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ON DUALS OF WEAKLY ACYCLIC (LF)-SPACES

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ABSTRACT. For countable inductive limits of Fréchet spaces ((LF)-spaces) the property of being weakly acyclic in the sense of Palamodov (or, equivalently, having condition (M_0) in the terminology of Retakh) is useful to avoid some important pathologies and in relation to the problem of well-located subspaces. In this note we consider if weak acyclicity is enough for a (LF)-space $E := \operatorname{ind} E_n$ to ensure that its strong dual is canonically homeomorphic to the projective limit of the strong duals of the spaces E_n. First we give an elementary proof of a known result by Vogt and obtain that the answer to this question is positive if the steps E_n are distinguished or weakly sequentially complete. Then we construct a weakly acyclic (LF)-space for which the answer is negative.

INTRODUCTION

Countable inductive limits of Fréchet spaces ((LF)-spaces) arise in many fields of functional analysis and its applications; e.g., in distribution theory, linear partial differential equations, convolution equations, Fourier analysis, complex analysis. Since their topological structure presents severe pathologies (see [11, 4], [9], [2, 3 and Appendix]) it is interesting to study those conditions which ensure a good behaviour with respect to some particular problems. Here we are concerned with deciding whether the property of being weakly acyclic in the sense of Palamodov (equivalent to the condition (M_0) of Retakh) is enough for a (LF)-space E to identify (in a canonical way) its strong dual with a projective limit of (DF)-spaces.

Let us fix some notation before going on with the introduction. For a locally convex space E let E' denote the topological dual and E^* its algebraic dual. Given a dual pair $\langle E, F\rangle$, let $\sigma(E,F)$ and $\beta(E,F)$ denote the corresponding weak and strong topologies, respectively. The bidual of E is $E'' := (E', \beta(E', E))'$, moreover we set $E'_\beta := (E', \beta(E', E))$ and $E'_{\beta^2} := (E', \beta(E', E''))$. The space E is said to be *distinguished* if every bounded subset of $(E'', \sigma(E'', E'))$ lies in the $\sigma(E'', E')$-closure of some bounded subset of E. This is equivalent to $\beta(E', E) = \beta(E', E'')$; it is also equivalent with requiring that E'_β is barrelled ([13, 23.7]). If E is metrizable then it is distinguished if and only if E'_β is bornological, cf. [13, 29.4].

A sequence $(E_n)_{n\in\mathbb{N}}$ of locally convex spaces is inductive if E_n is continuously included into E_{n+1} ($n \in \mathbb{N}$); then $E := \operatorname{ind} E_n$ is its inductive limit whenever $E = \bigcup_{n\in\mathbb{N}} E_n$ and E is endowed with the finest locally convex topology such that the injections $E_n \to E$ are continuous. E is an (LF)-space if the E_n's are Fréchet.

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(We refer to [2], [19] for more information about inductive limits.) Recall that $E := \text{ind } E_n$ is said to be *weakly acyclic* if the map

$$\oplus E_n \to \oplus E_n, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (x_n - x_{n-1})_{n\in\mathbb{N}},\ (x_0 := 0)$$

is a weak topological isomorphism onto its range. Weak acyclicity is equivalent to the property (M_0) introduced by Retakh in connection with the problem of well-located subspaces, which applies to the study of surjectivity and normal solvability of linear operators in duals of (LF)-spaces (in particular of linear partial differential operators in the space $\mathcal{D}'(\Omega)$ of distributions on Ω); see [14], [2, Appendix] and [9] for details.

