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COMBINATORIAL ASPECTS OF F_σ FILTERS WITH AN APPLICATION TO $\mathcal{N}$-SETS

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Abstract. We discuss F_σ filters and show that the minimum size of a filter base generating an undiagonalizable filter included in some F_σ filter is the better known bounded evasion number $\mathfrak{e}_{ubd}$. An application to $\mathcal{N}$-sets from trigonometric series is given by showing that if A is an $\mathcal{N}$-set and B has size less than $\mathfrak{e}_{ubd}$, then $A \cup B$ is again an $\mathcal{N}$-set.

1. Introduction

Our terminology is standard but we review the main concepts and notation. The set of natural numbers will be denoted by ω, $\mathcal{P}(\omega)$ denotes the collection of all its subsets. Given a set X, we write $[X]^\omega$ and $[X]^{<\omega}$ to denote the collection of infinite or finite subsets of X respectively; if we wish to be more specific, we write $[X]^n$ and $[X]^{\leq n}$ to denote the collection of subsets of size n or at most n respectively. We use the well known 'almost inclusion' ordering between members of $[\omega]^\omega$, i.e. $X \subseteq^* Y$ if $X \setminus Y$ is finite. We identify $\mathcal{P}(\omega)$ with ${}^\omega 2$ via characteristic functions. The space ${}^\omega 2$ is further equipped with the product topology of the discrete space $\{0, 1\}$; a basic neighbourhood is then a set of the form

$$\mathcal{O}_s = \{f \in {}^\omega 2 : s \subseteq f\}$$

where $s \in {}^{<\omega}2$, the collection of finite binary sequences. The terms "nowhere dense", "meager", "Baire property" and "F_σ" all refer to this topology. We also write ${}^\omega\omega$ to denote all functions on the natural numbers. The ordering of eventual dominance is defined by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. Without further mention, terminology with respect to families of functions all refer to this ordering; in particular a family $\mathcal{H} \subseteq {}^\omega\omega$ is said to be *bounded* if it is bounded by a single function in this ordering.

A filter is a collection of subsets of ω containing all cofinite sets and closed under finite intersections and supersets. It is called proper if it does not contain the empty set; thus the collection of cofinite sets is the smallest proper filter, it is called the *Fréchet* filter and is denoted by $\mathfrak{Fr}$. To avoid trivialities, we shall assume that all filters under discussion are proper. An infinite set $X \in [\omega]^\omega$ is said to zap (or diagonalize) a filter $\mathcal{F}$ if $X \subseteq^* Y$ for each $Y \in \mathcal{F}$. Given a collection of sets

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$\mathcal{X} \subseteq [\omega]^\omega$, we denote by $\langle \mathcal{X} \rangle$ the filter generated by $\mathcal{X}$, that is, the smallest filter containing each member of $\mathcal{X}$.

The Katětov ordering on filters is defined by

$$\mathcal{F} \leq_K \mathcal{G} \text{ if } (\exists f \in {}^\omega\omega)\, \mathcal{G} \supseteq \{f^{-1}\{X\} : X \in \mathcal{F}\}.$$

The following Lemma from [9] combinatorially describes F_σ filters.

Lemma 1.1. *Let $\mathcal{F}$ be an F_σ filter and $g \in {}^\omega\omega$. Then there is an increasing sequence of natural numbers $\langle n_k : k \in \omega \rangle$ and sets $a_i^k \subseteq [n_k, n_{k+1})$, $i < m_k$, such that*

(1) $(\forall x \in [m_k]^{\leq g(k)})\ \bigcap_{i \in x} a_i^k \neq \emptyset$,
(2) $(\forall X \in \mathcal{F})(\forall^\infty k)(\exists i < m_k)\ a_i^k \subseteq X$.

Proof. Let $\mathcal{F} = \bigcup_n \mathcal{C}_n$ where each $\mathcal{C}_n$ is closed and put $\mathcal{C} = \{X \cup n : n \in \omega \text{ and } X \in \mathcal{C}_n\}$. Then again $\mathcal{C}$ is a closed set and every member of $\mathcal{F}$ is almost equal to a member of $\mathcal{C}$.

