ON DUALS OF WEAKLY ACYCLIC $(LF)$-SPACES

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(Communicated by Dale E. Alspach)

Abstract. For countable inductive limits of Fréchet spaces ($(LF)$-spaces) the
property of being weakly acyclic in the sense of Palamodov (or, equivalently,
having condition $(M_0)$ in the terminology of Retakh) is useful to avoid some im-
portant pathologies and in relation to the problem of well-located subspaces. In
this note we consider if weak acyclicity is enough for a $(LF)$-space $E := \text{ind} E_n$
to ensure that its strong dual is canonically homeomorphic to the projective
limit of the strong duals of the spaces $E_n$. First we give an elementary proof of
a known result by Vogt and obtain that the answer to this question is positive
if the steps $E_n$ are distinguished or weakly sequentially complete. Then we
construct a weakly acyclic $(LF)$-space for which the answer is negative.

Introduction

Countable inductive limits of Fréchet spaces ($(LF)$-spaces) arise in many fields
of functional analysis and its applications; e.g., in distribution theory, linear partial
differential equations, convolution equations, Fourier analysis, complex analysis.
Since their topological structure presents severe pathologies (see [11, 4], [9], [2, 3
and Appendix]) it is interesting to study those conditions which ensure a good
behaviour with respect to some particular problems. Here we are concerned with
deciding whether the property of being weakly acyclic in the sense of Palamodov
(equivalent to the condition $(M_0)$ of Retakh) is enough for a $(LF)$-space $E$
to identify (in a canonical way) its strong dual with a projective limit of $(DF)$-spaces.

Let us fix some notation before going on with the introduction. For a locally
convex space $E$ let $E'$ denote the topological dual and $E^*$ its algebraic dual. Given
a dual pair $\langle E, F \rangle$, let $\sigma(E, F)$ and $\beta(E, F)$ denote the corresponding weak
and strong topologies, respectively. The bidual of $E$ is $E'' := (E', \beta(E', E))'$,
moreover we set $E''_\beta := (E', \beta(E', E))$ and $E''_{\beta_2} := (E', \beta(E', E''))$ . The space $E$ is said to
be distinguished if every bounded subset of $(E'', \sigma(E'', E'))$ lies in the $\sigma(E'', E')$-
closure of some bounded subset of $E$. This is equivalent to $\beta(E', E) = \beta(E', E'')$; it
is also equivalent with requiring that $E''_\beta$ is barrelled ([13, 23.7]). If $E$ is metrizable
then it is distinguished if and only if $E''_\beta$ is bornological, cf. [13, 29.4].

A sequence $(E_n)_{n \in \mathbb{N}}$ of locally convex spaces is inductive if $E_n$ is continuously
included into $E_{n+1}$ $(n \in \mathbb{N})$; then $E := \text{ind} E_n$ is its inductive limit whenever
$E = \bigcup_{n \in \mathbb{N}} E_n$ and $E$ is endowed with the finest locally convex topology such that
the injections $E_n \rightarrow E$ are continuous. $E$ is an $(LF)$-space if the $E_n$’s are Fréchet.
(We refer to [2], [19] for more information about inductive limits.) Recall that $E := \text{ind } E_n$ is said to be weakly acyclic if the map
\[ \oplus E_n \to \oplus E_n, \quad (x_n)_{n \in \mathbb{N}} \mapsto (x_n - x_{n-1})_{n \in \mathbb{N}}, \quad (x_0 := 0) \]
is a weak topological isomorphism onto its range. Weak acyclicity is equivalent to the property $(M_n)$ introduced by Retakh in connection with the problem of well-located subspaces, which applies to the study of surjectivity and normal solvability of linear operators in duals of $(LF)$-spaces (in particular of linear partial differential operators in the space $\mathcal{D}'(\Omega)$ of distributions on $\Omega$); see [14], [2, Appendix] and [9] for details.

