COMBINATORIAL ASPECTS OF $F_\sigma$ FILTERS WITH AN APPLICATION TO $\mathcal{N}$-SETS

CLAUDE LAFLAMME

(Communicated by Andreas R. Blass)

Abstract. We discuss $F_\sigma$ filters and show that the minimum size of a filter base generating an undiagonalizable filter included in some $F_\sigma$ filter is the better known bounded evasion number $e_{ubd}$. An application to $\mathcal{N}$-sets from trigonometric series is given by showing that if $A$ is an $\mathcal{N}$-set and $B$ has size less than $e_{ubd}$, then $A \cup B$ is again an $\mathcal{N}$-set.

1. Introduction

Our terminology is standard but we review the main concepts and notation. The set of natural numbers will be denoted by $\omega$, $\mathcal{P}(\omega)$ denotes the collection of all its subsets. Given a set $X$, we write $[X]^\omega$ and $[X]^{<\omega}$ to denote the collection of infinite or finite subsets of $X$ respectively; if we wish to be more specific, we write $[X]^n$ and $[X]^{\leq n}$ to denote the collection of subsets of size $n$ or at most $n$ respectively. We use the well known ‘almost inclusion’ ordering between members of $[\omega]^\omega$, i.e. $X \subset^* Y$ if $X \setminus Y$ is finite. We identify $\mathcal{P}(\omega)$ with $\omega^\omega$ via characteristic functions. The space $\omega^\omega$ is further equipped with the product topology of the discrete space $\{0, 1\}$; a basic neighbourhood is then a set of the form $\mathcal{O}_s = \{f \in \omega^\omega : s \subseteq f\}$ where $s \in <\omega^2$, the collection of finite binary sequences. The terms “nowhere dense”, “meager”, “Baire property” and “$F_\sigma$” all refer to this topology. We also write $\omega^\omega$ to denote all functions on the natural numbers. The ordering of eventual dominance is defined by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n$. Without further mention, terminology with respect to families of functions all refer to this ordering; in particular a family $\mathcal{H} \subseteq \omega^\omega$ is said to be bounded if it is bounded by a single function in this ordering.

A filter is a collection of subsets of $\omega$ containing all cofinite sets and closed under finite intersections and supersets. It is called proper if it does not contain the empty set; thus the collection of cofinite sets is the smallest proper filter, it is called the Fréchet filter and is denoted by $\mathcal{F}r$. To avoid trivialities, we shall assume that all filters under discussion are proper. An infinite set $X \in [\omega]^\omega$ is said to zap (or diagonalize) a filter $\mathcal{F}$ if $X \subseteq^* Y$ for each $Y \in \mathcal{F}$. Given a collection of sets
\( \mathcal{X} \subseteq [\omega]^\omega \), we denote by \( \langle \mathcal{X} \rangle \) the filter generated by \( \mathcal{X} \), that is, the smallest filter containing each member of \( \mathcal{X} \).

The Katětov ordering on filters is defined by
\[
\mathcal{F} \leq_K \mathcal{G} \text{ if } (\exists f \in {}^\omega \omega) \mathcal{G} \supseteq \{ f^{-1}\{X\} : X \in \mathcal{F} \}.
\]

The following Lemma from [9] combinatorially describes \( F_\sigma \) filters.

**Lemma 1.1.** Let \( \mathcal{F} \) be an \( F_\sigma \) filter and \( g \in {}^\omega \omega \). Then there is an increasing sequence of natural numbers \( \langle n_k : k \in \omega \rangle \) and sets \( a^k_i \subseteq [n_k, n_{k+1}) \), \( i < m_k \), such that
\[
\begin{align*}
(1) & \quad (\forall x \in [m_k]_{g(k)}) \bigcap_{i \in x} a^k_i \neq \emptyset, \\
(2) & \quad (\forall X \in \mathcal{F}) (\forall i \leq g(k)) (\exists i < m_k) a^k_i \subseteq X.
\end{align*}
\]

**Proof.** Let \( \mathcal{F} = \bigcup_n \mathcal{C}_n \) where each \( \mathcal{C}_n \) is closed and put \( \mathcal{C} = \{ X \cup n : n \in \omega \text{ and } X \in \mathcal{C}_n \} \). Then again \( \mathcal{C} \) is a closed set and every member of \( \mathcal{F} \) is almost equal to a member of \( \mathcal{C} \).