An important task on inductive limits, which goes back to Grothendieck, is the description of the strong dual of $E :=$ ind E_n as the projective limit of the duals of E_n ($n \in \mathbb{N}$). Indeed, the canonical linear map

$$j : E'_\beta \longrightarrow \text{proj } E'_{n,\beta}, \quad f \longmapsto (f|E_n)_{n\in\mathbb{N}}$$

(where the projective limit is formed with respect to the natural restrictions $E'_{n+1} \to E'_n,\ g \mapsto g|E_n$) is a continuous and surjective isomorphism, but j does not need to be open. By a standard duality argument, j is a topological isomorphism if and only if for every bounded subset B in E there is $n \in \mathbb{N}$ and a bounded subset A in E_n such that $B \subset \overline{A}^E$. We will call an inductive sequence $(E_n)_{n\in\mathbb{N}}$ *quasiregular* if it satisfies this (very weak regularity) condition. (It is known that the inductive sequence $(E_n)_{n\in\mathbb{N}}$ is said to be *regular* if for every bounded subset B in E there is $n \in \mathbb{N}$ such that B is contained and bounded in E_n. Regularity has been thoroughly studied in the theory of (LF)-spaces, cf. [2], [19].)

Next we give examples and further information about quasiregular (LF)-spaces: (i) Every inductive sequence consisting of DF-spaces E_n ($n \in \mathbb{N}$), in particular every (LB)-space, is quasiregular (by Grothendieck [10]). (ii) If an inductive sequence $(E_n)_{n\in\mathbb{N}}$ consists of semireflexive spaces E_n and if the inductive limit $E := \text{ind } E_n$ is Hausdorff, then $(E_n)_{n\in\mathbb{N}}$ is quasiregular if and only if it is regular. Consequently, an inductive sequence $(E_n)_{n\in\mathbb{N}}$ consisting of Fréchet Montel spaces with Hausdorff inductive limit is quasiregular if and only if it is complete (see Wengenroth [20, Theorem 3.9]). (A very easy example of an incomplete (LF)-space with Fréchet Montel steps is obtained in the following way: Take $X := \omega := \mathbb{K}^{\mathbb{N}}$ (where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let Y be any Fréchet Montel space continuously embedded into X with proper dense range. Then the inductive sequence $(\prod_{k<n} X \times \prod_{k\geq n} Y)_{n\in\mathbb{N}}$ consists of Fréchet Montel spaces and, according to Dierolf [8, Prop.], its inductive limit is topologically isomorphic to a proper dense linear subspace of $X^{\mathbb{N}}$ hence incomplete, metrizable and provided with a weak topology.) (iii) Quasiregular (LF)-spaces of generalized Moscatelli type of the shape $\oplus X + \ell^\infty(Y)$, where $Y,\ X$ are Fréchet spaces with continuous inclusion $Y \hookrightarrow X$, were characterized by Bonet-Dierolf-Fernández [6] .

It was demonstrated by Vogt ([19, 5], see also [15] and [3] for related results) that weakly acyclic (LF)-spaces need not be regular. So it is natural to ask whether they are quasiregular. This question was somehow considered by Vogt in [18] and [19]. The following theorem is a direct consequence of Vogt's Lemma 4.1 of [19], Theorem 5.6 of [18] and the fact that all E'_{n,β^2} are bornological by [13, 29.5]. (Note that this theorem extends a Grothendieck's result [12, 3.6, Theorem 2] asserting that a strict (LF)-space with distinguished steps is distinguished, from where it can be deduced that the space $\mathcal{D}'_\beta(\Omega)$ of distributions on Ω is barrelled and bornological. On the other hand, we refer the reader to [4] for negative results about the topological structure of strong duals of strict (LF)-spaces.)

Theorem A (Vogt)**.** *Let $E = \operatorname{ind} E_n$ be a weakly acyclic (LF)-space. Then the natural projective limit* $\operatorname{proj} E'_{n,\beta^2}$ *is bornological, hence the linear bijection*

$$j : E'_{\beta^2} \longrightarrow \operatorname{proj} E'_{n,\beta^2}, \quad f \longmapsto (f|E_n)_{n\in\mathbb{N}}$$

is a homeomorphism.