Let $n_0 = 0$ and having defined n_j for $j \leq k$, choose an $n_{k+1} > n_k$ such that

$$(\forall X_0, X_1, \cdots, X_{g(k)-1} \in \mathcal{C}) \bigcap_{i < g(k)} X_i \cap [n_k, n_{k+1}) \neq \emptyset.$$

The existence of such an n_{k+1} follows from the fact that $\mathcal{C}$ is closed and that $\mathcal{F}$ only contains infinite sets. Now enumerate $\{X \cap [n_k, n_{k+1}) : X \in \mathcal{C}\}$ as $\{a_i^k : i < m_k\}$ and this completes the proof. $\square$

It is worth noticing that conversely, given a family $\langle\langle a_i^k : i < m_k \rangle : k \in \omega, g\rangle$ satisfying conditions (1) and (2) above, then the collection

$$\{X : (\forall k)(\exists i < m_k)\ a_i^k \subseteq X\}$$

is a closed set generating an F_σ filter whenever $\lim_n g(n) = \infty$.

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2. F_σ FILTERS THAT CANNOT BE ZAPPED

We first present a combinatorial description of the smallest size of a family of sets generating a filter that cannot be zapped but which is included in some F_σ filter. This is a variation of some well known cardinals; indeed the cardinal $\mathfrak{p}$ is defined as the smallest size of a family of sets generating a filter that cannot be zapped and $\mathfrak{t}$ is defined as the smallest size of a well ordered (under almost inclusion) family of sets generating a filter that cannot be zapped. It turns out that these cardinals have a substantial impact on the set theory of the reals.

Definition 2.1.

$\mathfrak{f} = \min\{|\mathcal{X}| : \mathcal{X}$ generates a filter that cannot be zapped but which is included in some F_σ filter $\}$.

$\mathfrak{f}_1 = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega$ is bounded and for some $g \in {}^\omega\omega$ with $\lim_{n\to\infty} g(n) = \infty$, $(\forall X \in [\omega]^\omega)(\forall s_n \in [\omega]^{\leq g(n)})\ (\exists h \in \mathcal{H})(\exists^\infty n \in X)\ h(n) \notin s_n.\}$

$\mathfrak{f}_2 = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega$ is bounded and for some $g \in {}^\omega\omega$ with $\lim_{n\to\infty} g(n) = \infty$, $(\forall X \in [\omega]^\omega)(\forall \pi_n : {}^n\omega \to [\omega]^{\leq g(n)})\ (\exists h \in \mathcal{H})(\exists^\infty n \in X)\ h(n) \notin \pi_n(h \restriction n).\}$

$$\mathfrak{e}_{ubd} = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega \text{ is bounded and } (\forall X \in [\omega]^\omega)(\forall \pi_n : {}^n\omega \to \omega)\ (\exists h \in \mathcal{H})(\exists^\infty n \in X)\ h(n) \neq \pi_n(h \restriction n).\}$$

The cardinal $\mathfrak{e}_{ubd}$ is due to Brendle [5], and Brendle and Shelah [6]. Eisworth has shown (unpublished) that $\mathfrak{e}_{ubd} \leq \mathfrak{f}$ and an argument very similar to that of Blass in [4] shows that $\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$. We extend these results by showing that all four cardinals are equal.

Proposition 2.2. *The four cardinals $\mathfrak{f}$, $\mathfrak{f}_1$, $\mathfrak{f}_2$ and $\mathfrak{e}_{ubd}$ are equal.*

Proof. For simplicity, we prove $\mathfrak{f} \leq \mathfrak{f}_1 \leq \mathfrak{e}_{ubd} \leq \mathfrak{f}_2 \leq \mathfrak{f}_1 \leq \mathfrak{f}$.

$\mathfrak{f} \leq \mathfrak{f}_1$: Let $\mathcal{H} \subseteq {}^\omega\omega$ be given of size $|\mathcal{H}| < \mathfrak{f}$, without loss of generality bounded everywhere by $b \in {}^\omega\omega$, and fix $g \in {}^\omega\omega$ such that $\lim_n g(n) = \infty$.