Let \( n_0 = 0 \) and having defined \( n_j \) for \( j \leq k \), choose an \( n_{k+1} > n_k \) such that
\[
(\forall X_0, X_1, \ldots, X_{g(k)-1} \in \mathcal{C}) \bigcap_{i < g(k)} X_i \cap [n_k, n_{k+1}) \neq \emptyset.
\]

The existence of such an \( n_{k+1} \) follows from the fact that \( \mathcal{C} \) is closed and that \( \mathcal{F} \) only contains infinite sets. Now enumerate \( \{ X \cap [n_k, n_{k+1}) : X \in \mathcal{C} \} \) as \( \{ a^k_i : i < m_k \} \) and this completes the proof. \( \Box \)

It is worth noticing that conversely, given a family \( \langle a^k_i : i < m_k \rangle : k \in \omega, g \rangle \) satisfying conditions (1) and (2) above, then the collection
\[
\{ X : (\forall k)(\exists i < m_k) a^k_i \subseteq X \}
\]
is a closed set generating an \( F_\sigma \) filter whenever \( \lim_n g(n) = \infty \).

We thank Andreas Blass, Jörg Brendle, Juris Steprāns and the referee for valuable comments on the paper.

2. \( F_\sigma \) Filters That Cannot Be Zapped

We first present a combinatorial description of the smallest size of a family of sets generating a filter that cannot be zapped but which is included in some \( F_\sigma \) filter. This is a variation of some well known cardinals; indeed the cardinal \( p \) is defined as the smallest size of a family of sets generating a filter that cannot be zapped and \( t \) is defined as the smallest size of a well ordered (under almost inclusion) family of sets generating a filter that cannot be zapped. It turns out that these cardinals have a substantial impact on the set theory of the reals.

**Definition 2.1.**
\[
f = \min\{ |\mathcal{X}| : \mathcal{X} \text{ generates a filter that cannot be zapped but which is included in some } \mathcal{F}_\sigma \text{ filter } \}.
\]

\[
f_1 = \min\{ |\mathcal{H}| : \mathcal{H} \subseteq {}^\omega \omega \text{ is bounded and for some } g \in {}^\omega \omega \text{ with } \lim_{n \to \infty} g(n) = \infty, \}
\]
\[
(\forall X \in [\omega]^\omega) (\forall s_n \in [\omega]^{\leq g(n)}) (\exists h \in \mathcal{H}) (\exists s_n \in X) h(n) \notin s_n, \}
\]

\[
f_2 = \min\{ |\mathcal{H}| : \mathcal{H} \subseteq {}^\omega \omega \text{ is bounded and for some } g \in {}^\omega \omega \text{ with } \lim_{n \to \infty} g(n) = \infty, \}
\]
\[
(\forall X \in [\omega]^\omega) (\forall \pi_n : \omega \to [\omega]^{\leq g(n)}) (\exists h \in \mathcal{H}) (\exists s_n \in X) h(n) \notin \pi_n(h \upharpoonright n). \}
\]
\[ \epsilon_{ubd} = \min\{|H| : \mathcal{H} \subseteq {}^{<\omega} \omega \text{ is bounded and } \langle \forall X \in [\omega]^i (\forall \pi_n : i \omega \to \omega) (\exists h \in \mathcal{H}) (\exists^\infty n \in X) h(n) \neq \pi_n(h \upharpoonright n) \rangle \}
\]

The cardinal \( \epsilon_{ubd} \) is due to Brendle [5], and Brendle and Shelah [6]. Eisworth has shown (unpublished) that \( \epsilon_{ubd} \leq f \) and an argument very similar to that of Blass in [4] shows that \( f_1 \leq \epsilon_{ubd} \). We extend these results by showing that all four cardinals are equal.