Therefore, if the (LF)-space $E = \operatorname{ind} E_n$ is weakly acyclic and we consider the diagram

$$\begin{array}{ccc} E'_{\beta^2} & \xrightarrow{j} & \operatorname{proj} E'_{n,\beta^2} \\ \operatorname{id}\downarrow & & \operatorname{id}\downarrow \\ E'_{\beta} & \xrightarrow{j} & \operatorname{proj} E'_{n,\beta} \end{array}$$

of linear continuous maps, the upper horizontal arrow is a topological isomorphism. Moreover, if all E_n are distinguished, also the lower horizontal arrow is a topological isomorphism which means that $(E_n)_{n\in\mathbb{N}}$ is quasiregular. Thus the following result is an immediate consequence of Theorem A.

Corollary A (Vogt)**.** *A weakly acyclic (LF)-space E with distinguished steps is quasiregular.*

The distinguishedness hypothesis does not appear in the statement of [19, 4.2], but it is necessary to prove [18, Theorem 5.6] (see the introduction of [18, Chapter 4]). The problem remains open whether weakly acyclic (LF)-spaces are always quasiregular. In this paper we solve this question in the negative. Indeed we first give an elementary proof of Corollary A, which even provides additional information and a new insight into quasiregularity of weakly acyclic (LF)-spaces. Then we construct a non-quasiregular weakly acyclic (LF)-space.

We start with a technical lemma.

Lemma 1. *Let $(E_n)_{n\in\mathbb{N}}$ be an inductive sequence of metrizable locally convex spaces such that $E := \operatorname{ind} E_k$ is Hausdorff. For all $k, m \in \mathbb{N}$, $m \geq k$, let $j_{mk} : E_k \hookrightarrow E_m$ and $j_k : E_k \hookrightarrow E$ denote the natural inclusions. Moreover, let $n \in \mathbb{N}$ be given and an absolutely convex subset A of E_n such that*

$$\sigma(E_n, E'_n) \cap A = \sigma(E, E') \cap A.$$

Then

$$\overline{A}^E = \bigcup_{k\in\mathbb{N},\ k\geq n} \overline{A}^{E_k}$$

and there is $B \subset E''_n$, $A \subset B$, such that:

(i) *Each $\psi \in B$ is the $\sigma(E''_n, E'_n)$-limit of a sequence in A;*

(ii) *The bitranspose $j_n^{tt} : E''_n \to E''$ of j_n maps*

$$(B, \sigma(E''_n, E'_n) \cap B) \quad \textit{topologically onto} \quad (\overline{A}^E, \sigma(E, E') \cap \overline{A}^E).$$

Proof. Let $\overline{A}^n$ and $\overline{A}$ denote the closure of A in

$$(E'^*_n, \sigma(E'^*_n, E'_n)) =: (E'^*_n, \sigma'^*_n) \quad \text{and} \quad (E'^*, \sigma(E'^*, E')) =: (E'^*, \sigma'^*),$$

respectively. As $(\overline{A}^{n}, \sigma_n'^{*} \cap \overline{A}^{n})$ and $(\overline{A}, \sigma'^{*} \cap \overline{A})$ are Hausdorff completions of the uniform spaces $(A, \sigma(E_n, E_n') \cap A)$ and $(A, \sigma(E, E') \cap A)$, respectively, the bitranspose

$$j^{t*} : E_n'^{*} \longrightarrow E'^{*}, \quad \psi \mapsto (g \mapsto \psi(g \circ j_n))$$

induces a uniform equivalence from $(\overline{A}^{n}, \sigma_n'^{*} \cap \overline{A}^{n})$ onto $(\overline{A}, \sigma'^{*} \cap \overline{A})$. Let $\hat{A}^{n} := \{\psi \in \overline{A}^{n} : j_n^{t*}(\psi) \in E\}$. Then, clearly, j_n^{t*} induces a surjective uniform equivalence

$$\hat{j}_n : (\hat{A}^{n}, \sigma_n'^{*} \cap \hat{A}^{n}) \longrightarrow (\overline{A}, \sigma(E, E') \cap \overline{A}^{E}).$$