Consider $\mathcal{A}_n = [b(n)]^{\leq g(n)}$ and for $i < b(n)$, put $a_i^n = \{x \in \mathcal{A}_n : i \in x\}$. Notice that

$$(\forall x \in \mathcal{A}_n = [b(n)]^{\leq g(n)}) \quad \bigcap_{i \in x} a_i^n \neq \emptyset.$$

Identify $\bigcup_n \mathcal{A}_n$ with ω, and form the filter $\mathcal{F}$ generated by

$$\{\bigcup_n a_{h(n)}^n : h \in \mathcal{H}\}.$$

Then $\mathcal{F}$ is a filter generated by fewer than $\mathfrak{f}$ sets and included in the F_σ filter $\langle\langle a_i^n : i < b(n)\rangle : n \in \omega, g\rangle$. Therefore $\mathcal{F}$ must be zapped, which means here that for some $X \in [\omega]^\omega$ and $x_n \in \mathcal{A}_n$ for $n \in X$, $\{x_n : n \in X\}$ zaps $\mathcal{F}$. In particular, for $h \in \mathcal{H}$,

$$\{x_n : n \in X\} \subseteq^* \{a_{h(n)}^n : n \in \omega\},$$

and thus $h(n) \in x_n$ for all but finitely many $n \in X$.

$\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$: Let $\mathcal{H} \subseteq {}^\omega\omega$ be given of size $|\mathcal{H}| < \mathfrak{f}_1$, without loss of generality bounded everywhere by $b \in {}^\omega\omega$. Partition ω into consecutive intervals $\langle \mathcal{I}_n = [a_n, a_{n+1}) : n \in \omega\rangle$ such that $a_{n+1} - a_n > n^2$.

For $h \in \mathcal{H}$, define $\tilde{h}(n) = h \restriction \mathcal{I}_n$ and form $\tilde{\mathcal{H}} = \{\tilde{h} : h \in \mathcal{H}\}$.

Identifying $\prod_{a_n \leq i < a_{n+1}} b(i)$ with its cardinality, we have that $\tilde{\mathcal{H}}$ is a bounded family of size $|\tilde{\mathcal{H}}| < \mathfrak{f}_1$. Therefore there are $\tilde{X} \in [\omega]^\omega$ and $s_n \in [\ \prod_{a_n \leq i < a_{n+1}} b(i)]^{\leq n}$ such that

$$(\forall \tilde{h} \in \tilde{\mathcal{H}})(\forall^\infty n \in \tilde{X})\ \ \tilde{h}(n) \in s_n.$$

Now by the pigeonhole principle, there must be for each $n \in \tilde{X}$ an $i_n \in \mathcal{I}_n$ such that

$$(*) \qquad (\forall t, t' \in s_n)\ \ t \restriction i_n = t' \restriction i_n \to t(i_n) = t'(i_n);$$

this is where we use the fact that $|\mathcal{I}_n| > n^2$ while $|s_n| \leq n$.

Let $X = \{i_n : n \in \tilde{X}\}$ and define $\pi_i : {}^i\omega \to \omega$ as follows. If $i = i_n \in X$ and $t \in {}^i\omega$ is such that $t \restriction [a_n, i)$ is an initial segment of a member t' of s_n, then define $\pi_i(t) = t'(i_n)$; this is well defined by the choice of i_n. In all other cases define $\pi_i(t)$ arbitrarily.

Now for $h \in \mathcal{H}$, $\tilde{h} \in s_n$ for all but finitely many $n \in \tilde{X}$; for each such n, $i = i_n \in X$, $h \restriction [a_n, i)$ is an initial segment of a member of s_n, namely $\tilde{h}(n) = h \restriction \mathcal{I}_n$, and thus $\pi_i(h \restriction i) = h(i)$. This proves that $\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$ as desired.

$\mathfrak{e}_{ubd} \leq \mathfrak{f}_2$: This inequality is trivial.

$\mathfrak{f}_2 \leq \mathfrak{f}_1$: Let $\mathcal{H} \subseteq {}^\omega\omega$ be given of size $|\mathcal{H}| < \mathfrak{f}_2$, without loss of generality bounded everywhere by $b \in {}^\omega\omega$, and fix $g \in {}^\omega\omega$ such that $\lim_n g(n) = \infty$.