**Proposition 2.2.** The four cardinals \( f, f_1, f_2 \) and \( \epsilon_{ubd} \) are equal.

**Proof.** For simplicity, we prove \( f_2 \leq f_1 \leq \epsilon_{ubd} \leq f_2 \leq f_1 \leq f \).

\( f_1 \leq \epsilon_{ubd} \): Let \( \mathcal{H} \subseteq {}^{<\omega} \omega \) be given of size \( |\mathcal{H}| < f_1 \), without loss of generality bounded everywhere by \( b \in {}^{<\omega} \omega \), and fix \( g \in {}^{<\omega} \omega \) such that \( \lim_n g(n) = \infty \).

Consider \( A_n = [b(n)]^{\leq s(n)} \) and for \( i < b(n) \), put \( a^i_n = \{x \in A_n : i \in x\} \). Notice that
\[ (\forall x \in \mathcal{A}_n = [b(n)]^{\leq g(n)}) \cap_{i \in x} a^i_n \neq \emptyset. \]

Identify \( \bigcup_n A_n \) with \( \omega \), and form the filter \( \mathcal{F} \) generated by
\[ \{\bigcup_n a^i_n : h \in \mathcal{H}\}. \]

Then \( \mathcal{F} \) is a filter generated by fewer than \( f \) sets and included in the \( F_\sigma \) filter \( \langle \langle a^i_n : i < b(n) \rangle : n \in \omega, g \rangle \). Therefore \( \mathcal{F} \) must be zapped, which means here that for some \( X \in [\omega]^\omega \) and \( x_n \in A_n \) for \( n \in X \), \( \{x_n : n \in X\} \) zaps \( \mathcal{F} \). In particular, for \( h \in \mathcal{H} \),
\[ \{x_n : n \in X\} \subseteq^* \{a^i_n : n \in \omega\}, \]
and thus \( h(n) \in x_n \) for all but finitely many \( n \in X \).

\( f_1 \leq \epsilon_{ubd} \): Let \( \mathcal{H} \subseteq {}^{<\omega} \omega \) be given of size \( |\mathcal{H}| < f_1 \), without loss of generality bounded everywhere by \( b \in {}^{<\omega} \omega \). Partition \( \omega \) into consecutive intervals \( (I_n = [a_n, a_n+1) : n \in \omega) \) such that \( a_{n+1} - a_n > n^2 \).

For \( h \in \mathcal{H} \), define \( h(n) = h \upharpoonright I_n \) and form \( \mathcal{H} = \{h : h \in \mathcal{H}\} \).

Identifying \( \prod_{a_n \leq i < a_{n+1}} b(i) \) with its cardinality, we have that \( \mathcal{H} \) is a bounded family of size \( |\mathcal{H}| < f_1 \). Therefore there are \( X \in [\omega]^\omega \) and \( s_n \in \prod_{a_n \leq i < a_{n+1}} b(i) \) such that
\[ (\forall h \in \mathcal{H}) (\forall^\infty n \in X) \tilde{h}(n) \in s_n. \]

Now by the pigeonhole principle, there must be for each \( n \in X \) an \( i_n \in I_n \) such that
\[ (*): (\forall t, t' \in s_n) t \upharpoonright i_n = t' \upharpoonright i_n \rightarrow t(i_n) = t'(i_n); \]
this is where we use the fact that \( |I_n| > n^2 \) while \( |s_n| \leq n \).

Let \( X = \{i_n : n \in X\} \) and define \( \pi_i : i^1 \omega \to \omega \) as follows. If \( i = i_n \in X \) and \( t \in i^1 \omega \) is such that \( t \upharpoonright [a_n, i) \) is an initial segment of a member \( t' \) of \( s_n \), then define \( \pi_i(t) = t'(i_n) \); this is well defined by the choice of \( i_n \). In all other cases define \( \pi_i(t) \) arbitrarily.

Now for \( h \in \mathcal{H} \), \( \tilde{h} \in s_n \) for all but finitely many \( n \in X \): for each such \( n \), \( i = i_n \in X \), \( h \upharpoonright [a_n, i) \) is an initial segment of a member of \( s_n \), namely \( \tilde{h}(n) = h \upharpoonright I_n \), and thus \( \pi_i(h \upharpoonright i) = h(i) \). This proves that \( f_1 \leq \epsilon_{ubd} \) as desired.

