

## COMBINATORIAL ASPECTS OF $F_\sigma$ FILTERS WITH AN APPLICATION TO $\mathcal{N}$ -SETS

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ABSTRACT. We discuss  $F_\sigma$  filters and show that the minimum size of a filter base generating an undiagonalizable filter included in some  $F_\sigma$  filter is the better known bounded evasion number  $\epsilon_{ubd}$ . An application to  $\mathcal{N}$ -sets from trigonometric series is given by showing that if  $A$  is an  $\mathcal{N}$ -set and  $B$  has size less than  $\epsilon_{ubd}$ , then  $A \cup B$  is again an  $\mathcal{N}$ -set.

### 1. INTRODUCTION

Our terminology is standard but we review the main concepts and notation. The set of natural numbers will be denoted by  $\omega$ ,  $\mathcal{P}(\omega)$  denotes the collection of all its subsets. Given a set  $X$ , we write  $[X]^\omega$  and  $[X]^{<\omega}$  to denote the collection of infinite or finite subsets of  $X$  respectively; if we wish to be more specific, we write  $[X]^n$  and  $[X]^{\leq n}$  to denote the collection of subsets of size  $n$  or at most  $n$  respectively. We use the well known ‘almost inclusion’ ordering between members of  $[\omega]^\omega$ , i.e.  $X \subseteq^* Y$  if  $X \setminus Y$  is finite. We identify  $\mathcal{P}(\omega)$  with  ${}^\omega 2$  via characteristic functions. The space  ${}^\omega 2$  is further equipped with the product topology of the discrete space  $\{0, 1\}$ ; a basic neighbourhood is then a set of the form

$$\mathcal{O}_s = \{f \in {}^\omega 2 : s \subseteq f\}$$

where  $s \in {}^{<\omega} 2$ , the collection of finite binary sequences. The terms “nowhere dense”, “meager”, “Baire property” and “ $F_\sigma$ ” all refer to this topology. We also write  ${}^\omega \omega$  to denote all functions on the natural numbers. The ordering of eventual dominance is defined by  $f \leq^* g$  if  $f(n) \leq g(n)$  for all but finitely many  $n$ . Without further mention, terminology with respect to families of functions all refer to this ordering; in particular a family  $\mathcal{H} \subseteq {}^\omega \omega$  is said to be *bounded* if it is bounded by a single function in this ordering.

A filter is a collection of subsets of  $\omega$  containing all cofinite sets and closed under finite intersections and supersets. It is called proper if it does not contain the empty set; thus the collection of cofinite sets is the smallest proper filter, it is called the *Fréchet* filter and is denoted by  $\mathfrak{F}r$ . To avoid trivialities, we shall assume that all filters under discussion are proper. An infinite set  $X \in [\omega]^\omega$  is said to *zap* (or *diagonalize*) a filter  $\mathcal{F}$  if  $X \subseteq^* Y$  for each  $Y \in \mathcal{F}$ . Given a collection of sets

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$\mathcal{X} \subseteq [\omega]^\omega$ , we denote by  $\langle \mathcal{X} \rangle$  the filter generated by  $\mathcal{X}$ , that is, the smallest filter containing each member of  $\mathcal{X}$ .

The Katětov ordering on filters is defined by

$$\mathcal{F} \leq_K \mathcal{G} \text{ if } (\exists f \in {}^\omega\omega) \mathcal{G} \supseteq \{f^{-1}\{X\} : X \in \mathcal{F}\}.$$

The following Lemma from [9] combinatorially describes  $F_\sigma$  filters.

**Lemma 1.1.** *Let  $\mathcal{F}$  be an  $F_\sigma$  filter and  $g \in {}^\omega\omega$ . Then there is an increasing sequence of natural numbers  $\langle n_k : k \in \omega \rangle$  and sets  $a_i^k \subseteq [n_k, n_{k+1})$ ,  $i < m_k$ , such that*

- (1)  $(\forall x \in [m_k]^{\leq g(k)}) \bigcap_{i \in x} a_i^k \neq \emptyset$ ,
- (2)  $(\forall X \in \mathcal{F})(\forall^\infty k)(\exists i < m_k) a_i^k \subseteq X$ .

