or equal to a constant independent of δ. So if $1\le t\le 2$, then

$$\|K^{t,l}_\delta\|_\infty\le C.$$

Let $S^{t,l}_\delta$ be the convolution operator whose kernel is $K^{t,l}_\delta$. By Young's inequality, we have

$$\|S^{t,l}_\delta f\|_\infty\le C\|f\|_1,\ \ 1\le t\le 2,$$

which in turn gives that

$$\left\|\left(\int_1^2|S^{t,l}_\delta f|^2\frac{dt}{t}\right)^{1/2}\right\|_\infty\le C\|f\|_1. \tag{4}$$

Let $1<\gamma<n$. The observation of Carbery, Rubio de Francia and Vega [3] tells us that

$$\int_{\mathbb{R}^n}|S^{1,l}_\delta f(x)|^2|x|^{-\gamma}dx\le C2^{-l}\delta^{1-\gamma}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\gamma}dx.$$

This by homogeneity implies that the inequality

$$\int_{\mathbb{R}^n}|S^{t,l}_\delta f(x)|^2|x|^{-\gamma}dx\le C2^{-l}\delta^{1-\gamma}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\gamma}dx$$

holds uniformly in $1\le t\le 2$. By the same argument as in [3, page 516], it follows from the last inequality that

$$\int_1^2\int_{\mathbb{R}^n}|S^{t,l}_\delta f(x)|^2|x|^{-\gamma}dx\frac{dt}{t}\le C2^{-l}\delta^{2-\gamma}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\gamma}dx. \tag{5}$$

Using the interpolation theorem of Stein-Weiss (see [2, page 120]) we can obtain from inequalities (4) and (5) that for each q with $1<q<2$,

(6)

$$\int_{\mathbb{R}^n}\left(\int_1^2|S^{t,l}_\delta f(x)|^2\frac{dt}{t}\right)^{q'/2}|x|^{-\gamma}dx\le C2^{-l}\delta^{2-\gamma}\left(\int_{\mathbb{R}^n}|f(x)|^q|x|^{-\gamma(q-1)}dx\right)^{q'/q},$$

where $q'=q/(q-1)$.

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Now let $1<\alpha<n$. We turn our attention to $S^{t,l}_{\delta;\,b,k}$, the $k-$th order commutator of $S^{t,l}_{\delta}$. We decompose $\mathbb{R}^n$ into a grid of non-overlapping cubes with side length $2^l\delta^{-1}$, $\mathbb{R}^n=\bigcup_j Q_j$. Let χ_{Q_j} be the characteristic function of Q_j. Set $f_j=f\chi_{Q_j}$. Then

$$f=\sum_j f_j.$$

Since $\operatorname{supp} K^{t,l}_{\delta}\subset\{|x|\le C2^l\delta^{-1}\}$ when $1\le t\le 2$, it is obvious that the support of $S^{t,l}_{\delta}f_j$ is contained in a fixed multiple of Q_j, and that the supports of the various terms $S^{t,l}_{\delta;\,b,k}f_j$ have bounded overlaps. So we have the almost orthogonality property:

$$\int_1^2\int_{\mathbb{R}^n}|S^{t,l}_{\delta;\,b,k}f(x)|^2|x|^{-\alpha}dx\frac{dt}{t}\le C\sum_j\int_1^2\int_{\mathbb{R}^n}|S^{t,l}_{\delta;\,b,k}f_j(x)|^2|x|^{-\alpha}dx\frac{dt}{t}.$$