An important task on inductive limits, which goes back to Grothendieck, is the description of the strong dual of $E := \text{ind } E_n$ as the projective limit of the duals of $E_n \ (n \in \mathbb{N})$. Indeed, the canonical linear map
\[ j : E'_{\beta} \to \text{proj } E'_{n,\beta}, \quad f \mapsto (f|E_n)_{n \in \mathbb{N}} \]
(where the projective limit is formed with respect to the natural restrictions $E'_{n+1} \to E'_n$, $g \mapsto g|E_n$) is a continuous and surjective isomorphism, but $j$ does not need to be open. By a standard duality argument, $j$ is a topological isomorphism if and only if for every bounded subset $B$ in $E$ there is $n \in \mathbb{N}$ and a bounded subset $A$ in $E_n$ such that $B \subset \overline{A}^E$. We will call an inductive sequence $(E_n)_{n \in \mathbb{N}}$ quasiregular if it satisfies this (very weak regularity) condition. It is known that the inductive sequence $(E_n)_{n \in \mathbb{N}}$ is said to be regular if for every bounded subset $B$ in $E$ there is $n \in \mathbb{N}$ such that $B$ is contained and bounded in $E_n$. Regularity has been thoroughly studied in the theory of $(LF)$-spaces, cf. [2], [19].

Next we give examples and further information about quasiregular $(LF)$-spaces:
(i) Every inductive sequence consisting of $DF$-spaces $E_n \ (n \in \mathbb{N})$, in particular every $(LB)$-space, is quasiregular (by Grothendieck [10]).
(ii) If an inductive sequence $(E_n)_{n \in \mathbb{N}}$ consists of semireflexive spaces $E_n$ and if the inductive limit $E := \text{ind } E_n$ is Hausdorff, then $(E_n)_{n \in \mathbb{N}}$ is quasiregular if and only if it is regular. Consequently, an inductive sequence $(E_n)_{n \in \mathbb{N}}$ consisting of Fréchet Montel spaces with Hausdorff inductive limit is quasiregular if and only if it is complete (see Wengenroth [20, Theorem 3.9]).

(A very easy example of an incomplete $(LF)$-space with Fréchet Montel steps is obtained in the following way: Take $X := \omega := K^N$ (where $K \in \{\mathbb{R}, \mathbb{C}\}$) and let $Y$ be any Fréchet Montel space continuously embedded into $X$ with proper dense range. Then the inductive sequence $(\prod_{k<n} X \times \prod_{k \geq n} Y)_{n \in \mathbb{N}}$ consists of Fréchet Montel spaces and, according to Dierolf [8, Prop.], its inductive limit is topologically isomorphic to a proper dense linear subspace of $X^N$ hence incomplete, metrizable and provided with a weak topology.)
(iii) Quasiregular $(LF)$-spaces of generalized Moscatelli type of the shape $\oplus X + \ell^\infty(Y)$, where $Y$, $X$ are Fréchet spaces with continuous inclusion $Y \hookrightarrow X$, were characterized by Bonet-Dierolf-Fernández [6].

It was demonstrated by Vogt ([19, 5], see also [15] and [3] for related results) that weakly acyclic $(LF)$-spaces need not be regular. So it is natural to ask whether they are quasiregular. This question was somehow considered by Vogt in [18] and [19]. The following theorem is a direct consequence of Vogt’s Lemma 4.1 of [19], Theorem 5.6 of [18] and the fact that all $E'_{n,\beta}$ are bornological by [13, 29.5]. (Note that this theorem extends a Grothendieck’s result [12, 3.6, Theorem 2] asserting that a strict $(LF)$-space with distinguished steps is distinguished, from where it can be deduced that the space $\mathcal{D}'_{\beta}(\Omega)$ of distributions on $\Omega$ is barrelled and bornological. On the
other hand, we refer the reader to [4] for negative results about the topological structure of strong duals of strict \((L\phi)\)-spaces.

**Theorem A** (Vogt). Let \(E = \text{ind } E_n\) be a weakly acyclic \((L\phi)\)-space. Then the natural projective limit \(\text{proj } E_{n,\beta}^{r}\) is bornological, hence the linear bijection

\[ j : E_{\beta}^{r} \to \text{proj } E_{n,\beta}^{r}, \quad f \mapsto (f|_{E_{n}})_{n \in \mathbb{N}} \]

is a homeomorphism.