*Proof.* Let  $\mathcal{F} = \bigcup_n \mathcal{C}_n$  where each  $\mathcal{C}_n$  is closed and put  $\mathcal{C} = \{X \cup n : n \in \omega \text{ and } X \in \mathcal{C}_n\}$ . Then again  $\mathcal{C}$  is a closed set and every member of  $\mathcal{F}$  is almost equal to a member of  $\mathcal{C}$ .

Let  $n_0 = 0$  and having defined  $n_j$  for  $j \leq k$ , choose an  $n_{k+1} > n_k$  such that

$$(\forall X_0, X_1, \dots, X_{g(k)-1} \in \mathcal{C}) \bigcap_{i < g(k)} X_i \cap [n_k, n_{k+1}) \neq \emptyset.$$

The existence of such an  $n_{k+1}$  follows from the fact that  $\mathcal{C}$  is closed and that  $\mathcal{F}$  only contains infinite sets. Now enumerate  $\{X \cap [n_k, n_{k+1}) : X \in \mathcal{C}\}$  as  $\{a_i^k : i < m_k\}$  and this completes the proof.  $\square$

It is worth noticing that conversely, given a family  $\langle \langle a_i^k : i < m_k \rangle : k \in \omega, g \rangle$  satisfying conditions (1) and (2) above, then the collection

$$\{X : (\forall k)(\exists i < m_k) a_i^k \subseteq X\}$$

is a closed set generating an  $F_\sigma$  filter whenever  $\lim_n g(n) = \infty$ .

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## 2. $F_\sigma$ FILTERS THAT CANNOT BE ZAPPED

We first present a combinatorial description of the smallest size of a family of sets generating a filter that cannot be zapped but which is included in some  $F_\sigma$  filter. This is a variation of some well known cardinals; indeed the cardinal  $\mathfrak{p}$  is defined as the smallest size of a family of sets generating a filter that cannot be zapped and  $\mathfrak{t}$  is defined as the smallest size of a well ordered (under almost inclusion) family of sets generating a filter that cannot be zapped. It turns out that these cardinals have a substantial impact on the set theory of the reals.

**Definition 2.1.**

$\mathfrak{f} = \min\{|\mathcal{X}| : \mathcal{X} \text{ generates a filter that cannot be zapped but which is included in some } F_\sigma \text{ filter}\}.$

$$\mathfrak{f}_1 = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega \text{ is bounded and for some } g \in {}^\omega\omega \text{ with } \lim_{n \rightarrow \infty} g(n) = \infty, (\forall X \in [\omega]^\omega)(\forall s_n \in [\omega]^{\leq g(n)}) (\exists h \in \mathcal{H})(\exists^\infty n \in X) h(n) \notin s_n.\}$$

$$\mathfrak{f}_2 = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega \text{ is bounded and for some } g \in {}^\omega\omega \text{ with } \lim_{n \rightarrow \infty} g(n) = \infty, (\forall X \in [\omega]^\omega)(\forall \pi_n : {}^n\omega \rightarrow [\omega]^{\leq g(n)}) (\exists h \in \mathcal{H})(\exists^\infty n \in X) h(n) \notin \pi_n(h \upharpoonright n).\}$$

$$\mathfrak{e}_{ubd} = \min\{|\mathcal{H}| : \mathcal{H} \subseteq {}^\omega\omega \text{ is bounded and } (\forall X \in [\omega]^\omega)(\forall \pi_n : {}^n\omega \rightarrow \omega) (\exists h \in \mathcal{H})(\exists^\infty n \in X) h(n) \neq \pi_n(h \upharpoonright n)\}.$$

The cardinal  $\mathfrak{e}_{ubd}$  is due to Brendle [5], and Brendle and Shelah [6]. Eisworth has shown (unpublished) that  $\mathfrak{e}_{ubd} \leq \mathfrak{f}$  and an argument very similar to that of Blass in [4] shows that  $\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$ . We extend these results by showing that all four cardinals are equal.