Choose integers $\delta_0 = 0 < \delta_1 < \dots$ such that

$$(\forall n) \quad \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}).$$

Now for $n \in \omega$, define

$$\tilde{b}(n) = \prod_{\delta_n \leq i \leq \delta_{n+1}} b(i),$$

which we identify with the cartesian product. For $h \in \mathcal{H}$, define

$$\tilde{h}(n) = h \restriction [\delta_n, \delta_{n+1}] \in \tilde{b}(n)$$

and put $\tilde{\mathcal{H}} = \{\tilde{h}; h \in \mathcal{H}\}$, a bounded family of size less than $\mathfrak{f}_2$. Therefore

$$(\exists \tilde{X} \in [\omega]^\omega)(\exists \pi_n : {}^n\omega \to [\tilde{b}(n)]^{\leq g(\delta_n)})(\forall \tilde{h} \in \tilde{\mathcal{H}})(\forall^\infty n \in \tilde{X}) \;\; \tilde{h}(n) \in \pi_n(\tilde{h} \restriction n).$$

For $n \in \tilde{X}$, let

$$s_{\delta_{n+1}} = \{u(\delta_{n+1}) : u \in \pi_n(t) \text{ for } t \in \prod_{i<n} \tilde{b}(i)\}.$$

Then

$$|s_{\delta_{n+1}}| \leq \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}),$$

and so $s_{\delta_{n+1}} \in [b(\delta_{n+1})]^{\leq g(\delta_{n+1})}$. Finally, given $h \in \mathcal{H}$, and thus $\tilde{h} \in \tilde{\mathcal{H}}$,

$$(\forall^\infty n \in \tilde{X}) \;\; \tilde{h}(n) \in \pi_n(\tilde{h} \restriction n),$$

and so

$$(\forall^\infty n \in \tilde{X}) \;\; h(\delta_{n+1}) \in s_{\delta_{n+1}}.$$

Since g was arbitrary, $|\mathcal{H}| < \mathfrak{f}_1$ and we conclude that $\mathfrak{f}_2 \leq \mathfrak{f}_1$.

$\mathbf{\mathfrak{f}_1 \leq \mathfrak{f}}$: Let $\mathcal{F}$ be a filter generated by $\langle A_\alpha : \alpha < \kappa \rangle$, $\kappa < \mathfrak{f}_1$, and included in the F_σ filter $\langle \langle a_i^k : i < m_k \rangle : k \in \omega, g \rangle$.

For each $\alpha < \kappa$, define a function $f_\alpha \in {}^\omega\omega$ such that for all but finitely many k, $a_{f_\alpha(k)}^k \subseteq A_\alpha$.

Then $\{f_\alpha : \alpha < \kappa\}$ is a bounded family of size less than $\mathfrak{f}_1$, and therefore

$$(\exists X \in [\omega]^\omega)(\exists s_k \in [m_k]^{\leq g(k)})(\forall \alpha)(\forall^\infty k \in X) \;\; f_\alpha(k) \in s_k.$$

We conclude that $\bigcup_{k \in X} \bigcap_{i \in s_k} a_i^k$ zaps the filter $\mathcal{F}$. □

We conclude this section by giving a small perspective on these cardinals (see [12] for a description of cardinals not defined here). If one removes the boundedness restriction on $\mathcal{H}$, the cardinal $\mathfrak{e}_{ubd}$ becomes the evasion number, known as $\mathfrak{e}$ ([4]); clearly $\mathfrak{e} \leq \mathfrak{e}_{ubd}$ and it has been proved consistent by Shelah that $\mathfrak{e} < \mathfrak{e}_{ubd}$ [6]. Removing the boundedness restriction on $\mathcal{H}$ and fixing $X = \omega$, the cardinal $\mathfrak{f}_1$ is the additivity of measure, $add(\mathcal{N})$ ([1]); keeping the boundedness and using $X = \omega$ yields the so-called transitive additivity of measure (due to Pawlikovski), and finally removing the boundedness condition but keeping X arbitrary yields the cardinal $\mathfrak{s}e$ (see [4]). Thus we have $add(\mathcal{N}) \leq \mathfrak{e} \leq \mathfrak{e}_{ubd} = \mathfrak{f}$, and $\mathfrak{s}e, trans - add(\mathcal{N}) \leq \mathfrak{e}_{ubd}$.