Therefore, if the \((L\phi)\)-space \(E = \text{ind } E_n\) is weakly acyclic and we consider the diagram

\[
\begin{array}{ccc}
E_{\beta}^{r} & \xrightarrow{j} & \text{proj } E_{n,\beta}^{r} \\
\downarrow \text{id} & & \downarrow \text{id} \\
E_{\beta} & \xrightarrow{j} & \text{proj } E_{n,\beta}^{r}
\end{array}
\]

of linear continuous maps, the upper horizontal arrow is a topological isomorphism. Moreover, if all \(E_n\) are distinguished, also the lower horizontal arrow is a topological isomorphism which means that \((E_n)_{n \in \mathbb{N}}\) is quasiregular. Thus the following result is an immediate consequence of Theorem A.

**Corollary A** (Vogt). A weakly acyclic \((L\phi)\)-space \(E\) with distinguished steps is quasiregular.

The distinguishedness hypothesis does not appear in the statement of [19, 4.2], but it is necessary to prove [18, Theorem 5.6] (see the introduction of [18, Chapter 4]). The problem remains open whether weakly acyclic \((L\phi)\)-spaces are always quasiregular. In this paper we solve this question in the negative. Indeed we first give an elementary proof of Corollary A, which even provides additional information and a new insight into quasiregularity of weakly acyclic \((L\phi)\)-spaces. Then we construct a non-quasiregular weakly acyclic \((L\phi)\)-space.

We start with a technical lemma.

**Lemma 1.** Let \((E_n)_{n \in \mathbb{N}}\) be an inductive sequence of metrizable locally convex spaces such that \(E := \text{ind } E_k\) is Hausdorff. For all \(k,m \in \mathbb{N}\), \(m \geq k\), let \(j_{mk} : E_k \hookrightarrow E_m\) and \(j_k : E_k \hookrightarrow E\) denote the natural inclusions. Moreover, let \(n \in \mathbb{N}\) be given and an absolutely convex subset \(A\) of \(E_n\) such that

\[
\sigma(E_n, E_n^{*}) \cap A = \sigma(E, E') \cap A.
\]

Then

\[
\overline{A}^{E} = \bigcup_{k \in \mathbb{N}, \ k \geq n} \overline{A}^{E_k}
\]

and there is \(B \subset E_n'^{r}, \ A \subset B\), such that:

(i) Each \(\psi \in B\) is the \(\sigma(E_n'^{r}, E_n')\)-limit of a sequence in \(A\);

(ii) The bitranspose \(j_{n}^{*} : E_{n}'' \to E^{''}\) of \(j_{n}\) maps

\[
(B, \sigma(E_{n}''', E_{n}') \cap B) \text{ topologically onto } (\overline{A}^{E}, \sigma(E, E') \cap \overline{A}^{E}).
\]

**Proof.** Let \(\overline{A}^{*}\) and \(\overline{A}\) denote the closure of \(A\) in

\[
(E_{n}^{*, \sigma(E_{n}', E_{n}')}) =: (E_{n}^{*, \sigma_{n}'}) \quad \text{and} \quad (E_{n}^{*, \sigma(E_{n}', E')}) =: (E_{n}^{*, \sigma'})
\]
respectively. As $(\overline{A}^n, \sigma^n \cap \overline{A}^n)$ and $(\overline{A}, \sigma^n \cap \overline{A})$ are Hausdorff completions of the uniform spaces $(A, \sigma(E_n, E'_n) \cap A)$ and $(A, \sigma(E, E') \cap A)$, respectively, the bitranspose
\[ j^* : E''_n \rightarrow E' ', \quad \psi \mapsto (g \mapsto \psi(g \circ j_n)) \]
induces a uniform equivalence from $(\overline{A}^n, \sigma^n \cap \overline{A}^n)$ onto $(\overline{A}, \sigma^n \cap \overline{A})$. Let $\hat{A}^n := \{ \psi \in \overline{A}^n : j^*_n(\psi) \in E \}$. Then, clearly, $j^*_n$ induces a surjective uniform equivalence
\[ \hat{j}_n : (\hat{A}^n, \sigma^n \cap \hat{A}^n) \rightarrow (\overline{A}, \sigma(E, E') \cap \overline{A}^E). \]
Analogously, for $m \geq n$, the bitranspose $\tilde{j}^*_n : E''_n \rightarrow E''_m$ induces a surjective uniform equivalence $\tilde{j}_m, n : (\hat{A}^m, \sigma^m \cap \hat{A}^m) \rightarrow (\hat{A}^n, \sigma^n \cap \hat{A}^n)$ and $\tilde{j}_m \circ \tilde{j}^*_n = \tilde{j}_n$. Let $z \in \overline{A}^E$ be given. Then there is $m \in \mathbb{N}$, $m \geq n$ such that $z \in E_m$, and hence
\[ z = (\tilde{j}_n)^{-1}(z) \in \hat{A}^m \cap E_m \subset \overline{A}^m = E''_m. \]
Consequently, there is a sequence $(z_k)_{k \in \mathbb{N}}$ in $A$ which converges to $z$ in the Fréchet space topology of $E_m$, hence in $\hat{A}^m$. Because of the properties of $\tilde{j}_m, n$, the sequence $(z_k)_{k \in \mathbb{N}}$ converges to $(\tilde{j}_n)^{-1}(z)$ in $(\hat{A}^n, \sigma^n \cap \hat{A}^n)$. This implies $(\tilde{j}_n)^{-1}(z) \in E''_n$, hence because of $\tilde{j}_n(\hat{A}^n) = \overline{A}^E$, $\hat{A}^n \subset E''_n$.