Thus we may assume that $\operatorname{supp} f\subset Q$ for some cube Q with side length $2^l\delta^{-1}$. Choose $\eta\in C_0^\infty(\mathbb{R}^n)$, $0\le\eta\le 1$, η is identically one on $50nQ$ and vanishes outside $100nQ$. Set $\bar{Q}=200nQ$, and $\bar{b}(x)=(b(x)-b_{\bar{Q}})\eta(x)$, where $b_{\bar{Q}}$ is the mean value of b on $\bar{Q}$, i.e., $b_{\bar{Q}}=|\bar{Q}|^{-1}\int_{\bar{Q}}b(y)dy$. It is not difficult to see that

$$S^{t,l}_{\delta;\,b,k}f(x)=\sum_{m=0}^{k}C_k^m\bar{b}^m(x)S^{t,l}_{\delta}(\bar{b}^{k-m}f)(x).$$

Let $2<q_1,\,q_2<\infty$ such that $1/q_1+1/q_2=1/2$ and $\alpha q_2<2n$. By Hölder's inequality and (6), we can get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_1^2\left|\bar{b}^m(x)S^{t,l}_{\delta}(\bar{b}^{k-m}f)(x)\right|^2\frac{dt}{t}|x|^{-\alpha}dx\\
&\le\|\bar{b}^m\|^2_{q_1}\left(\int_{\mathbb{R}^n}\left(\int_1^2\left|S^{t,l}_{\delta}(\bar{b}^{k-m}f)(x)\right|^2\frac{dt}{t}\right)^{q_2/2}|x|^{-\alpha q_2/2}dx\right)^{2/q_2}\\
&\le C\left(2^{-l}\delta^{2-\alpha q_2/2}\right)^{2/q_2}\|\bar{b}^m\|^2_{q_1}\left(\int_{\mathbb{R}^n}\left|\bar{b}^{k-m}(x)f(x)\right|^{q_2'}|x|^{-\alpha(q_2'-1)q_2/2}dx\right)^{2/q_2'}\\
&\le C\left(2^{-l}\delta^{2-\alpha q_2/2}\right)^{2/q_2}\|\bar{b}^m\|^2_{q_1}\|\bar{b}^{k-m}\|^2_{2q_2/(q_2-2)}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}dx\\
&\le C\left(2^{-l}\delta^{2-\alpha q_2/2}\right)^{2/q_2}(2^l\delta^{-1})^{2n(1-2/q_2)}\|b\|^{2k}_{\mathrm{BMO}}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}dx,
\end{aligned}$$

where in the last inequality we have invoked the fact that (see [9, Chapter IV])

$$\|\bar{b}^m\|_{q_1}\le C(q_1,m)\|b\|^m_{\mathrm{BMO}}|Q|^{1/q_1}.$$

For each fixed $1<\alpha<n$ and $\beta>0$, we choose q_2 such that

$$2<q_2<\min\left\{2+1/n,\,2n/\alpha,\,4(1+n)\Big/\Big(2(1+n)-\beta\Big)\right\}.$$

Then we have that for some positive constant ε,

$$\int_{\mathbb{R}^n}\int_1^2|S^{t,l}_{\delta;\,b,k}f(x)|^2\frac{dt}{t}|x|^{-\alpha}dx\le C2^{-\varepsilon l}\delta^{2-\alpha-\beta}\|b\|^{2k}_{\mathrm{BMO}}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}dx.$$

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The summation of the last inequality over all $l \geq 0$ then finishes the proof of Lemma 3. $\square$

Proof of Theorem 1. By Theorem A and the interpolation theorem of Stein-Weiss, it suffices to prove Theorem 1 for the case of $1 < \alpha < 1 + 2\lambda$. For each fixed $\delta > 0$, let m_δ and S_δ^t be the same as in Lemma 3. Set

$$S^*_{\delta;\,b,k} f(x) = \sup_{t>0} |S^t_{\delta;\,b,k} f(x)|.$$

As in [3] and [4], we have the following decomposition:

$$(1 - |\xi|^2)_+^\lambda = \sum_{i=0}^{\infty} 2^{-i\lambda} m_{2^{-i}}(|\xi|).$$

Therefore

$$T^*_{\lambda;\,b,k} f(x) \leq C \sum_{i=0}^{\infty} 2^{-i\lambda} S^*_{2^{-i};\,b,k}.$$