For upper bounds, it is already known that $\mathfrak{e}_{ubd}$ is less than or equal to the uniformity of the null and meager ideals; these are easy to prove through the cardinal $\mathfrak{f}_1$.

It is also known that the cardinal $\mathfrak{e}_{ubd}$ is not provably equal to any of the standard cardinals $\mathfrak{b}, \mathfrak{d}, \mathfrak{t}$ or additivity, uniformity, cofinality and covering of the null or meager ideals (see [9], [11], [6]).

A different but provable lower bound however for the number $\mathfrak{e}_{ubd}$ (and thus $\mathfrak{f}$) is $\mathfrak{t}$; the idea of the proof is from [2]. We shall use the cardinal $\mathfrak{b}$, the minimum size of an unbounded family in ${}^\omega\omega$, and the well-known inequality $\mathfrak{t} \leq \mathfrak{b}$ (see [12]).

Brendle remarks that from $\mathfrak{t} \leq \mathfrak{e}_{ubd}$ and his result with Shelah ([5, 6]) that $\mathfrak{s}e = \min\{\mathfrak{e}, \mathfrak{b}\} = \min\{\mathfrak{e}_{ubd}, \mathfrak{b}\}$, one concludes that $\mathfrak{t} \leq \mathfrak{s}e$, thus improving Blass' result $\mathfrak{p} \leq \mathfrak{s}e$ [4]. Brendle also noticed that this can be proved directly from the following result as any family of functions of size less than $\mathfrak{t}$ is necessarily bounded.

Proposition 2.3. $\mathfrak{t} \leq \mathfrak{e}_{ubd} \leq 2^{\aleph_0}$.

Proof. We show that $\mathfrak{t} \leq \mathfrak{f}_1$. Let $\mathcal{H} = \langle h_\alpha : \alpha < \kappa \rangle$ be a family of size $\kappa < \mathfrak{t}$, bounded everywhere by $b \in {}^\omega\omega$, and fix $g \in {}^\omega\omega$ such that $\lim_n g(n) = \infty$.

We construct a sequence $\langle \phi_\alpha : \alpha \leq \kappa \rangle$ such that:

(1) $\phi_\alpha : dom(\phi_\alpha) \to [\omega]^{<\omega}$,
   - $dom(\phi_\alpha) \in [\omega]^\omega$,
   - $\phi_\alpha(k) \in [b(k)]^{\leq g(k)}$,
   - $\lim_{k \in dom(\phi_\alpha)} g(k) - |\phi_\alpha(k)| = +\infty$,

(2) $(\forall \beta < \alpha)\ dom(\phi_\alpha) \subseteq^* dom(\phi_\beta)$, and $(\forall^\infty k \in dom(\phi_\alpha))\ \phi_\beta(k) \subseteq \phi_\alpha(k)$,

(3) $(\forall \beta < \alpha)(\forall^\infty k \in dom(\phi_\alpha))\ h_\beta(k) \in \phi_\alpha(k)$.

Once we have obtained ϕ_κ, then clearly $s_k = \phi_\kappa(k)$ for $k \in dom(\phi_\kappa)$ is as desired.

Now to construct the sequence, assume that we already have $\{\phi_\beta : \beta < \alpha\}$ for some $\alpha \leq \kappa$.

If $\alpha = \beta+1$ is a successor ordinal, define $\phi_\alpha(k) = \phi_\beta(k) \cup \{h_\alpha(k)\}$ for $k \in dom(\phi_\beta)$ such that $|\phi_\beta(k)| < g(k)$.

For α a limit ordinal, first choose a $\tilde{g} \in {}^\omega\omega$ such that $\lim_k g(k) - \tilde{g}(k) = \infty$ and

$$(\forall \beta < \alpha)(\forall^\infty k \in dom(\phi_\beta))\ \ |\phi_\beta(k)| \leq \tilde{g}(k).$$

This is possible as $\alpha \leq \kappa < \mathfrak{t} \leq \mathfrak{b}$; I am not sure where this idea comes from.