Putting $B := \hat{A}^n$ finishes the proof of the lemma. \hfill \Box

Next we state and prove our main result about quasiregularity of weakly acyclic (LF)-spaces.

**Theorem 2.** Let $E = \text{ind } E_n$ be a weakly acyclic (LF)-space.

(i) For every separable bounded subset $C$ of $E$ there exists $r \in \mathbb{N}$ and a bounded subset $B$ in $E_r$ such that $C \subset \overline{B}^E$.

(ii) Let us assume that each $E_n$ (or at least infinitely many of the $E_n$’s) satisfies the following condition: For each bounded set $B \subset E''_n$, whose elements are $\sigma(E''_n, E'_n)$-limits of sequences in $E_n$, there is a bounded set $D$ in $E_0$, such that $B$ is contained in the $\sigma(E''_n, E'_n)$-closure of $D$. Then the inductive sequence $(E_n)_{n \in \mathbb{N}}$ is quasiregular. In particular, weakly acyclic (LF)-spaces whose steps are distinguished or weakly sequentially complete, are quasiregular.

**Proof.** Since $E = \text{ind } E_n$, satisfies condition $(M_0)$ of Retakh (see e.g. Vogt [19, Theorem 2.11]), we may assume without loss of generality that there exists an increasing sequence $(U_n)_{n \in \mathbb{N}}$ of absolutely convex 0-nbds $U_n$ in $E_n$ such that for all $n \in \mathbb{N}$ one has $\sigma(E_{n+1}, E''_{n+1}) \cap U_n = \sigma(E, E') \cap U_n$. Now let $C \subset E$ be bounded. As $E$ is in particular barrelled, a standard argument (see [16, 8.1.23]) shows that $C \subset \overline{nU_n^E}$ for a suitable $n \in \mathbb{N}$. Applying the lemma to $A := nU_n \subset E_{n+1}$, we obtain that there is a subset
\[ D \subset \{ \psi \in E''_{n+1} : \psi \text{ is the } \sigma(E''_{n+1}, E''_{n+1})-\text{limit of a sequence in } A \} \]
such that the bitranspose $j^{tt}_{n+1} : E''_{n+1} \rightarrow E''$ of the inclusion $j_{n+1} : E_{n+1} \hookrightarrow E$ maps $(D, \sigma(E''_{n+1}, E''_{n+1}) \cap D)$ topologically onto $(C, \sigma(E, E') \cap C)$. Clearly, $D$ is bounded in $E''_{n+1}$.

If $C$ is separable, also $D$ is separable and there is $B \subset E_{n+1}$ bounded and absolutely convex such that $D$ is contained in the $\sigma(E''_{n+1}, E''_{n+1})$-closure of $B$, whence $C \subset \overline{B}^{\sigma(E''_n, E'_n)} \cap E = \overline{B}^E$. This proves (i).
If the hypothesis in (ii) is satisfied, then there is $B \subset E_{n+1}$ bounded and absolutely convex, such that $D \subset \overline{B}(E_{n+1}^{'},E_{n+1}^{'})$ and we obtain $C \subset \overline{B}^E$.  