Clearly, if $i = 0,\ 1$, then

$$S^*_{2^{-i};\,b,k} f(x) \leq C M_{b,k} f(x),$$

where $M_{b,k}$ is the $k-$th order commutator of the Hardy-Littlewood maximal operator. Recalling that $|x|^{-\gamma} \in A_2$ if and only if $-n < \gamma < n$, we thus have that by Lemma 1

$$\int_{\mathbb{R}^n} |S^*_{2^{-i};\,b,k} f(x)|^2 |x|^{-\alpha} dx \leq C \|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx,\ \ i = 0,\ 1. \tag{7}$$

Following along the same line as in [3], we now introduce the quadric operators

$$G_\delta f(x) = \left(\int_0^\infty |S^t_{\delta;\,b,k} f(x)|^2 \frac{dt}{t} \right)^{1/2}$$

and

$$\tilde{G}_\delta f(x) = \left(\int_0^\infty |\tilde{S}^t_{\delta;\,b,k} f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\tilde{S}^t_{\delta;\,b,k}$ is the $k-$th order commutator of the following multiplier operator,

$$(\tilde{S}^t_\delta f)^\wedge(\xi) = \tilde{m}_\delta(t\xi) \hat{f}(\xi),$$

and

$$\tilde{m}_\delta(\xi) = \tilde{m}_\delta(|\xi|) = \delta|\xi|(\frac{d}{dr} m_\delta(r))(|\xi|).$$

We want to show that if $1 < \alpha < n$, $\beta > 0$ and $0 < \delta < 1/2$, then

$$\int_{\mathbb{R}^n} |G_\delta f(x)|^2 |x|^{-\alpha} dx \leq C(n, k, \alpha, \beta) \delta^{2-\alpha-\beta} \|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx. \tag{8}$$

If we can do this, we can finish the proof of Theorem 1. In fact, it is easy to see that $\tilde{m}_\delta$ enjoys the same properties as those of m_δ; thus $\tilde{G}_\delta$ satisfies the same estimate (8), as G_δ. By Schwartz's inequality, straightforward computation yields that

$$(S^*_{2^{-i};\,b,k} f(x))^2 \leq C 2^i G_{2^{-i}} f(x) \tilde{G}_{2^{-i}} f(x).$$

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Consequently, if $i \geq 2$,

$$\int_{\mathbb{R}^n} |S^*_{2^{-i};\, b,k} f(x)|^2 |x|^{-\alpha} dx \leq C 2^{i(\alpha+\beta-1)} \|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx. \tag{9}$$

For each fixed $\lambda > 0$ and $1 < \alpha < 1 + 2\lambda$, we choose $\beta > 0$ such that $1 < \alpha + \beta < 1 + 2\lambda$. Combining the inequalities (7) and (9) establishes our desired estimate.

The proof of Theorem 1 is now reduced to showing (8). Observe that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then for any $t > 0$, $b_t(x) = b(tx)$ also belongs to $\mathrm{BMO}(\mathbb{R}^n)$ and $\|b_t\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}}$. By dilation-invariance, it follows from Lemma 3 that, for any $l \in \mathbb{Z}$,

$$\int_{\mathbb{R}^n} \int_{2^{-l}}^{2^{-l+1}} |S^t_{\delta;\, b,k} f(x)|^2 \frac{dt}{t} |x|^{-\alpha} dx \leq C \delta^{2-\alpha-\beta} \|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx. \tag{10}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ as in Lemma 2. Write

$$\begin{aligned} S^t_{\delta;\, b,k} f(x) &= \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \Big(\varphi(2^{-l}\cdot) m_\delta(t\cdot) \Big)^\wedge (x-y)(b(x)-b(y))^k f(y) dy \\ &= \sum_{l \in \mathbb{Z}} (S^t_\delta g_l)_{b,k} f(x), \end{aligned}$$

where g_l is the multiplier operator defined in Lemma 2. With the aid of the formula

$$(b(x)-b(y))^k = \sum_{m=0}^{k} C_k^m \Big(b(x) - b(z) \Big)^m \Big(b(z) - b(y) \Big)^{k-m}, \ z \in \mathbb{R}^n,$$

we have that

$$(S^t_\delta g_l)_{b,k} f(x) = \sum_{m=0}^{k} C_k^m S^t_{\delta;\, b,m} (g_{l;\, b,k-m} f)(x).$$