Analogously, for $m \geq n$, the bitranspose $j_{m,n}^{t*} : E_n'^{*} \to E_m'^{*}$ induces a surjective uniform equivalence $\hat{j}_{m,n} : (\hat{A}^n, \sigma_n'^{*} \cap \hat{A}^n) \to (\hat{A}^m, \sigma_m'^{*} \cap \hat{A}^m)$ and $\hat{j}_m \circ \hat{j}_{m,n} = \hat{j}_n$. Let $z \in \overline{A}^{E}$ be given. Then there is $m \in \mathbb{N}$, $m \geq n$ such that $z \in E_m$, and hence

$$z = (\hat{j}_m)^{-1}(z) \in \hat{A}^m \cap E_m \subset \overline{A}^{\,\sigma(E_m, E_m')} = \overline{A}^{E_m}.$$

Consequently, there is a sequence $(z_k)_{k\in\mathbb{N}}$ in A which converges to z in the Fréchet space topology of E_m, hence in $(\hat{A}^m, \sigma_m'^{*} \cap \hat{A}^m)$. Because of the properties of $\hat{j}_{m,n}$, the sequence $(z_k)_{k\in\mathbb{N}}$ converges to $(\hat{j}_n)^{-1}(z)$ in $(\hat{A}^n, \sigma_n'^{*} \cap \hat{A}^n)$. This implies $(\hat{j}_n)^{-1}(z) \in E_n''$, hence because of $\hat{j}_n(\hat{A}^n) = \overline{A}^{E}$, $\hat{A}^n \subset E_n''$.

Putting $B := \hat{A}^n$ finishes the proof of the lemma. □

Next we state and prove our main result about quasiregularity of weakly acyclic (LF)-spaces.

Theorem 2. *Let $E = \operatorname{ind} E_n$ be a weakly acyclic (LF)-space.*

(i) *For every separable bounded subset C of E there exists $r \in \mathbb{N}$ and a bounded subset B in E_r such that $C \subset \overline{B}^{E}$.*

(ii) *Let us assume that each E_n (or at least infinitely many of the E_n's) satisfies the following condition: For each bounded set $B \subset E_n''$, whose elements are $\sigma(E_n'', E_n')$-limits of sequences in E_n, there is a bounded set D in E_n such that B is contained in the $\sigma(E_n'', E_n')$-closure of D. Then the inductive sequence $(E_n)_{n\in\mathbb{N}}$ is quasiregular. In particular, weakly acyclic (LF)-spaces whose steps are distinguished or weakly sequentially complete, are quasiregular.*

Proof. Since $E = \operatorname{ind} E_n$ satisfies condition (M_0) of Retakh (see e.g. Vogt [19, Theorem 2.11]), we may assume without loss of generality that there exists an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex 0-nbhds U_n in E_n such that for all $n \in \mathbb{N}$ one has $\sigma(E_{n+1}, E_{n+1}') \cap U_n = \sigma(E, E') \cap U_n$. Now let $C \subset E$ be bounded. As E is in particular barrelled, a standard argument (see [16, 8.1.23]) shows that $C \subset n\overline{U}_n^{E}$ for a suitable $n \in \mathbb{N}$. Applying the lemma to $A := nU_n \subset E_{n+1}$, we obtain that there is a subset

$$D \subset \{\psi \in E_{n+1}'' : \ \psi \text{ is the } \sigma(E_{n+1}'', E_{n+1}')\text{-limit of a sequence in } A\}$$

such that the bitranspose $j_{n+1}^{tt} : E_{n+1}'' \longrightarrow E''$ of the inclusion $j_{n+1} : E_{n+1} \hookrightarrow E$ maps $(D, \sigma(E_{n+1}'', E_{n+1}') \cap D)$ topologically onto $(C, \sigma(E, E') \cap C)$. Clearly, D is bounded in E_{n+1}''.

If C is separable, also D is separable and there is $B \subset E_{n+1}$ bounded and absolutely convex such that D is contained in the $\sigma(E_{n+1}'', E_{n+1}')$-closure of B, whence $C \subset \overline{B}^{\sigma(E'', E')} \cap E = \overline{B}^{E}$. This proves (i).