The remainder of the note is devoted to show that the hypotheses of Theorem 2 cannot be dropped. We first give a technical result where conditions to construct a weakly acyclic not quasiregular $(LF)$-space are settled.

**Proposition 3.** Assume that there exist Fréchet spaces $Y$, $X$ with continuous inclusion $j : Y \hookrightarrow X$ satisfying the following three conditions:

(i) There is a 0-nbhd. $U$ in $Y$ such that $\sigma(Y,Y') \cap U = \sigma(X,X') \cap U$.

(ii) The bitranspose $j'' : Y'' \rightarrow X''$ maps $Y''$ into $X$.

(iii) $Y$ is not quasiregular.

Then there exists a weakly acyclic $(LF)$-space which is not quasiregular.

**Proof.** For every $n \in \mathbb{N}$,

$$E_n := \prod_{h \leq n} X \times c_0((Y)_{h \geq n}) := \{(x_h)_{h \in \mathbb{N}} \in \prod_{h \leq n} X \times \prod_{h \geq n} Y : x_h^{h \geq n}_0 \in Y\}$$

provided with its natural Fréchet space topology is continuously included into $E_{n+1}$. The $(LF)$-space $E = \text{ind} \ E_n$ is weakly acyclic by Müller-Dierolf-Frerick [15, Theorem (2)]. We will prove that $(E_n)_{n \in \mathbb{N}}$ is not quasiregular. In fact, since $Y$ is not distinguished, there is a bounded set $B$ in $Y''$ such that for all bounded subsets $A$ in $Y$, $B$ is not contained in $A^{\sigma(Y'',Y')}$.

We first check that $C$ is bounded in $E$. Let $V$ be a closed 0-nbhd in $E$. Then there is an absolutely convex 0-nbhd $W$ in $Y$ such that $W^N \cap c_0(Y) \subset V$. As $B$ is $\sigma(Y'',Y')$-bounded, there is $\rho > 0$ such that $B \subset \rho W^{\sigma(Y'',Y')}$ which implies

$$C = \bigoplus_{h \in \mathbb{N}} j''(B) \subset \bigoplus_{h \in \mathbb{N}} j''(W^{\sigma(Y'',Y')}) \subset C_{(ii)} \rho \bigoplus_{h \in \mathbb{N}} W^X \subset C_{(i)} \rho \bigoplus_{h \in \mathbb{N}} \overline{W}^E \subset \rho V$$

where $(*)$ follows from the fact that the direct sum $\bigoplus_{h \in \mathbb{N}} X$ is continuously embedded into $E$. Now it remains to show that $C$ is not contained in the $E$-closure of any bounded set of a step. Assuming the contrary, there is a bounded set $A$ in $Y$ and $n \in \mathbb{N}$ such that

$$C \subset \left(\prod_{h \leq n} X \times \prod_{h \geq n} A\right) \cap E_n^E.$$ 

Looking at the $n$-th component one obtains that $j''(B) \subset \overline{A}^X$. On the other hand, by (i) and (ii) the bitranspose $j'' : Y'' \rightarrow X$ maps $U^{\sigma(Y'',Y')}$ provided with $\sigma(Y'',Y') \cap U^{\sigma(Y'',Y')}$ topologically onto $(U^X, \sigma(X,X') \cap U^X)$. In particular, $j''$ is injective. Since $A$ is absorbed by $U$, we may assume that $A \subset U$ and obtain that $j''(\overline{A}^{\sigma(Y'',Y')}) = \overline{A}^{\sigma(X,X')}$. From the above $j''(B) \subset \overline{A}^X$, we now deduce that $B \subset \overline{A}^{\sigma(Y'',Y')}$, which is a contradiction to the choice of $B$.  

**Observations 4.** 1) The statement of Proposition 3 holds if $Y$, $X$ are assumed to be just metrizable locally convex spaces. With the same proof we get a corresponding inductive sequence $(E_n)_{n \in \mathbb{N}}$ of metrizable locally convex spaces which is again weakly acyclic (by [15, Theorem (2)]) but not quasiregular. For this (less interest-
ing) situation suitable entries \( Y \hookrightarrow X \) are obtained quite easily. According to [4, p. 208], see also [1], there exists a reflexive Fréchet space \( X \) containing a dense linear subspace \( Y \) which is not distinguished. In that case the corresponding \( LM \)-space \( (E_n)_{n \in \mathbb{N}} \) (of generalized Moscattelli type) is even strict without being quasiregular.

