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WEAK TYPE BOUNDS FOR A CLASS OF ROUGH OPERATORS WITH POWER WEIGHTS

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Abstract. In this note we show that $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$, the fractional integral and maximal operators with rough kernel respectively, are bounded operators from $L^1(|x|^{\beta(n-\alpha)/n},\mathbb{R}^n)$ to $L^{n/(n-\alpha),\infty}(|x|^\beta,\mathbb{R}^n)$, where $0<\alpha<n$ and $-1<\beta<0$.

§1. Introduction

Suppose that $0<\alpha<n$, and $\Omega\in L^s(S^{n-1})$ $(s\ge 1)$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. Moreover, Ω is homogeneous of degree zero. We define the fractional maximal operator by

$$M_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\,dy$$

and the fractional integral operator by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy.$$

In 1971, B.Muckenhoupt and R.Wheeden [2] studied the weighted norm inequalities for $T_{\Omega,\alpha}$ with power weight. On the other hand, in 1993, S.Chanillo, D.Watson and R.Wheeden [1] proved that $T_{\Omega,\alpha}$ is of weak type $(1,n/(n-\alpha))$ under the restriction of $s\ge n/(n-\alpha)$. In this note , we use the idea in [3] to extend the result about weighted weak type $(1,n/(n-\alpha))$ for $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ to power weights.

Let us now formulate our results as follows.

Theorem 1. *Let $0<\alpha<n$, $-1<\beta<0$, $n/(n-\alpha)\le s\le\infty$ and $\Omega\in L^s(S^{n-1})$. Then $T_{\Omega,\alpha}$ is a bounded operator from $L^1(|x|^{\beta(n-\alpha)/n},\mathbb{R}^n)$ to $L^{n/(n-\alpha),\infty}(|x|^\beta,\mathbb{R}^n)$. That is , for any $\lambda>0$ and any $f\in L^1(|x|^{\beta(n-\alpha)/n},\mathbb{R}^n)$,*

$$\int_{\{x:|T_{\Omega,\alpha}f(x)|>\lambda\}}|x|^\beta\,dx\le C\left(\frac{1}{\lambda}\int_{\mathbb{R}^n}|f(x)||x|^{\beta(n-\alpha)/n}\,dx\right)^{n/(n-\alpha)},$$

where C is independent of λ and f.

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Theorem 2. *Let $0 < \alpha < n$, $-1 < \beta < 0$, $n/(n-\alpha) \le s \le \infty$ and $\Omega \in L^s(S^{n-1})$. Then $M_{\Omega,\alpha}$ is a bounded operator from $L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$ to $L^{n/(n-\alpha),\infty}(|x|^\beta, \mathbb{R}^n)$. That is, for any $\lambda > 0$ and any $f \in L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$,*

$$\int_{\{x:|M_{\Omega,\alpha}f(x)|>\lambda\}} |x|^\beta\,dx \le C\left(\frac{1}{\lambda}\int_{\mathbb{R}^n}|f(x)||x|^{\beta(n-\alpha)/n}\,dx\right)^{n/(n-\alpha)},$$

where C is independent of λ and f.

§2. Proof of the theorems

Let us begin by giving some lemmas.

Lemma 1. *Let $q > 1$ and T be a sublinear operator satisfying for each $a > 0$ the estimate*

(2.1)
$$\left|\left\{\frac{a}{2} \le |x| \le a : |T(f\chi_{\{|x|>2a\}})(x)| > \lambda\right\}\right| \le C\left(\frac{1}{\lambda}\int_{|y|>2a}|f(y)|(\frac{a}{|y|})^{1/q}\,dy\right)^q.$$

Then , if T is of weak type $(1,q)$, it is also of weak type $\left(L^1(|x|^{\beta/q}), L^{q,\infty}(|x|^\beta)\right)$ for $-1 < \beta < 0$.

Proof. Given f, we now define, for each $k \in \mathbb{Z}, f_{k,0} = f\chi_{\{|x|\le 2^{k+1}\}}$ and $f_{k,1} = f - f_{k,0}$. Then we can write, as usual,

$$|Tf(x)| \le \sum_k |Tf_{k,0}|\chi_{I_k} + \sum_k |Tf_{k,1}|\chi_{I_k} = T_0 f(x) + T_1 f(x),$$

where $I_k = \{x \in \mathbb{R}^n : 2^{k-1} \le |x| < 2^k\}$ for each $k \in \mathbb{Z}$.

If we call $\omega_\beta(x) = |x|^\beta$, we have

$$\begin{aligned}
\omega_\beta\{x : T_0 f > \lambda\} &= \int_{\{x:T_0 f>\lambda\}} \omega_\beta(x)\,dx \\
&\le C\sum_k \omega_\beta(2^k)|\{x \in I_k : T_0 f_{k,0} > \lambda\}| \\
&\le \frac{C}{\lambda^q}\sum_k \omega_\beta(2^k)\left(\int_{|y|\le 2^{k+1}}|f(y)|\,dy\right)^q \\
&= \frac{C}{\lambda^q}\sum_k \omega_\beta(2^k)\left(\sum_{j\le k+1}\int_{I_j}|f(y)|\,dy\right)^q \\
&\le \frac{C}{\lambda^q}\left\{\sum_j\left[\sum_{k\ge j-1}\left(\int_{I_j}|f(y)|\,dy\right)^q\omega_\beta(2^k)\right]^{1/q}\right\}^q \\
&= \frac{C}{\lambda^q}\left\{\sum_j\int_{I_j}|f(y)|\,dy\left[\sum_{k\ge j-1}\omega_\beta(2^k)\right]^{1/q}\right\}^q \\
&\le \frac{C}{\lambda^q}\left\{\sum_j\int_{I_j}|f(y)|\omega_\beta(2^j)^{1/q}\,dy\right\}^q \le C\left\{\frac{1}{\lambda}\int_{\mathbb{R}^n}|f(y)|\omega_\beta(y)^{1/q}\,dy\right\}^q.
\end{aligned}$$