Note that for each fixed t, the number of l's for which $\operatorname{supp}(\varphi(2^{-l}\cdot)) \cap \operatorname{supp} m_\delta(t\cdot)$ is non-empty is at most 100. Thus

$$\begin{aligned} \int_0^\infty |S^t_{\delta;\, b,k} f(x)|^2 \frac{dt}{t} &\leq C \sum_{l \in \mathbb{Z}} \int_0^\infty \Big| (S^t_\delta g_l)_{b,k} f(x) \Big|^2 \frac{dt}{t} \\ &\leq C \sum_{m=0}^{k} \sum_{l \in \mathbb{Z}} \int_0^\infty \Big| S^t_{\delta;\, b,m} (g_{l;b,k-m} f)(x) \Big|^2 \frac{dt}{t} \\ &\leq C \sum_{m=0}^{k} \sum_{l \in \mathbb{Z}} \int_{2^{-l}}^{2^{-l+1}} \Big| S^t_{\delta;\, b,m} (g_{\delta;\, b,k-m} f)(x) \Big|^2 \frac{dt}{t}. \end{aligned}$$

By Lemma 2 and the inequality (10), we finally obtain that

$$\begin{aligned} \int_{\mathbb{R}^n} |G_\delta f(x)|^2 |x|^{-\alpha} dx &\leq C \sum_{m=0}^{k} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^{-l}}^{2^{-l+1}} \Big| S^t_{\delta;\, b,m} (g_{l;\, b,k-m} f)(x) \Big|^2 \frac{dt}{t} |x|^{-\alpha} dx \\ &\leq C \delta^{2-\alpha-\beta} \sum_{m=0}^{k} \|b\|^{2m}_{\mathrm{BMO}} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} |g_{l;\, b,k-m} f(x)|^2 |x|^{-\alpha} dx \\ &\leq C \delta^{2-\alpha-\beta} \|b\|^{2k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx. \end{aligned}$$

□

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Proof of Theorem 2. Note that if $f \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f \subset Q$ for some cube Q, then $(b - b_Q)^m f \in L^2(\mathbb{R}^n)$ for any positive integer m. Write

$$T^r_{\lambda;\,b,k} f(x) = \sum_{m=0}^{k} C_k^m (b(x) - b_Q)^m T^r_\lambda \Big((b_Q - b(\cdot))^{k-m} f \Big)(x).$$

By the almost everywhere convergence of the Bochner-Riesz means (see [9, Chapter IX]), we have

$$\begin{aligned}\lim_{r\to\infty} T^r_{\lambda;\,b,k} f(x) &= \sum_{m=0}^{k} C_k^m (b(x) - b_Q)^m \lim_{r\to\infty} T^r_\lambda \left((b_Q - b(\cdot))^{k-m} f \right)(x) \\ &= \sum_{m=0}^{k} C_k^m (b(x) - b_Q)^m (b_Q - b(x))^{k-m} f(x) = 0.\end{aligned}$$

For each given p with $2 \le p < 2n/(n-1-2\lambda)$, we can choose α such that

$$n(1 - 2/p) < \alpha < 1 + 2\lambda.$$

It is easy to see that $L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) + L^2(\mathbb{R}^n, |x|^{-\alpha} dx)$. Thus Theorem 2 follows from Theorem A and Theorem 1. □

Acknowledgement

The authors would like to thank the referee for some corrections and suggestions.

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