2) It is also easy to present a pair \( Y \hookrightarrow X \) even of Banach spaces satisfying (i) and (ii) of the proposition: Take \( Y := c_0 \) and \( X := \{ (x_n)_{n \in \mathbb{N}} : (\frac{1}{n} x_n)_{n \in \mathbb{N}} \in c_0 \} \). Then the Moscattelli construction leads to a weakly acyclic but not regular \( LB \)-space (see [15, p. 158]).

3) In order to find Fréchet entries \( Y \hookrightarrow X \) satisfying (i), (ii), (iii) of Proposition 3, one must at least find a Fréchet space \( Y \) which is nondistinguished to such an extent that its bidual \( Y''' \) contains a bounded set whose elements are limits of weak Cauchy sequences in \( Y \) and which is not contained in the \( \sigma(Y'',Y') \)-closure of a bounded set in \( Y \). A Fréchet space with that special property had been constructed by Díaz [7]. In fact, this space is weak*-sequentially dense in its bidual. An appropriate use of that construction leads to the following example:

**Example 5.** (Fréchet spaces \( Y \hookrightarrow X \) satisfying (i), (ii) and (iii) of Proposition 3.) We start with some Banach background. The classical quasireflexive James space \( J \) is defined as

\[
J := \{ (x_i)_{i \geq 1} : \sup_{0=0, n_1 < \cdots < n_k} \left( \sum_{j=0}^{k-1} \left( \sum_{i=n_j+1}^{n_{j+1}} x_i \right)^2 \right)^{1/2} < \infty \}.
\]

If \( e_i \) denotes the sequence taking the value 1 in the \( i \)-th component and 0 elsewhere then \( (e_i)_{i \geq 1} \) is a boundedly complete basis (by basis we mean Schauder basis) of \( J \). The dual space \( J' \) has a basis given by \( (e_i^*)_{i \geq 0} \) where \( (e_i^*)_{i \geq 1} \) is the sequence of bidualic coefficient functionals associated to \( (e_i)_{i \geq 1} \) and \( e_0^* \) is defined by

\[
e_0^* (\sum_{i=1}^\infty x_i e_i) := \sum_{i=1}^\infty x_i, \quad \forall x = (x_i) \in J.
\]

(This element should be denoted \( e_0^* \) but we prefer to use 0 instead of \( \omega \) in our context.) Thus \( J' \equiv \text{sp}\{e_0^* \} \oplus \text{sp}\{e_i^* : i \geq 1 \} \). The dual of \( \text{sp}\{e_i^* : i \geq 1 \} \) is \( J \); therefore if \( e_0 \) denotes the element in \( J'' \) such that \( e_0(e_i^*) = \delta_{0,i}, \quad (i \geq 0) \), then \( J'' \equiv \text{sp}\{e_0 \} \oplus J \) and \( (e_i)_{i \geq 0} \) is a basis of \( J'' \). On account of this information if we consider the continuous linear map

\[
\phi : J \rightarrow \mathbb{K} \oplus \ell_2, \quad (x_i)_{i \geq 1} \mapsto \left( \sum_{i=1}^\infty x_i, (x_i)_{i \geq 1} \right),
\]

then an easy computation shows that the transpose \( \phi^t \) and bitranspose \( \phi^{tt} \) are given by

\[
\phi^t : \mathbb{K} \oplus \ell_2 \rightarrow J', \quad (\alpha, (y_n)) \mapsto \alpha e_0^* + \sum_{n=1}^\infty y_n e_n^*,
\]

\[
\phi^{tt} : J'' \rightarrow \mathbb{K} \oplus \ell_2, \quad (x_n)_{n \geq 0} \mapsto \left( \sum_{n=0}^\infty x_n, (x_n)_{n \geq 1} \right);
\]

in particular \( \phi^{tt} \) is injective.