If the hypothesis in (ii) is satisfied, then there is $B \subset E_{n+1}$ bounded and absolutely convex, such that $D \subset \overline{B}^{\sigma(E''_{n+1},E'_{n+1})}$ and we obtain $C \subset \overline{B}^{E}$. $\square$

The remainder of the note is devoted to show that the hypotheses of Theorem 2 cannot be dropped. We first give a technical result where conditions to construct a weakly acyclic not quasiregular (LF)-space are settled.

Proposition 3. *Assume that there exist Fréchet spaces Y, X with continuous inclusion $j : Y \hookrightarrow X$ satisfying the following three conditions:*
(i) *There is a 0-nbhd. U in Y such that $\sigma(Y, Y') \cap U = \sigma(X, X') \cap U$.*
(ii) *The bitranspose $j^{tt} : Y'' \to X''$ maps Y'' into X.*
(iii) *Y is not distinguished.*
Then there exists a weakly acyclic (LF)-space which is not quasiregular.

Proof. For every $n \in \mathbb{N}$,

$$E_n := \prod_{h<n} X \times c_0((Y)_{h\geq n}) := \{(x_h)_{h\in\mathbb{N}} \in \prod_{h<n} X \times \prod_{h\geq n} Y : \; x_h \overset{h\geq n}{\longrightarrow} 0 \text{ in } Y\}$$

provided with its natural Fréchet space topology is continuously included into E_{n+1}. The (LF)-space $E = \operatorname{ind} E_n$ is weakly acyclic by Müller-Dierolf-Frerick [15, Theorem (2)]. We will prove that $(E_n)_{n\in\mathbb{N}}$ is not quasiregular. In fact, since Y is not distinguished, there is a bounded set B in Y'' such that for all bounded subsets A in Y, B is not contained in $\overline{A}^{\sigma(Y'',Y')}$. We will construct a bounded set C in E for which the quasiregularity property fails. By hypothesis (ii) the set $C := \oplus_{h\in\mathbb{N}} j^{tt}(B)$ is a subset of E.

We first check that C is bounded in E. Let V be a closed 0-nbhd in E. Then there is an absolutely convex 0-nbhd W in Y such that $W^{\mathbb{N}} \cap c_0(Y) \subset V$. As B is $\sigma(Y'', Y')$-bounded, there is $\rho > 0$ such that $B \subset \rho \overline{W}^{\sigma(Y'',Y')}$ which implies

$$C = \oplus_{h\in\mathbb{N}} j^{tt}(B) \subset \rho \oplus_{h\in\mathbb{N}} j^{tt}(\overline{W}^{\sigma(Y'',Y')}) \subset_{(ii)} \rho \oplus_{h\in\mathbb{N}} \overline{W}^{X} \subset_{(*)} \rho \overline{\oplus_{h\in\mathbb{N}} W}^{E} \subset \rho V$$

where $(*)$ follows from the fact that the direct sum $\oplus_{h\in\mathbb{N}} X$ is continuously embedded into E. Now it remains to show that C is not contained in the E-closure of any bounded set of a step. Assuming the contrary, there is a bounded set A in Y and $n \in \mathbb{N}$ such that

$$C \subset \overline{(\prod_{h<n} X \times \prod_{h\geq n} A) \cap E_n}^{E}.$$

Looking at the n-th component one obtains that $j^{tt}(B) \subset \overline{A}^{X}$. On the other hand, by (i) and (ii) the bitranspose $j^{tt} : Y'' \to X$ maps $\overline{U}^{\sigma(Y'',Y')}$ provided with $\sigma(Y'', Y') \cap \overline{U}^{\sigma(Y'',Y')}$ topologically onto $(\overline{U}^{X}, \sigma(X, X') \cap \overline{U}^{X})$. In particular, j^{tt} is injective. Since A is absorbed by U, we may assume that $A \subset U$ and obtain that $j^{tt}(\overline{A}^{\sigma(Y'',Y')}) = \overline{A}^{\sigma(X,X')}$. From the above $j^{tt}(B) \subset \overline{A}^{X}$, we now deduce that $B \subset \overline{A}^{\sigma(Y'',Y')}$, which is a contradiction to the choice of B. $\square$