**Proposition 2.2.** *The four cardinals  $\mathfrak{f}$ ,  $\mathfrak{f}_1$ ,  $\mathfrak{f}_2$  and  $\mathfrak{e}_{ubd}$  are equal.*

*Proof.* For simplicity, we prove  $\mathfrak{f} \leq \mathfrak{f}_1 \leq \mathfrak{e}_{ubd} \leq \mathfrak{f}_2 \leq \mathfrak{f}_1 \leq \mathfrak{f}$ .

$\mathfrak{f} \leq \mathfrak{f}_1$ : Let  $\mathcal{H} \subseteq {}^\omega\omega$  be given of size  $|\mathcal{H}| < \mathfrak{f}$ , without loss of generality bounded everywhere by  $b \in {}^\omega\omega$ , and fix  $g \in {}^\omega\omega$  such that  $\lim_n g(n) = \infty$ .

Consider  $\mathcal{A}_n = [b(n)]^{\leq g(n)}$  and for  $i < b(n)$ , put  $a_i^n = \{x \in \mathcal{A}_n : i \in x\}$ . Notice that

$$(\forall x \in \mathcal{A}_n = [b(n)]^{\leq g(n)}) \bigcap_{i \in x} a_i^n \neq \emptyset.$$

Identify  $\bigcup_n \mathcal{A}_n$  with  $\omega$ , and form the filter  $\mathcal{F}$  generated by

$$\left\{ \bigcup_n a_{h(n)}^n : h \in \mathcal{H} \right\}.$$

Then  $\mathcal{F}$  is a filter generated by fewer than  $\mathfrak{f}$  sets and included in the  $F_\sigma$  filter  $\langle \langle a_i^n : i < b(n) \rangle : n \in \omega, g \rangle$ . Therefore  $\mathcal{F}$  must be zapped, which means here that for some  $X \in [\omega]^\omega$  and  $x_n \in \mathcal{A}_n$  for  $n \in X$ ,  $\{x_n : n \in X\}$  zaps  $\mathcal{F}$ . In particular, for  $h \in \mathcal{H}$ ,

$$\{x_n : n \in X\} \subseteq^* \{a_{h(n)}^n : n \in \omega\},$$

and thus  $h(n) \in x_n$  for all but finitely many  $n \in X$ .

$\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$ : Let  $\mathcal{H} \subseteq {}^\omega\omega$  be given of size  $|\mathcal{H}| < \mathfrak{f}_1$ , without loss of generality bounded everywhere by  $b \in {}^\omega\omega$ . Partition  $\omega$  into consecutive intervals  $\langle \mathcal{I}_n = [a_n, a_{n+1}) : n \in \omega \rangle$  such that  $a_{n+1} - a_n > n^2$ .

For  $h \in \mathcal{H}$ , define  $\tilde{h}(n) = h \upharpoonright \mathcal{I}_n$  and form  $\tilde{\mathcal{H}} = \{\tilde{h} : h \in \mathcal{H}\}$ .

Identifying  $\prod_{a_n \leq i < a_{n+1}} b(i)$  with its cardinality, we have that  $\tilde{\mathcal{H}}$  is a bounded family of size  $|\tilde{\mathcal{H}}| < \mathfrak{f}_1$ . Therefore there are  $\tilde{X} \in [\omega]^\omega$  and  $s_n \in [\prod_{a_n \leq i < a_{n+1}} b(i)]^{\leq n}$  such that

$$(\forall \tilde{h} \in \tilde{\mathcal{H}})(\forall^\infty n \in \tilde{X}) \tilde{h}(n) \in s_n.$$

Now by the pigeonhole principle, there must be for each  $n \in \tilde{X}$  an  $i_n \in \mathcal{I}_n$  such that

$$(*) \quad (\forall t, t' \in s_n) t \upharpoonright i_n = t' \upharpoonright i_n \rightarrow t(i_n) = t'(i_n);$$

this is where we use the fact that  $|\mathcal{I}_n| > n^2$  while  $|s_n| \leq n$ .

Let  $X = \{i_n : n \in \tilde{X}\}$  and define  $\pi_i : {}^i\omega \rightarrow \omega$  as follows. If  $i = i_n \in X$  and  $t \in {}^i\omega$  is such that  $t \upharpoonright [a_n, i)$  is an initial segment of a member  $t'$  of  $s_n$ , then define  $\pi_i(t) = t'(i_n)$ ; this is well defined by the choice of  $i_n$ . In all other cases define  $\pi_i(t)$  arbitrarily.