\( \epsilon_{ubd} \leq f_2 \): This inequality is trivial.

\( f_2 \leq f_1 \): Let \( \mathcal{H} \subseteq {}^{<\omega} \omega \) be given of size \( |\mathcal{H}| < f_2 \), without loss of generality bounded everywhere by \( b \in {}^{<\omega} \omega \), and fix \( g \in {}^{<\omega} \omega \) such that \( \lim_n g(n) = \infty \).
Choose integers $\delta_0 = 0 < \delta_1 < \ldots$ such that
\[(\forall n) \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}). \]

Now for $n \in \omega$, define
\[
\tilde{b}(n) = \prod_{\delta_n \leq i \leq \delta_{n+1}} b(i),
\]
which we identify with the cartesian product. For $h \in \mathcal{H}$, define
\[
\tilde{h}(n) = h \upharpoonright [\delta_n, \delta_{n+1}] \in \tilde{b}(n)
\]
and put $\tilde{\mathcal{H}} = \{\tilde{h}; h \in \mathcal{H}\}$, a bounded family of size less than $f_2$. Therefore
\[
(\exists X \in [\omega]^\omega)(\exists \pi_n : \omega \rightarrow [b(n)] \leq g(\delta_n))(\forall \tilde{h} \in \tilde{\mathcal{H}})(\forall^* n \in X) \tilde{h}(n) \in \pi_n(\tilde{h} \upharpoonright n).
\]
For $n \in X$, let
\[
s_{\delta_{n+1}} = \{u(\delta_{n+1}) : u \in \pi_n(t) \text{ for } t \in \prod_{i \leq n} \tilde{b}(i)\}.
\]
Then
\[
|s_{\delta_{n+1}}| \leq \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}),
\]
and so $s_{\delta_{n+1}} \in [b(\delta_{n+1})] \leq g(\delta_{n+1})$. Finally, given $h \in \mathcal{H}$, and thus $\tilde{h} \in \tilde{\mathcal{H}}$,
\[(\forall^* n \in X) \tilde{h}(n) \in \pi_n(\tilde{h} \upharpoonright n),\]
and so
\[(\forall^* n \in X) h(\delta_{n+1}) \in s_{\delta_{n+1}}.
\]

Since $g$ was arbitrary, $|\tilde{\mathcal{H}}| < f_1$ and we conclude that $f_2 \leq f_1$.

$f_1 \leq f$: Let $\mathcal{F}$ be a filter generated by $\langle A_\alpha : \alpha < \kappa \rangle$, $\kappa < f_1$, and included in the $\mathcal{F}_\sigma$ filter $\langle \langle a_i^k : i < m_k \rangle : k \in \omega, g \rangle$.

For each $\alpha < \kappa$, define a function $f_\alpha \in [\omega]^\omega$ such that for all but finitely many $k$, $a_{f_\alpha(k)} \subseteq A_\alpha$.

Then $\{f_\alpha : \alpha < \kappa\}$ is a bounded family of size less than $f_1$, and therefore
\[(\exists X \in [\omega]^\omega)(\exists s_k \in [m_k] \leq g(k))(\forall^* \alpha)(\forall^* k \in X) f_\alpha(k) \in s_k.
\]
We conclude that $\bigcup_{k \in X} \bigcap_{i \in s_k} a_i^k$ zaps the filter $\mathcal{F}$.

We conclude this section by giving a small perspective on these cardinals (see [12] for a description of cardinals not defined here). If one removes the boundedness restriction on $\mathcal{H}$, the cardinal $\varepsilon_{ubd}$ becomes the evasion number, known as $\varepsilon$ ([4]); clearly $\varepsilon \leq \varepsilon_{ubd}$ and it has been proved consistent by Shelah that $\varepsilon < \varepsilon_{ubd}$ [6].