We now come to the Fréchet space framework. Denote by \( \Gamma \) the set of all real valued increasing sequences \( (\gamma(n))_{n \in \mathbb{N}} \) with \( \gamma(1) \geq 1 \). Given the space \( \mathbb{K}^{\Gamma \times \mathbb{N}} \) and \( k \in \mathbb{N} \), \( P_k \) denotes the canonical linear projection onto the \( n \geq k \) rows. We also
denote by $P_k$ the projection in $J$ defined by $P_k(\sum_{i=1}^{\infty} x_i e_i) := \sum_{i=k}^{\infty} x_i e_i$ ($x = (x_i) \in J$); note that $\|P_k\| = 1$, $k \in \mathbb{N}$. The following space was introduced in [7]:

$$Y := \{(z_\gamma) = (z_{\gamma,j})_{\gamma,j \geq 1} \in J^\Gamma;$$

$$(\|z_\gamma\|)_k := \left( \sum_{j=1}^{k-1} (\sum_{\gamma} |z_{\gamma,j}|^2 \gamma(j))^{1/2} + \sum_{\gamma} \|P_k(z_\gamma)\|_{\gamma,j}^{2/2} \right) < \infty, k \in \mathbb{N}.$$\]

Observe that the $k$-th local Banach space $Y_k := (Y/\|\cdot\|_k^{-1}(0), \|\cdot\|)$ is isometric to

$$\ell_2(\Gamma, \gamma(1)) \oplus \cdots \oplus \ell_2(\Gamma, \gamma(k-1)) \oplus P_k(\ell_2(\Gamma, J))$$

where

$$\ell_2(\Gamma, \gamma(i)) := \{(x_\gamma) \in \mathbb{K}^\Gamma; \|x_\gamma\| = \left( \sum_{\gamma \in \Gamma} |x_\gamma|^2 \gamma(i) \right)^{1/2} < \infty,$$

$$\ell_2(\Gamma, J) := \{(z_\gamma) \in J^\Gamma; \|z_\gamma\| = \left( \sum_{\gamma \in \Gamma} |z_\gamma|_{\gamma,j}^2 \right)^{1/2} < \infty.$$\]

Moreover for every $k \in \mathbb{N}$ the linking map $I_{k,k+1} : Y_{k+1} \to Y_k$ is defined, when restricted to the $k$-th row, as the canonical continuous inclusion from $\ell_2(\Gamma, \gamma(k))$ into $(P_k - P_{k-1})(\ell_2(\Gamma, J))$ and it is the identity on the rest. The space $Y \equiv \text{proj}(Y_k, I_{k,k+1})$ is not distinguished (see [7]) but every linear form defined on $Y'_{\beta}$ and bounded on the bounded sets is continuous: In fact, if $f$ is a linear form defined on $Y'$ and continuous on $Y'_{\beta}$, there is countable subset $N \subset \Gamma$ such that, if $Y_N := \{(z_\gamma) \in Y : z_\gamma = 0$ if $\gamma \notin N\}$ then $f$ annihilates outside $(Y_N)'$. Now $(Y_N)'_{\beta}$ is separable (cf. [7, Theorem 5.1(i)]) hence barrelled and bornological ([13, 29.3 and 29.4]). Thus $f$ is continuous on $(Y_N)'_{\beta}$; since $(Y_N)'_{\beta}$ is complemented in $Y'_{\beta}$ it follows that $f$ is continuous. This in particular implies that the bidual of $Y$ is $Y'' \equiv \text{proj}(Y'_{\beta}, I_{k,k+1}').$ Also observe that $I_{k,k+1}'$ is injective for every $k \in \mathbb{N}$ whence the seminorm induced by $Y''_{\beta}$ is a norm on $Y''.$

We now construct the space $X$. First we need the following non-separable Köthe sequence space of order 2 on a Köthe matrix $A$ implicitly defined below,

$$\lambda_2(\Gamma \times \mathbb{N}, A) := \{(x_{\gamma,n}) \in \mathbb{R}^{\Gamma \times \mathbb{N}};$$

$$(\|x_{\gamma,n}\|)_k := \left( \sum_{\gamma, (n < k)} |x_{\gamma,n}|^2 \gamma(n) + \sum_{\gamma, (n \geq k)} |x_{\gamma,n}|^2 \right)^{1/2} < \infty, k \in \mathbb{N}.$$\]