Now for  $h \in \mathcal{H}$ ,  $\tilde{h} \in s_n$  for all but finitely many  $n \in \tilde{X}$ ; for each such  $n$ ,  $i = i_n \in X$ ,  $h \upharpoonright [a_n, i)$  is an initial segment of a member of  $s_n$ , namely  $\tilde{h}(n) = h \upharpoonright \mathcal{I}_n$ , and thus  $\pi_i(h \upharpoonright i) = h(i)$ . This proves that  $\mathfrak{f}_1 \leq \mathfrak{e}_{ubd}$  as desired.

$\mathfrak{e}_{ubd} \leq \mathfrak{f}_2$ : This inequality is trivial.

$\mathfrak{f}_2 \leq \mathfrak{f}_1$ : Let  $\mathcal{H} \subseteq {}^\omega\omega$  be given of size  $|\mathcal{H}| < \mathfrak{f}_2$ , without loss of generality bounded everywhere by  $b \in {}^\omega\omega$ , and fix  $g \in {}^\omega\omega$  such that  $\lim_n g(n) = \infty$ .

Choose integers  $\delta_0 = 0 < \delta_1 < \dots$  such that

$$(\forall n) \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}).$$

Now for  $n \in \omega$ , define

$$\tilde{b}(n) = \prod_{\delta_n \leq i \leq \delta_{n+1}} b(i),$$

which we identify with the cartesian product. For  $h \in \mathcal{H}$ , define

$$\tilde{h}(n) = h \upharpoonright [\delta_n, \delta_{n+1}] \in \tilde{b}(n)$$

and put  $\tilde{\mathcal{H}} = \{\tilde{h}; h \in \mathcal{H}\}$ , a bounded family of size less than  $\mathfrak{f}_2$ . Therefore

$$(\exists \tilde{X} \in [\omega]^\omega)(\exists \pi_n : {}^n\omega \rightarrow [\tilde{b}(n)]^{\leq g(\delta_n)})(\forall \tilde{h} \in \tilde{\mathcal{H}})(\forall^\infty n \in \tilde{X}) \tilde{h}(n) \in \pi_n(\tilde{h} \upharpoonright n).$$

For  $n \in \tilde{X}$ , let

$$s_{\delta_{n+1}} = \{u(\delta_{n+1}) : u \in \pi_n(t) \text{ for } t \in \prod_{i < n} \tilde{b}(i)\}.$$

Then

$$|s_{\delta_{n+1}}| \leq \prod_{i \leq \delta_n} b(i) \times g(\delta_n) \leq g(\delta_{n+1}),$$

and so  $s_{\delta_{n+1}} \in [b(\delta_{n+1})]^{\leq g(\delta_{n+1})}$ . Finally, given  $h \in \mathcal{H}$ , and thus  $\tilde{h} \in \tilde{\mathcal{H}}$ ,

$$(\forall^\infty n \in \tilde{X}) \tilde{h}(n) \in \pi_n(\tilde{h} \upharpoonright n),$$

and so

$$(\forall^\infty n \in \tilde{X}) h(\delta_{n+1}) \in s_{\delta_{n+1}}.$$

Since  $g$  was arbitrary,  $|\mathcal{H}| < \mathfrak{f}_1$  and we conclude that  $\mathfrak{f}_2 \leq \mathfrak{f}_1$ .

$\mathfrak{f}_1 \leq \mathfrak{f}$ : Let  $\mathcal{F}$  be a filter generated by  $\langle A_\alpha : \alpha < \kappa \rangle$ ,  $\kappa < \mathfrak{f}_1$ , and included in the  $F_\sigma$  filter  $\langle \langle a_i^k : i < m_k \rangle : k \in \omega, g \rangle$ .

For each  $\alpha < \kappa$ , define a function  $f_\alpha \in {}^\omega\omega$  such that for all but finitely many  $k$ ,  $a_{f_\alpha(k)}^k \subseteq A_\alpha$ .