Removing the boundedness restriction on $\mathcal{H}$ and fixing $X = \omega$, the cardinal $f_1$ is the additivity of measure, $\text{add}(\mathcal{N})$ ([1]); keeping the boundedness and using $X = \omega$ yields the so-called transitive additivity of measure (due to Pawlikowski), and finally removing the boundedness condition but keeping $X$ arbitrary yields the cardinal $\varepsilon$ (see [4]). Thus we have $\text{add}(\mathcal{N}) \leq \varepsilon \leq \varepsilon_{ubd} = 1$, and $\varepsilon$, $\text{trans} - \text{add}(\mathcal{N}) \leq \varepsilon_{ubd}$.

For upper bounds, it is already known that $\varepsilon_{ubd}$ is less than or equal to the uniformity of the null and meager ideals; these are easy to prove through the cardinal $f_1$.

It is also known that the cardinal $\varepsilon_{ubd}$ is not provably equal to any of the standard cardinals $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{t}$ or additivity, uniformity, cofinality and covering of the null or meager ideals (see [9], [11], [6]).
A different but provable lower bound however for the number \( \varepsilon_{ubd} \) (and thus \( f \)) is \( t \); the idea of the proof is from [2]. We shall use the cardinal \( b \), the minimum size of an unbounded family in \( \omega^\omega \), and the well-known inequality \( t \leq b \) (see [12]).

Brendle remarks that from \( t \leq \varepsilon_{ubd} \) and his result with Shelah ([5, 6]) that \( se = \min\{t, b\} = \min\{\varepsilon_{ubd}, b\} \), one concludes that \( t \leq se \), thus improving Blass’ result \( p \leq se \) [4]. Brendle also noticed that this can be proved directly from the following result as any family of functions of size less than \( t \) is necessarily bounded.

**Proposition 2.3.** \( t \leq \varepsilon_{ubd} \leq 2^{\aleph_0} \).

**Proof.** We show that \( t \leq f_1 \). Let \( \mathcal{H} = \langle h_\alpha : \alpha < \kappa \rangle \) be a family of size \( \kappa < t \), bounded everywhere by \( b \in \omega^\omega \), and fix \( g \in \omega^\omega \) such that \( \lim_n g(n) = \infty \).

We construct a sequence \( \langle \phi_\alpha : \alpha \leq \kappa \rangle \) such that:

1. \( \phi_\alpha : \text{dom}(\phi_\alpha) \to [\omega \leftarrow \omega] \),
2. \( \text{dom}(\phi_\alpha) \subseteq [\omega \leftarrow \omega] \),
3. \( \lim_{k \in \text{dom}(\phi_\alpha)} g(k) - |\phi_\alpha(k)| = +\infty \),
4. \( (\forall \beta < \alpha) \text{dom}(\phi_\beta) \subseteq^* \text{dom}(\phi_\alpha) \), and \( (\forall \beta < \alpha) k \in \text{dom}(\phi_\alpha) \implies \phi_\beta(k) \subseteq \phi_\alpha(k) \),
5. \( (\forall \beta < \alpha) (\forall k \in \text{dom}(\phi_\beta)) \text{dom}(\phi_k) \subseteq \phi_\alpha(k) \).

Once we have obtained \( \phi_\kappa \), then clearly \( s_k = \phi_\kappa(k) \) for \( k \in \text{dom}(\phi_\kappa) \) is as desired. Now to construct the sequence, assume that we already have \( \langle \phi_\beta : \beta < \alpha \rangle \) for some \( \alpha \leq \kappa \).

If \( \alpha = \beta + 1 \) is a successor ordinal, define \( \phi_\alpha(k) = \phi_\beta(k) \cup \{h_\alpha(k)\} \) for \( k \in \text{dom}(\phi_\beta) \) such that \( |\phi_\beta(k)| < g(k) \).

For \( \alpha \) a limit ordinal, first choose a \( \tilde{g} \in \omega^\omega \) such that \( \lim_k g(k) - \tilde{g}(k) = \infty \) and \( (\forall \beta < \alpha) (\forall k \in \text{dom}(\phi_\beta)) |\phi_\beta(k)| \leq \tilde{g}(k) \).