Here we have used that T is a weak type $(1,q)$ bounded operator and $\beta<0$. In order to estimate T_1, we make use of (2.1):

$$\begin{aligned}
\omega_\beta\{x:T_1f>\lambda\} &= \int_{\{x:T_1f>\lambda\}}\omega_\beta(x)\,dx\\
&\le C\sum_k \omega_\beta(2^k)|\{x\in I_k : T_1 f_{k,1}>\lambda\}|\\
&= \frac{C}{\lambda^q}\sum_k \omega_\beta(2^k)\bigg(\int_{|y|>2^{k+1}}|f(y)|\Big(\frac{2^k}{|y|}\Big)^{1/q}dy\bigg)^q\\
&= \frac{C}{\lambda^q}\sum_k \omega_\beta(2^k)\,2^k\bigg(\sum_{j\ge k}\int_{I_j}|f(y)|\Big(\frac{1}{|y|}\Big)^{1/q}dy\bigg)^q\\
&\le \frac{C}{\lambda^q}\bigg\{\sum_j\bigg[\sum_{k\le j}\omega_\beta(2^k)2^k\bigg(\int_{I_j}|f(y)|(\frac{1}{|y|})^{1/q}\,dy\bigg)^q\bigg]^{1/q}\bigg\}^q\\
&= \frac{C}{\lambda^q}\bigg\{\sum_j\int_{I_j}|f(y)|(\frac{1}{|y|})^{1/q}\,dy\bigg[\sum_{k\le j}\omega_\beta(2^k)2^k\bigg]^{1/q}\bigg\}^q\\
&\le \frac{C}{\lambda^q}\bigg\{\sum_j\int_{I_j}|f(y)|\,dy\frac{1}{2^{j/q}}\omega_\beta(2^j)^{1/q}2^{j/q}\bigg\}^q\\
&\le C\bigg\{\frac{1}{\lambda}\int_{\mathbb{R}^n}|f(y)|\omega_\beta(y)^{1/q}\,dy\bigg\}^q,
\end{aligned}$$

where we have used that $\beta>-1$. This finishes the proof of Lemma 1. □

Lemma 2. *Let $0<\alpha<n,\Omega\in L^s(S^{n-1})$ and $s\ge 1$. Then there is a $C>0$ depending only on n and α, such that*

$$M_{\Omega,\alpha}f(x)\le CT_{|\Omega|,\alpha}(|f|)(x). \tag{2.2}$$

Proof. Fix $r>0$; then we have

$$\begin{aligned}
T_{|\Omega|,\alpha}(|f|)(x) &\ge \int_{|x-y|<r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy\\
&\ge \frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\,dy.
\end{aligned}\tag{2.3}$$

Taking the supremum for $r>0$ on two sides of (2.3), we get

$$T_{|\Omega|,\alpha}(|f|)(x)\ge \sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\,dy.$$

This is (2.2). □

Now, let us turn to the proof of the theorems. In the first we consider Theorem 1. Since $T_{\Omega,\alpha}$ is weak type $(1,n/(n-\alpha))$, by Lemma 1 we only need to show that

$T_{\Omega,\alpha}$ satisfies (2.1) for $q = n/(n-\alpha)$ and any $a > 0$. In fact,

$$
\begin{aligned}
&|\{\frac{a}{2} \le |x| \le a : |T_{\Omega,\alpha}(f\chi_{\{|x|>2a\}})(x)| > \lambda\}| \\
&\le \frac{1}{\lambda^q}\int_{|x|\le a} |T_{\Omega,\alpha}(f\chi_{\{|x|>2a\}})(x)|^q \, dx \\
&= \frac{1}{\lambda^q}\int_{|x|\le a} \left| \int_{|y|>2a} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy \right|^q dx \\
&\le \frac{1}{\lambda^q}\left\{ \int_{|y|>2a} |f(y)| \left(\int_{|x|\le a} \frac{|\Omega(x-y)|^q}{|x-y|^{(n-\alpha)q}}\, dx \right)^{1/q} dy \right\}^q \\
&\le \frac{1}{\lambda^q}\left\{ \int_{|y|>2a} |f(y)| \left(\int_{|x-y|\le a} \frac{|\Omega(x)|^q}{|x|^{(n-\alpha)q}}\, dx \right)^{1/q} dy \right\}^q \\
&\le \frac{C}{\lambda^q}\left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \left(\int_{|x-y|\le a} |\Omega(x)|^q \, dx \right)^{1/q} dy \right\}^q \\
&\le \frac{C}{\lambda^q}\left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \left(\int_{|y|-a}^{|y|+a} \int_{S^{n-1}} |\Omega(\theta)|^q \, d\theta r^{n-1}\, dr \right)^{1/q} dy \right\}^q \\
&\le \frac{C}{\lambda^q}\left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \|\Omega\|_q (a|y|^{n-1})^{1/q}\, dy \right\}^q \\
&= C\|\Omega\|_q^q \left\{ \frac{1}{\lambda} \int_{|y|>2a} |f(y)| (\frac{a}{|y|})^{1/q}\, dy \right\}^q,
\end{aligned}
$$

where $\|\Omega\|_q^q = \int_{S^{n-1}} |\Omega(\theta)|^q \, d\theta$. Thus, the conclusion of Theorem 1 immediately follows from Lemma 1.

It is easy to see that the conclusion of Theorem 2 is a direct consequence of Theorem 1 and Lemma 2.

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