Then  $\{f_\alpha : \alpha < \kappa\}$  is a bounded family of size less than  $\mathfrak{f}_1$ , and therefore

$$(\exists X \in [\omega]^\omega)(\exists s_k \in [m_k]^{\leq g(k)})(\forall \alpha)(\forall^\infty k \in X) f_\alpha(k) \in s_k.$$

We conclude that  $\bigcup_{k \in X} \bigcap_{i \in s_k} a_i^k$  zaps the filter  $\mathcal{F}$ .  $\square$

We conclude this section by giving a small perspective on these cardinals (see [12] for a description of cardinals not defined here). If one removes the boundedness restriction on  $\mathcal{H}$ , the cardinal  $\mathfrak{e}_{ubd}$  becomes the evasion number, known as  $\mathfrak{e}$  ([4]); clearly  $\mathfrak{e} \leq \mathfrak{e}_{ubd}$  and it has been proved consistent by Shelah that  $\mathfrak{e} < \mathfrak{e}_{ubd}$  [6]. Removing the boundedness restriction on  $\mathcal{H}$  and fixing  $X = \omega$ , the cardinal  $\mathfrak{f}_1$  is the additivity of measure,  $add(\mathcal{N})$  ([1]); keeping the boundedness and using  $X = \omega$  yields the so-called transitive additivity of measure (due to Pawlikovski), and finally removing the boundedness condition but keeping  $X$  arbitrary yields the cardinal  $\mathfrak{se}$  (see [4]). Thus we have  $add(\mathcal{N}) \leq \mathfrak{e} \leq \mathfrak{e}_{ubd} = \mathfrak{f}$ , and  $\mathfrak{se}, trans - add(\mathcal{N}) \leq \mathfrak{e}_{ubd}$ .

For upper bounds, it is already known that  $\mathfrak{e}_{ubd}$  is less than or equal to the uniformity of the null and meager ideals; these are easy to prove through the cardinal  $\mathfrak{f}_1$ .

It is also known that the cardinal  $\mathfrak{e}_{ubd}$  is not provably equal to any of the standard cardinals  $\mathfrak{b}$ ,  $\mathfrak{d}$ ,  $\mathfrak{t}$  or additivity, uniformity, cofinality and covering of the null or meager ideals (see [9], [11], [6]).

A different but provable lower bound however for the number  $\epsilon_{ubd}$  (and thus  $\mathfrak{f}$ ) is  $\mathfrak{t}$ ; the idea of the proof is from [2]. We shall use the cardinal  $\mathfrak{b}$ , the minimum size of an unbounded family in  ${}^\omega\omega$ , and the well-known inequality  $\mathfrak{t} \leq \mathfrak{b}$  (see [12]).

Brendle remarks that from  $\mathfrak{t} \leq \epsilon_{ubd}$  and his result with Shelah ([5, 6]) that  $\mathfrak{se} = \min\{\mathfrak{e}, \mathfrak{b}\} = \min\{\epsilon_{ubd}, \mathfrak{b}\}$ , one concludes that  $\mathfrak{t} \leq \mathfrak{se}$ , thus improving Blass' result  $\mathfrak{p} \leq \mathfrak{se}$  [4]. Brendle also noticed that this can be proved directly from the following result as any family of functions of size less than  $\mathfrak{t}$  is necessarily bounded.

**Proposition 2.3.**  $\mathfrak{t} \leq \epsilon_{ubd} \leq 2^{\aleph_0}$ .

*Proof.* We show that  $\mathfrak{t} \leq \mathfrak{f}_1$ . Let  $\mathcal{H} = \langle h_\alpha : \alpha < \kappa \rangle$  be a family of size  $\kappa < \mathfrak{t}$ , bounded everywhere by  $b \in {}^\omega\omega$ , and fix  $g \in {}^\omega\omega$  such that  $\lim_n g(n) = \infty$ .