This is possible as \( \alpha \leq \kappa < t \leq b \); I am not sure where this idea comes from.

Now for \( \beta < \alpha \) define

\[
A_\beta = \{ (k, x) : k \in \text{dom}(\phi_\beta) \text{ and } \phi_\beta(k) \subseteq x \subseteq b(k) \text{ and } |x| \leq \tilde{g}(k) \}.
\]

Clearly \( \beta < \gamma < \alpha \implies A_\gamma \subseteq^* A_\beta \) and, identifying \( A_0 \) with \( \omega \) and as \( \alpha < t \), we can find some infinite \( A_\alpha \subseteq^* A_\beta \) for each \( \beta < \alpha \). Finally we put

\[
\phi_\alpha(k) = \text{any } x \text{ such that } (k, x) \in A_\alpha
\]

and undefined if there is no such \( x \). Clearly \( \phi_\alpha \) is as desired. \( \square \)

### 3. Trigonometric series and \( \mathcal{N} \)-sets

In this section we prove that if \( A \) is an \( \mathcal{N} \)-set and \( |B| < \varepsilon_{ubd} \), then \( A \cup B \) is also an \( \mathcal{N} \)-set. A similar result was earlier proved with \( p \) in the role of \( \varepsilon_{ubd} \) in [7] on which we modeled our proof, and then with \( t \) in [2] on which we modeled the above proof of \( t \leq \varepsilon_{ubd} \).

It is known that the collection of \( \mathcal{N} \)-sets is not in general closed under unions; in [3] two \( \mathcal{N} \)-sets of cardinality \( \varepsilon \) are constructed whose union is not an \( \mathcal{N} \)-set.

**Definition 3.1.** A set \( A \subseteq \mathbb{R} \) is called an \( \mathcal{N} \)-set ([3]) if there is a sequence of non-negative reals \( \langle a_n : n \in \omega \rangle \) such that:

1. \( \sum_{n=0}^{\infty} a_n = +\infty \),
2. \( (\forall a \in A) \sum_{n=0}^{\infty} a_n |\sin \pi na| < \infty \).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proposition 3.2. If \( A \subseteq \mathbb{R} \) is an \( \mathcal{N} \)-set and \( |B| < \varepsilon_{abd} \), then \( A \cup B \) is also an \( \mathcal{N} \)-set.

Proof. Fix a sequence of nonnegative reals \( \langle a_n : n \in \omega \rangle \) as in the definition for the \( \mathcal{N} \)-set \( A \). As is now standard procedure (see [7] and [2]), we put \( s_n = \sum_{i=0}^n a_i \) and \( b_n = a_n/s_n \); then again \( \sum_{n=0}^\infty b_n = +\infty \).

As in [3], find an unbounded, nondecreasing sequence of natural numbers \( \langle q_n : n \in \omega \rangle \) such that

\[
\sum_{n=0}^\infty a_n/s_n^{1+\frac{1}{q_n^2}} < \infty;
\]

as we may replace the sequence \( \langle q_n : n \in \omega \rangle \) by any slower but unbounded and monotonic sequence, we may as well assume that \( q_{\pi_n} \leq n \). Let \( \varepsilon_n = s_n^{-1/q_n} \).

Choose an increasing sequence of natural numbers \( \langle \pi_n : n \in \omega \rangle \) such that

\[
(1) \quad (\forall k) \sum_{i=\pi_n}^{\pi_{n+1}-1} b_i \geq 1,
(2) \quad (\forall m)(\forall n \geq \pi_n) q_m \geq n^2.
\]

Finally for \( T \subseteq B \) and \( m \in \omega \), define

\[ a^n_T = \{ k \in \omega : 0 \leq k \leq s_m \text{ and } (\forall t \in T) |\sin \pi km| \leq 2\pi \varepsilon_m \}, \]

and for each \( n, m \geq \pi_n \) and \( |T| \leq q_{\pi_n}/n \), put

\[ b^n_T = \{ a^n_T : T \subseteq T' \text{ and } |T'| \leq q_{\pi_n} \}. \]

The following claim is from standard number theory and follows from the pigeonhole principle.