Now for $\beta < \alpha$ define

$$A_\beta = \{\langle k, x \rangle : k \in dom(\phi_\beta) \text{ and } \phi_\beta(k) \subseteq x \subseteq b(k) \text{ and } |x| \leq \tilde{g}(k)\}.$$

Clearly $\beta < \gamma < \alpha \implies A_\gamma \subseteq^* A_\beta$ and, identifying A_0 with ω and as $\alpha < \mathfrak{t}$, we can find some infinite $A_\alpha \subseteq^* A_\beta$ for each $\beta < \alpha$. Finally we put

$$\phi_\alpha(k) = \text{any } x \text{ such that } \langle k, x \rangle \in A_\alpha$$

and undefined if there is no such x. Clearly ϕ_α is as desired. $\square$

3. Trigonometric series and $\mathcal{N}$-sets

In this section we prove that if A is an $\mathcal{N}$-set and $|B| < \mathfrak{e}_{ubd}$, then $A \cup B$ is also an $\mathcal{N}$-set. A similar result was earlier proved with $\mathfrak{p}$ in the role of $\mathfrak{e}_{ubd}$ in [7] on which we modeled our proof, and then with $\mathfrak{t}$ in [2] on which we modeled the above proof of $\mathfrak{t} \leq \mathfrak{e}_{ubd}$.

It is known that the collection of $\mathcal{N}$-sets is not in general closed under unions; in [3] two $\mathcal{N}$-sets of cardinality $\mathfrak{c}$ are constructed whose union is not an $\mathcal{N}$-set.

Definition 3.1. A set $A \subseteq \mathbb{R}$ is called an $\mathcal{N}$-set ([3]) if there is a sequence of non-negative reals $\langle a_n : n \in \omega \rangle$ such that:

$$\begin{array}{ll} (1) & \sum_{n=0}^\infty a_n = +\infty, \\ (2) & (\forall a \in A)\ \sum_{n=0}^\infty a_n |\sin \pi n a| < \infty. \end{array}$$

Proposition 3.2. *If $A \subseteq \mathbb{R}$ is an $\mathcal{N}$-set and $|B| < \mathfrak{e}_{ubd}$, then $A \cup B$ is also an $\mathcal{N}$-set.*

Proof. Fix a sequence of nonnegative reals $\langle a_n : n \in \omega \rangle$ as in the definition for the $\mathcal{N}$-set A. As is now standard procedure (see [7] and [2]), we put $s_n = \sum_{i=0}^{n} a_i$ and $b_n = a_n/s_n$; then again $\sum_{n=0}^{\infty} b_n = +\infty$.

As in [3], find an unbounded, nondecreasing sequence of natural numbers $\langle q_n : n \in \omega \rangle$ such that

$$\sum_{n=0}^{\infty} a_n / s_n^{1+\frac{1}{q_n}} < \infty;$$

as we may replace the sequence $\langle q_n : n \in \omega \rangle$ by any slower but unbounded and monotonic sequence, we may as well assume that $q_{\pi_n} \leq n$. Let $\epsilon_n = s_n^{-1/q_n}$.

Choose an increasing sequence of natural numbers $\langle \pi_n : n \in \omega \rangle$ such that

$$\begin{array}{ll} (1) & (\forall k)\ \sum_{i=\pi_k}^{\pi_{k+1}-1} b_i \geq 1, \\ (2) & (\forall n)(\forall m \geq \pi_n)\ q_m \geq n^2. \end{array}$$

Finally for $T \subseteq B$ and $m \in \omega$, define

$$a_T^m = \{k \in \omega : 0 \leq k \leq s_m \text{ and } (\forall t \in T)\ |\sin \pi k m t| \leq 2\pi\epsilon_m\},$$

and for each n, $m \geq \pi_n$ and $|T| \leq q_{\pi_n}/n$, put

$$b_T^m = \{a_{T'}^m : T \subseteq T' \text{ and } |T'| \leq q_{\pi_n}\}.$$

The following claim is from standard number theory and follows from the pigeon hole principle.