We construct a sequence  $\langle \phi_\alpha : \alpha \leq \kappa \rangle$  such that:

- (1)  $\phi_\alpha : \text{dom}(\phi_\alpha) \rightarrow [\omega]^{<\omega}$ ,
  - $\text{dom}(\phi_\alpha) \in [\omega]^\omega$ ,
  - $\phi_\alpha(k) \in [b(k)]^{\leq g(k)}$ ,
  - $\lim_{k \in \text{dom}(\phi_\alpha)} g(k) - |\phi_\alpha(k)| = +\infty$ ,
- (2)  $(\forall \beta < \alpha) \text{dom}(\phi_\alpha) \subseteq^* \text{dom}(\phi_\beta)$ , and  $(\forall^\infty k \in \text{dom}(\phi_\alpha)) \phi_\beta(k) \subseteq \phi_\alpha(k)$ ,
- (3)  $(\forall \beta < \alpha)(\forall^\infty k \in \text{dom}(\phi_\alpha)) h_\beta(k) \in \phi_\alpha(k)$ .

Once we have obtained  $\phi_\kappa$ , then clearly  $s_k = \phi_\kappa(k)$  for  $k \in \text{dom}(\phi_\kappa)$  is as desired.

Now to construct the sequence, assume that we already have  $\{\phi_\beta : \beta < \alpha\}$  for some  $\alpha \leq \kappa$ .

If  $\alpha = \beta + 1$  is a successor ordinal, define  $\phi_\alpha(k) = \phi_\beta(k) \cup \{h_\alpha(k)\}$  for  $k \in \text{dom}(\phi_\beta)$  such that  $|\phi_\beta(k)| < g(k)$ .

For  $\alpha$  a limit ordinal, first choose a  $\tilde{g} \in {}^\omega\omega$  such that  $\lim_k g(k) - \tilde{g}(k) = \infty$  and

$$(\forall \beta < \alpha)(\forall^\infty k \in \text{dom}(\phi_\beta)) |\phi_\beta(k)| \leq \tilde{g}(k).$$

This is possible as  $\alpha \leq \kappa < \mathfrak{t} \leq \mathfrak{b}$ ; I am not sure where this idea comes from.

Now for  $\beta < \alpha$  define

$$A_\beta = \{\langle k, x \rangle : k \in \text{dom}(\phi_\beta) \text{ and } \phi_\beta(k) \subseteq x \subseteq b(k) \text{ and } |x| \leq \tilde{g}(k)\}.$$

Clearly  $\beta < \gamma < \alpha \implies A_\gamma \subseteq^* A_\beta$  and, identifying  $A_0$  with  $\omega$  and as  $\alpha < \mathfrak{t}$ , we can find some infinite  $A_\alpha \subseteq^* A_\beta$  for each  $\beta < \alpha$ . Finally we put

$$\phi_\alpha(k) = \text{any } x \text{ such that } \langle k, x \rangle \in A_\alpha$$

and undefined if there is no such  $x$ . Clearly  $\phi_\alpha$  is as desired. □

### 3. TRIGONOMETRIC SERIES AND $\mathcal{N}$ -SETS

In this section we prove that if  $A$  is an  $\mathcal{N}$ -set and  $|B| < \epsilon_{ubd}$ , then  $A \cup B$  is also an  $\mathcal{N}$ -set. A similar result was earlier proved with  $\mathfrak{p}$  in the role of  $\epsilon_{ubd}$  in [7] on which we modeled our proof, and then with  $\mathfrak{t}$  in [2] on which we modeled the above proof of  $\mathfrak{t} \leq \epsilon_{ubd}$ .

It is known that the collection of  $\mathcal{N}$ -sets is not in general closed under unions; in [3] two  $\mathcal{N}$ -sets of cardinality  $\mathfrak{c}$  are constructed whose union is not an  $\mathcal{N}$ -set.

**Definition 3.1.** A set  $A \subseteq \mathbb{R}$  is called an  $\mathcal{N}$ -set ([3]) if there is a sequence of non-negative reals  $\langle a_n : n \in \omega \rangle$  such that:

- (1)  $\sum_{n=0}^\infty a_n = +\infty$ ,
- (2)  $(\forall a \in A) \sum_{n=0}^\infty a_n |\sin \pi na| < \infty$ .