Observations 4. 1) The statement of Proposition 3 holds if Y, X are assumed to be just metrizable locally convex spaces. With the same proof we get a corresponding inductive sequence $(E_n)_{n\in\mathbb{N}}$ of metrizable locally convex spaces which is again weakly acyclic (by [15, Theorem (2)]) but not quasiregular. For this (less interesting) situation suitable entries $Y \hookrightarrow X$ are obtained quite easily. According to [4, p. 208], see also [1], there exists a reflexive Fréchet space X containing a dense linear subspace Y which is not distinguished. In that case the corresponding LM-space $(E_n)_{n\in\mathbb{N}}$ (of generalized Moscatelli type) is even strict without being quasiregular.

2) It is also easy to present a pair $Y \hookrightarrow X$ even of Banach spaces satisfying (i) and (ii) of the proposition: Take $Y := c_0$ and $X := \{(x_n)_{n\in\mathbb{N}} : (\frac{1}{n}x_n)_{n\in\mathbb{N}} \in c_0\}$. Then the Moscatelli construction leads to a weakly acyclic but not regular LB-space (see [15, p. 158]).

3) In order to find Fréchet entries $Y \hookrightarrow X$ satisfying (i), (ii), (iii) of Proposition 3, one must at least find a Fréchet space Y which is nondistinguished to such an extent that its bidual Y'' contains a bounded set whose elements are limits of weak Cauchy sequences in Y and which is not contained in the $\sigma(Y'', Y')$-closure of a bounded set in Y. A Fréchet space with that special property had been constructed by Díaz [7] . In fact, this space is weak*-sequentially dense in its bidual. An appropriate use of that construction leads to the following example:

Example 5. (Fréchet spaces $Y \hookrightarrow X$ satisfying (i), (ii) and (iii) of Proposition 3.) We start with some Banach background. The classical quasireflexive James space J is defined as

$$J := \{(x_i)_{i\geq 1}; \sup_{0=n_0<n_1<\cdots<n_k} \Big(\sum_{j=0}^{k-1}\Big(\sum_{i=n_j+1}^{n_{j+1}} x_i\Big)^2\Big)^{1/2} < \infty\}.$$

If e_i denotes the sequence taking the value 1 in the i-th component and 0 elsewhere then $(e_i)_{i\geq 1}$ is a boundedly complete basis (by basis we mean Schauder basis) of J. The dual space J' has a basis given by $(e_i^*)_{i\geq 0}$ where $(e_i^*)_{i\geq 1}$ is the sequence of biorthogonal coefficient functionals associated to $(e_i)_{i\geq 1}$ and e_0^* is defined by

$$e_0^*(\sum_{i=1}^{\infty} x_i e_i) := \sum_{i=1}^{\infty} x_i, \quad \forall\, x = (x_i) \in J.$$

(This element should be denoted e_ω^* but we prefer to use 0 instead of ω in our context.) Thus $J' \equiv \mathrm{sp}[e_0^*] \oplus \overline{\mathrm{sp}}[e_i^*;\ i \geq 1]$. The dual of $\overline{\mathrm{sp}}[e_i^*;\ i \geq 1]$ is J; therefore if e_0 denotes the element in J'' such that $e_0(e_i^*) = \delta_{0,i}$, $(i \geq 0)$, then $J'' \equiv \mathrm{sp}[e_0] \oplus J$ and $(e_i)_{i\geq 0}$ is a basis of J''. On account of this information if we consider the continuous linear map

$$\phi : J \to \mathbb{K} \oplus \ell_2, \quad (x_i)_{i\geq 1} \mapsto (\sum_{i=1}^{\infty} x_i, (x_i)_{i\geq 1}),$$

then an easy computation shows that the transpose ϕ^t and bitranspose ϕ^{tt} are given by

$$\phi^t : \mathbb{K} \oplus \ell_2 \to J', \quad (\alpha, (y_n)) \mapsto \alpha e_0^* + \sum_{n=1}^{\infty} y_n e_n^*,$$

$$\phi^{tt} : J'' \to \mathbb{K} \oplus \ell_2, \quad (x_n)_{n\geq 0} \mapsto (\sum_{n=0}^{\infty} x_n, (x_n)_{n\geq 1});$$

in particular ϕ^{tt} is injective.