Claim 3.3. If $T \subseteq B$ and $|T| \leq q_{\pi_n}$, then $a_T^m \neq \emptyset$ for any $m \geq \pi_n$.

Proof. Fix n, m and we let $T = \{t_i : i < |T|\}$. Define a map

$$c : (1/\epsilon_m)^{|T|} + 1 \to \prod_{i \in |T|} 1/\epsilon_m$$

by $c(j) = (\ell_0, \cdots, \ell_{|T|-1})$ if

$$(\forall i < |T|)(\exists p \in \omega)\ p + \ell_i \epsilon_m \leq jmt_i < p + (\ell_i + 1)\epsilon_m.$$

There must then be two integers $j_1 < j_2 < (1/\epsilon_m)^{|T|} + 1$ such that $c(j_1) = c(j_2)$ and let $k = j_2 - j_1$.

Then $k \leq (1/\epsilon_m)^{|T|} \leq (1/\epsilon_m)^{q_{\pi_n}} \leq (1/\epsilon_m)^{q_m} = s_m$. Now using $[\![x]\!]$ to denote the distance from x to the nearest integer, we have, for $t \in T$,

$$|\sin \pi k m t| \leq \pi [\![kmt]\!] \leq 2\pi\epsilon_m$$

as desired. □

Therefore $a_{T'}^m \neq \emptyset$ for each $a_{T'}^m \in b_T^m$ and of course, if $T_i \subseteq B$, $i < n$, and each $|T_i| \leq q_{\pi_n}/n$, then $\bigcap_{i<n} b_{T_i}^m \neq \emptyset$ whenever $m \geq \pi_n$.

Now consider the set

$$\mathcal{W} = \{\langle a_{T_m}^m : \pi_n \leq m < \pi_{n+1} \rangle : n \in \omega \text{ and } |T_m| \leq q_{\pi_n}\},$$

which we may identify with ω since it is a countable set, and the filter

$$\begin{aligned} \mathcal{G} = \langle \{X \subseteq \mathcal{W} : (\forall n)(\forall \pi_n \leq m < \pi_{n+1}) \\ (\exists T_m \subseteq B)\ |T_m| \leq q_{\pi_n}/n \text{ and } \prod_{\pi_n \leq m < \pi_{n+1}} b_{T_n}^m \subseteq X\} \rangle. \end{aligned}$$

Then $\mathcal{G}$ is an F_σ filter as it is generated by a closed set and contains the filter

$$\mathcal{F} = \langle \{X \subseteq \mathcal{W} : (\exists T \subseteq B)\ |T| < \infty \\ \text{and for all } n \text{ such that } |T| \leq q_{\pi_n}/n, \prod_{\pi_n \leq m < \pi_{n+1}} b_T^m \subseteq X\}\rangle.$$

As $\mathcal{F}$ is generated by fewer than $\mathfrak{e}_{ubd} = \mathfrak{f}$ sets, it must be diagonalized by an infinite set $\mathcal{X} \subseteq \mathcal{W}$ which, without loss of generality, is of the form

$$\mathcal{X} = \{\langle a_{T_m}^m : \pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}\rangle : \ell \in \omega\}$$

where $|T_m| \leq q_{\pi_{n_\ell}}$ and $n_\ell < n_{\ell+1}$. For each ℓ and $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}$, pick $k_m \in a_{T_m}^m$.

Now clearly $\sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}} b_m = \infty$ and it remains to show that

$$\sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}} b_m |\sin mk_m \pi x| < \infty$$

for each $x \in A \cup B$. For $x \in A$,

$$b_m |\sin mk_m \pi x| \leq b_m k_m |\sin m\pi x| \leq b_m s_m |\sin m\pi x| = a_m |\sin m\pi x|.$$

Finally for $x \in B$, let $T = \{x\}$ and thus for all but finitely many ℓ, for all $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}$,

$$|\sin mk_m \pi x| \leq 2\pi\epsilon_m$$

and therefore

$$b_m |\sin mk_m \pi x| \leq 2\pi a_m / s_m^{1+1/q_m}.$$

This completes the proof. □

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