**Proposition 3.2.** *If  $A \subseteq \mathbb{R}$  is an  $\mathcal{N}$ -set and  $|B| < \epsilon_{ubd}$ , then  $A \cup B$  is also an  $\mathcal{N}$ -set.*

*Proof.* Fix a sequence of nonnegative reals  $\langle a_n : n \in \omega \rangle$  as in the definition for the  $\mathcal{N}$ -set  $A$ . As is now standard procedure (see [7] and [2]), we put  $s_n = \sum_{i=0}^n a_i$  and  $b_n = a_n/s_n$ ; then again  $\sum_{n=0}^\infty b_n = +\infty$ .

As in [3], find an unbounded, nondecreasing sequence of natural numbers  $\langle q_n : n \in \omega \rangle$  such that

$$\sum_{n=0}^\infty a_n/s_n^{1+\frac{1}{q_n}} < \infty;$$

as we may replace the sequence  $\langle q_n : n \in \omega \rangle$  by any slower but unbounded and monotonic sequence, we may as well assume that  $q_{\pi_n} \leq n$ . Let  $\epsilon_n = s_n^{-1/q_n}$ .

Choose an increasing sequence of natural numbers  $\langle \pi_n : n \in \omega \rangle$  such that

- (1)  $(\forall k) \sum_{i=\pi_k}^{\pi_{k+1}-1} b_i \geq 1$ ,
- (2)  $(\forall n)(\forall m \geq \pi_n) q_m \geq n^2$ .

Finally for  $T \subseteq B$  and  $m \in \omega$ , define

$$a_T^m = \{k \in \omega : 0 \leq k \leq s_m \text{ and } (\forall t \in T) |\sin \pi kmt| \leq 2\pi\epsilon_m\},$$

and for each  $n, m \geq \pi_n$  and  $|T| \leq q_{\pi_n}/n$ , put

$$b_{T'}^m = \{a_{T'}^m : T \subseteq T' \text{ and } |T'| \leq q_{\pi_n}\}.$$

The following claim is from standard number theory and follows from the pigeon hole principle.

*Claim 3.3.* If  $T \subseteq B$  and  $|T| \leq q_{\pi_n}$ , then  $a_T^m \neq \emptyset$  for any  $m \geq \pi_n$ .

*Proof.* Fix  $n, m$  and we let  $T = \{t_i : i < |T|\}$ . Define a map

$$c : (1/\epsilon_m)^{|T|} + 1 \rightarrow \prod_{i \in |T|} 1/\epsilon_m$$

by  $c(j) = (\ell_0, \dots, \ell_{|T|-1})$  if

$$(\forall i < |T|)(\exists p \in \omega) p + \ell_i \epsilon_m \leq jmt_i < p + (\ell_i + 1)\epsilon_m.$$

There must then be two integers  $j_1 < j_2 < (1/\epsilon_m)^{|T|} + 1$  such that  $c(j_1) = c(j_2)$  and let  $k = j_2 - j_1$ .

Then  $k \leq (1/\epsilon_m)^{|T|} \leq (1/\epsilon_m)^{q_{\pi_n}} \leq (1/\epsilon_m)^{q_m} = s_m$ . Now using  $\llbracket x \rrbracket$  to denote the distance from  $x$  to the nearest integer, we have, for  $t \in T$ ,

$$|\sin \pi kmt| \leq \pi \llbracket kmt \rrbracket \leq 2\pi\epsilon_m$$

as desired. □

Therefore  $a_{T'}^m \neq \emptyset$  for each  $a_{T'}^m \in b_{T'}^m$  and of course, if  $T_i \subseteq B$ ,  $i < n$ , and each  $|T_i| \leq q_{\pi_n}/n$ , then  $\bigcap_{i < n} b_{T_i}^m \neq \emptyset$  whenever  $m \geq \pi_n$ .