Then we consider the space $X := \ell_2(\Gamma) \oplus \lambda_2(\Gamma \times \mathbb{N}, A)$ and define the following linear inclusion:

$$\Phi : Y \hookrightarrow X, \quad (x_\gamma) \mapsto \left(\sum_{n=1}^{\infty} x_{\gamma,n} \right), (x_{\gamma,n} \gamma, (n \geq 1)).$$\]

Let us check that $\Phi$ is continuous.

$$\|\Phi(x)\|_k = \left( \sum_{\gamma} \left( \sum_{n=1}^{\infty} |x_{\gamma,n}|^2 \right)^{1/2} \right)^{1/2} + \left( \sum_{j=1}^{k-1} \left( \sum_{\gamma} |x_{\gamma,j}|^2 \gamma(j) \right)^{1/2} + \sum_{\gamma} \sum_{j \geq k} |x_{\gamma,j}|^2 \right)^{1/2}$$

$$\leq \left( \sum_{\gamma} |x_\gamma|_{\gamma,j}^2 \right)^{1/2} + \left( \sum_{j=1}^{k-1} \left( \sum_{\gamma} |x_{\gamma,j}|^2 \gamma(j) \right)^{1/2} + \sum_{\gamma} \|P_k(x_\gamma)\|_{\gamma,j}^{2/2} \right)^{1/2}$$

$$\leq \|x\|_1 + T_k \|x\|_k$$

for some constant $T_k$ that only depends on $k.$

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Condition (ii) of Proposition 3 is obvious and condition (iii) was proved in [7] as already stated. To show (i) we will check the following two properties:

(i′) Let $U_1$ denote the unit ball of $\|\cdot\|_1$ in $Y$. If $B \subset U_1$ and $\Phi(B)$ is bounded in $X$ then $B$ is bounded in $Y$.

(i′′) The bitranspose $\Phi^{tt}$ is injective.

(It is readily checked that (i′) + (i′′) is equivalent to (i) with $U = U_1$.)

**Proof of (i′).** Take any $B \subset Y$ with $B \subset U_1$, such that $\Phi(B)$ is bounded, and let us show that $B$ is bounded in $Y$. Indeed let $M_k := \sup \{\|\Phi(z)\|_k; z \in B\}$ and let us estimate $\|z\|_k, z = (z_γ) ∈ B$.

$$\|z\|_k = \sum_{j=1}^{k-1} \left( \sum_γ |z_{γ,j}|^2 \right)^{1/2} + \left( \sum_γ \|P_k(z_γ)\|_j^2 \right)^{1/2} \leq \|\Phi(z)\|_k + \left( \sum_γ \|z_γ\|^2_j \right)^{1/2} \leq M_k + \|z\|_1 \leq M_k + 1.$$ 

**Proof of (i′′).** From the proof of continuity it follows that $\Phi$ factorizes through the $k$-th local Banach spaces; in particular we have

$$\begin{align*}
Y & \xrightarrow{\Phi} X \\
\downarrow & \quad \downarrow \\
Y_1 & \xrightarrow{\phi} X_1
\end{align*}$$

Bidualizing this diagram we get

$$\begin{align*}
Y'' & \xrightarrow{\Phi^{tt}} X \\
\downarrow & \quad \downarrow \\
Y''_1 & \xrightarrow{\phi^{tt}} X_1
\end{align*}$$

Since the first seminorm is a norm on $Y''$ it suffices to show that $\phi^{tt}$ is injective. In this case we have (recall the Banach case)

$$\phi^{tt} : \ell_2(Γ, J'') \to \ell_2(Γ) \oplus \ell_2(Γ × N),$$

$$(x_γ) = (x_γ,n)_{γ,n ≥ 0} \mapsto \left( \left( \sum_{n=0}^{∞} x_γ,n \right)_γ, (x_γ,n)_γ, (n ≥ 1) \right),$$

which is clearly injective. This finishes the construction of the example. \hfill \Box

**Acknowledgements**

The authors are indebted to J. Bonet for many valuable comments, in particular he pointed out that a Fréchet space which is weak*-sequentially dense in its bidual need not be distinguished (as follows from results by Valdivia [17] and Díaz [7]) and indicated that the spaces introduced in [7] should be good candidates to construct the counterexample given at the end of this note. This result was obtained during a stay of the first author at the University of Trier. He gratefully thanks S. Dierolf and J. Wengenroth for their kind hospitality.
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