We now come to the Fréchet space framework. Denote by Γ the set of all real valued increasing sequences $(\gamma(n))_{n\in\mathbb{N}}$ with $\gamma(1) \geq 1$. Given the space $\mathbb{K}^{\Gamma\times\mathbb{N}}$ and $k \in \mathbb{N}$, P_k denotes the canonical linear projection onto the $n \geq k$ rows. We also denote by P_k the projection in J defined by $P_k(\sum_{i=1}^{\infty} x_i e_i) := \sum_{i=k}^{\infty} x_i e_i$ $(x = (x_i) \in J)$; note that $\|P_k\| = 1$, $k \in \mathbb{N}$. The following space was introduced in [7] :

$$Y := \{(z_\gamma) = (z_{\gamma,j})_{\gamma,(j\geq 1)} \in J^\Gamma;$$
$$\|(z_\gamma)\|_k := \sum_{j=1}^{k-1} (\sum_\gamma |z_{\gamma,j}|^2 \gamma(j))^{1/2} + (\sum_\gamma \|P_k(z_\gamma)\|_J^2)^{1/2} < \infty,\ k \in \mathbb{N}\}.$$

Observe that the k-th local Banach space $Y_k := (Y/\|.\|_k^{-1}(0), \|.\|_k)\hat{}$ is isometric to

$$\ell_2(\Gamma, \gamma(1)) \oplus \cdots \oplus \ell_2(\Gamma, \gamma(k-1)) \oplus P_k(\ell_2(\Gamma, J))$$

where

$$\ell_2(\Gamma, \gamma(i)) := \{(x_\gamma) \in \mathbb{K}^\Gamma;\ \|(x_\gamma)\| = (\sum_{\gamma\in\Gamma} |x_\gamma|^2 \gamma(i))^{1/2} < \infty\},$$
$$\ell_2(\Gamma, J) := \{(z_\gamma) \in J^\Gamma;\ \|(z_\gamma)\| = (\sum_{\gamma\in\Gamma} \|z_\gamma\|_J^2)^{1/2} < \infty\}.$$

Moreover for every $k \in \mathbb{N}$ the linking map $I_{k,k+1} : Y_{k+1} \to Y_k$ is defined, when restricted to the k-th row, as the canonical continuous inclusion from $\ell_2(\Gamma, \gamma(k))$ into $(P_k - P_{k-1})(\ell_2(\Gamma, J))$ and it is the identity on the rest. The space $Y \equiv \operatorname{proj}\,(Y_k, I_{k,k+1})$ is not distinguished (see [7]) but every linear form defined on Y'_β and bounded on the bounded sets is continuous: In fact, if f is a linear form defined on Y' and continuous on Y'_k $(k \in \mathbb{N})$, there is countable subset $N \subset \Gamma$ such that, if $Y_N := \{(z_\gamma) \in Y :\ z_\gamma = 0 \text{ if } \gamma \notin N\}$ then f annihilates outside $(Y_N)'$. Now $(Y_N)'_\beta$ is separable (cf. [7, Theorem 5.(i)]) hence barrelled and bornological ([13, 29.3 and 29.4]). Thus f is continuous on $(Y_N)'_\beta$; since $(Y_N)'_\beta$ is complemented in Y'_β it follows that f is continuous. This in particular implies that the bidual of Y is $Y'' \equiv \operatorname{proj}(Y''_k, I^{tt}_{k+1,k})$. Also observe that $I^{tt}_{k+1,k}$ is injective for every $k \in \mathbb{N}$ whence the seminorm induced by Y''_1 is a norm on Y''.