Now consider the set

$$\mathcal{W} = \{ \langle a_{T_m}^m : \pi_n \leq m < \pi_{n+1} \rangle : n \in \omega \text{ and } |T_m| \leq q_{\pi_n} \},$$

which we may identify with  $\omega$  since it is a countable set, and the filter

$$\mathcal{G} = \langle \{ X \subseteq \mathcal{W} : (\forall n)(\forall \pi_n \leq m < \pi_{n+1}) (\exists T_m \subseteq B) |T_m| \leq q_{\pi_n}/n \text{ and } \prod_{\pi_n \leq m < \pi_{n+1}} b_{T_m}^m \subseteq X \} \rangle.$$

Then  $\mathcal{G}$  is an  $F_\sigma$  filter as it is generated by a closed set and contains the filter

$$\mathcal{F} = \langle \{X \subseteq \mathcal{W} : (\exists T \subseteq B) |T| < \infty$$

$$\text{and for all } n \text{ such that } |T| \leq q_{\pi_n}/n, \prod_{\pi_n \leq m < \pi_{n+1}} b_T^m \subseteq X \rangle.$$

As  $\mathcal{F}$  is generated by fewer than  $\mathfrak{c}_{ubd} = \mathfrak{f}$  sets, it must be diagonalized by an infinite set  $\mathcal{X} \subseteq \mathcal{W}$  which, without loss of generality, is of the form

$$\mathcal{X} = \{ \langle a_{T_m}^m : \pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1} \rangle : \ell \in \omega \}$$

where  $|T_m| \leq q_{\pi_{n_\ell}}$  and  $n_\ell < n_{\ell+1}$ . For each  $\ell$  and  $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}$ , pick  $k_m \in a_{T_m}^m$ .

Now clearly  $\sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}} b_m = \infty$  and it remains to show that

$$\sum_\ell \sum_{\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}} b_m |\sin mk_m \pi x| < \infty$$

for each  $x \in A \cup B$ . For  $x \in A$ ,

$$b_m |\sin mk_m \pi x| \leq b_m k_m |\sin m \pi x| \leq b_m s_m |\sin m \pi x| = a_m |\sin m \pi x|.$$

Finally for  $x \in B$ , let  $T = \{x\}$  and thus for all but finitely many  $\ell$ , for all  $\pi_{n_\ell} \leq m < \pi_{n_{\ell+1}-1}$ ,

$$|\sin mk_m \pi x| \leq 2\pi \epsilon_m$$

and therefore

$$b_m |\sin mk_m \pi x| \leq 2\pi a_m / s_m^{1+1/q_m}.$$

This completes the proof. □

#### REFERENCES

1. T. Bartoszyński, *Combinatorial Aspects of Measure and Category*, Fund. Math. **127** (1987), 225-239. MR **88m**:04001
2. T. Bartoszyński and M. Scheepers, *Remarks on Sets related to Trigonometric Series*, Topology and Appl. **64** (1995), 133-140. MR **96h**:42004
3. N.K. Bary, *A Treatise on Trigonometric Series*, MacMillan, New York, 1964. MR **30**:1347
4. A. Blass, *Cardinal Characteristics and the Product of Countably many Infinite Cyclic groups*, Journal of Algebra **169** (1994), 512-540. MR **95h**:20069
5. J. Brendle, *Evasion and Prediction, the Specker Phenomenon and Gross Spaces*, Forum Math. **7** (1995), no. 5, 513-541. MR **96i**:03042
6. J. Brendle and S. Shelah, *Evasion and Prediction II*, J. London Math. Soc. (2) **53** (1996), no. 1, 19-27. CMP 96:04
7. Z. Bukovská and L. Bukovský, *Adding Small Sets to an  $N$ -set*, Proc. Amer. Math. Soc. (to appear, 1994).
8. M. Dagenet, *Emploi des Filtres sur  $\mathbb{N}$  dans l'Etude Descriptive des Fonctions*, Fund. Math. **XCIV** (1977), 11-33. MR **57**:10671
9. C. Laflamme, *Zapping Small Filters*, Proc. Amer. Math. Soc. **114** (1992), 535-544. MR **92e**:03076
10. S. Shelah, *Proper Forcing*, Lecture Notes in Mathematics, Springer-Verlag, Berlin 1982. MR **84h**:03002
11. S. Shelah, *Vive la Différence I*, Set Theory of the Continuum (H. Judah, W. Just and H. Woodin, eds.), Mathematical Sciences Research Institute Publications, Springer-Verlag, 1992, 357-405. MR **94g**:03068
12. J. Vaughan, *Small Uncountable Cardinals and Topology*, Open problems in Topology (J. van Mill and G. Reed, eds.), North-Holland, 1990, 195-218. CMP 91:03

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