Claim 3.3. If \( T \subseteq B \) and \( |T| \leq q_{\pi_n} \), then \( a^n_T \neq \emptyset \) for any \( m \geq \pi_n \).

Proof. Fix \( n, m \) and we let \( T = \{ t_i : i < |T| \} \). Define a map

\[ c : (1/\varepsilon_m)^{|T|} + 1 \rightarrow \prod_{i \in |T|} 1/\varepsilon_m \]

by \( c(j) = (\ell_0, \cdots, \ell_{|T|-1}) \) if

\[ (\forall i < |T|)(\exists p \in \omega) p + \ell_i \varepsilon_m \leq j m \ell_i < p + (\ell_i + 1) \varepsilon_m. \]

There must then be two integers \( j_1 < j_2 < (1/\varepsilon_m)^{|T|} + 1 \) such that \( c(j_1) = c(j_2) \) and let \( k = j_2 - j_1 \).

Then \( k \leq (1/\varepsilon_m)^{|T|} \leq (1/\varepsilon_m)^{q_{\pi_n}} \leq (1/\varepsilon_m)^{q_m} = s_m \). Now using \( [x] \) to denote the distance from \( x \) to the nearest integer, we have, for \( t \in T \),

\[ |\sin \pi km| \leq \pi[km] \leq 2\pi \varepsilon_m \]

as desired. \( \square \)

Therefore \( a^n_{T_i} \neq \emptyset \) for each \( a^n_{T_i} \in b^n_{T_i} \) and of course, if \( T_i \subseteq B, i < n, \) and each \( |T_i| \leq q_{\pi_n}/n, \) then \( \bigcap_{i<n} b^n_{T_i} \neq \emptyset \) whenever \( m \geq \pi_n \).

Now consider the set

\[ W = \{ \langle a^n_{T_m} : \pi_n \leq m < \pi_{n+1} \rangle : n \in \omega \text{ and } |T_m| \leq q_{\pi_n} \}, \]

which we may identify with \( \omega \) since it is a countable set, and the filter

\[ \mathcal{G} = \{ (X \subseteq \omega) : (\forall \pi_n \leq m < \pi_{n+1}) (\exists T_m \subseteq B) |T_m| \leq q_{\pi_n}/n \text{ and } \prod_{\pi_n \leq m < \pi_{n+1}} b^n_{T_m} \subseteq X \}. \]
Then $G$ is an $F_{\sigma}$ filter as it is generated by a closed set and contains the filter
\[ F = \{ \{ X \subseteq W : (\exists T \subseteq B) |T| < \infty \} \mid |T| \leq q_{\pi_n}/n, \prod_{\pi_n \leq \pi_{n+1}} b_T^m \subseteq X \}. \]

As $F$ is generated by fewer than $\epsilon_{abd} = f$ sets, it must be diagonalized by an infinite set $X \subseteq W$ which, without loss of generality, is of the form
\[ X = \{ \langle a_{T_m}^m : \pi_{n_1} \leq m < \pi_{n_{1+1}} \rangle : \ell \in \omega \} \]
where $|T_m| \leq q_{\pi_{n_\ell}}$ and $n_\ell < n_{\ell+1}$. For each $\ell$ and $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}}$, pick $k_m \in a_{T_m}^m$.

Now clearly $\sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}}} b_m = \infty$ and it remains to show that
\[ \sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}}} b_m |\sin mk_m \pi x| < \infty \]
for each $x \in A \cup B$. For $x \in A$,
\[ b_m |\sin mk_m \pi x| \leq b_m k_m |\sin m \pi x| \leq b_m s_m |\sin m \pi x| = a_m |\sin m \pi x|. \]
Finally for $x \in B$, let $T = \{ x \}$ and thus for all but finitely many $\ell$, for all $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}}$,
\[ |\sin mk_m \pi x| \leq 2\pi \epsilon_m \]
and therefore
\[ b_m |\sin mk_m \pi x| \leq 2\pi a_m / s_m^{1+1/q_m}. \]
This completes the proof. \qed

References


Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4
E-mail address: laflamme@acs.ucalgary.ca