We now construct the space X. First we need the following non-separable Köthe sequence space of order 2 on a Köthe matrix A implicitly defined below,

$$\lambda_2(\Gamma \times \mathbb{N}, A) := \{(x_{\gamma,n}) \in \mathbb{K}^{\Gamma\times\mathbb{N}};$$
$$\|(x_{\gamma,n})\|_k :=(\sum_{\gamma,(n<k)} |x_{\gamma,n}|^2 \gamma(n) + \sum_{\gamma,(n\geq k)} |x_{\gamma,n}|^2)^{1/2} < \infty,\ k \in \mathbb{N}\}.$$

Then we consider the space $X := \ell_2(\Gamma) \oplus \lambda_2(\Gamma \times \mathbb{N}, A)$ and define the following linear inclusion:

$$\Phi : Y \hookrightarrow X, \quad (x_\gamma) \mapsto ((\sum_{n=1}^{\infty} x_{\gamma,n})_\gamma, (x_{\gamma,n})_{\gamma,(n\geq 1)}).$$

Let us check that Φ is continuous.

$$\begin{aligned}\|\Phi(x)\|_k &= (\sum_\gamma (\sum_{n=1}^{\infty} x_{\gamma,n})^2)^{1/2} + (\sum_{j=1}^{k-1} (\sum_\gamma |x_{\gamma,j}|^2 \gamma(j)) + \sum_\gamma \sum_{j\geq k} |x_{\gamma,j}|^2)^{1/2} \\ &\leq (\sum_\gamma \|x_\gamma\|_J^2)^{1/2} + (\sum_{j=1}^{k-1} (\sum_\gamma |x_{\gamma,j}|^2 \gamma(j)) + \sum_\gamma \|P_k(x_\gamma)\|_J^2)^{1/2} \\ &\leq \|x\|_1 + T_k \|x\|_k\end{aligned}$$

for some constant T_k that only depends on k.

Condition (ii) of Proposition 3 is obvious and condition (iii) was proved in [7] as already stated. To show (i) we will check the following two properties:

(i′) Let U_1 denote the unit ball of $\|.\|_1$ in Y. If $B \subset U_1$ and $\Phi(B)$ is bounded in X then B is bounded in Y.

(i″) The bitranspose Φ^{tt} is injective.

(It is readily checked that (i′) + (i″) is equivalent to (i) with $U = U_1$.)

Proof of (i′). Take any $B \subset Y$ with $B \subset U_1$, such that $\Phi(B)$ is bounded, and let us show that B is bounded in Y. Indeed let $M_k := \sup \{\|\Phi(z)\|_k;\ z \in B\}$ and let us estimate $\|z\|_k$, $z = (z_\gamma) \in B$.

$$\begin{aligned}\|z\|_k &= \sum_{j=1}^{k-1}(\sum_\gamma |z_{\gamma,j}|^2\gamma(j))^{1/2} + (\sum_\gamma \|P_k(z_\gamma)\|_J^2)^{1/2}\\ &\le \|\Phi(z)\|_k + (\sum_\gamma \|z_\gamma\|_J^2)^{1/2}\\ &\le M_k + \|z\|_1 \le M_k + 1.\end{aligned}$$

□

Proof of (i″). From the proof of continuity it follows that Φ factorizes through the k-th local Banach spaces; in particular we have

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & X \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{\phi} & X_1 \end{array}$$

Bidualizing this diagram we get

$$\begin{array}{ccc} Y'' & \xrightarrow{\Phi^{tt}} & X \\ \downarrow & & \downarrow \\ Y_1'' & \xrightarrow{\phi^{tt}} & X_1 \end{array}$$

Since the first seminorm is a norm on Y'' it suffices to show that ϕ^{tt} is injective. In this case we have (recall the Banach case)

$$\phi^{tt} : \ell_2(\Gamma, J'') \to \ell_2(\Gamma) \oplus \ell_2(\Gamma \times \mathbb{N}),$$

$$(x_\gamma) = (x_{\gamma,n})_{\gamma,(n\ge 0)} \mapsto ((\sum_{n=0}^{\infty} x_{\gamma,n})_\gamma, (x_{\gamma,n})_{\gamma,(n\ge 1)}),$$

which is clearly injective. This finishes the construction of the example. □

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