PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 3107–3117
S 0002-9939(97)03932-4

# ON THE UNITARY DUAL OF THE CLASSICAL LIE GROUPS, REPRESENTATIONS OF $Sp(p,q)$

SUSANA A. SALAMANCA–RIBA

(Communicated by Roe Goodman)

Abstract. In this paper we prove that a unitary representation of $Sp(p,q)$ whose infinitesimal character satisfies some regularity condition is infinitesimally isomorphic to an $A_{\mathfrak{q}}\ (\lambda)$ module. This is done using similar techniques as those used by the author in earlier work.

## 1. Introduction

Suppose that $G$ is a real reductive Lie group with complexified Lie algebra $\mathfrak{g}$, Cartan involution $\theta$, maximal compact subgroup $K$ and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. For any given real Lie subgroup $L$ of $G$ that we will consider, we will denote its complexified Lie algebra by the corresponding German lower case letter $\mathfrak{l}$, and its real Lie algebra Lie$(L)$, by $\mathfrak{l}_0$. Let $T$ be a Cartan subgroup of $K$ and $(\ ,\ )$ be a non-degenerate invariant symmetric bilinear form on $G$. Finite dimensional irreducible modules of $K$, $T$, $\mathfrak{k}$, $L \cap K$, etc., will be identified with their highest weights (with respect to the appropriate subalgebra). The equivalence class of such modules is usually denoted by $\hat{K}$, $\hat{T}$, etc.

The problem that we have been working on is the following (see [4] and [5] for a proof of this conjecture for some groups)

*Conjecture* 1.1 (Vogan–Zuckerman). Suppose $X$ is a unitary Harish–Chandra module of $G$ with infinitesimal character $\gamma$ satisfying

$$(\gamma - \rho, \alpha) \geq 0 \tag{1.1}$$

for all positive roots $\alpha$ in $\Delta(\mathfrak{g}, \mathfrak{t})$ and $\rho$, half the sum of these positive roots.

Then $X$ is an $A_{\mathfrak{q}}\ (\lambda)$ module.

The theorem that we prove in this paper is the following

**Theorem 1.2.** *Let $G$ be $Sp(p,q)$ and $X$ be a Harish–Chandra module of $G$ with a hermitian form $\langle\ ,\ \rangle$ and infinitesimal character $\gamma$ satisfying condition* (1.1). *Then either*

1. *$X$ is an $A_{\mathfrak{q}}\ (\lambda)$ module, or*
2. *There are two $K$–types $(\mu, V_\mu)$ and $(\eta, V_\eta)$, of $X$ where $\langle\ ,\ \rangle|_{V_\mu \oplus V_\eta}$ is indefinite.*

---

Received by the editors October 10, 1995 and, in revised form, April 19, 1996.
1991 *Mathematics Subject Classification.* Primary 22E46, 22D10; Secondary 22E47, 20G05.
*Key words and phrases.* Real semisimple Lie groups, unitary representations, Zuckerman modules.
Supported by NSF grant DMS–9108990.

The above conjecture is part of the larger program of the classification of all irreducible unitary representations of $G$ and the ideas behind the proofs of all the special cases done so far, including those in this paper, are inspired by the Barbasch–Vogan program (they were suggested to the author by them and she gladly acknowledges them). This program in turn is modeled after Vogan's classification of all irreducible admissible Harish–Chandra modules of $G$ (see [6]).

Vogan's classification consists of reducing the problem to quasisplit subgroups of $G$; that is, in attaching to a given representation $\mu \in \hat{K}$ the following set of parameters:

1. A $\theta$-stable parabolic subalgebra $\mathfrak{q}_{_V} = \mathfrak{l}_{_V} + \mathfrak{u}_{_V}$ (see Definition 2.2), whose Levi subgroup $L_{_V}$ is quasisplit, and
2. A representation $\mu_{_{L_V}}$ of $\widehat{L_{_V} \cap K}$

such that the following is given

> (a) A list of Langlands parameters for the set of all irreducible Harish–Chandra modules of the (quasisplit) group $L_{_V}$ containing the representation $\mu_{_{L_V}}$ as lowest $(L_{_V} \cap K)$ type.
> (b) A one-to-one correspondence between the above list and the set of irreducible admissible Harish–Chandra modules of $G$ containing $\mu$ as a lowest $K$-type.

The goal of the Barbasch–Vogan program is then very similar –with the added ingredient of so-called "Bottom–layer arguments" to detect non-unitarity (see [8]): given $\mu \in \hat{K}$, find

1. $\mathfrak{q}_{_u} = \mathfrak{l}_{_u} + \mathfrak{u}_{_u} \supseteq \mathfrak{q}_{_V}$, $\theta$–stable,
2. a representation $\mu_{_{L_u}} \in \widehat{L_u \cap K}$

such that

> (a) Irreducible unitary representations of $G$ with lowest $K$-type $\mu$ are (via Vogan's classification) in one-to-one correspondence with irreducible unitary representations of $L_u$ with lowest $(L_u \cap K)$ type $\mu_{_{L_u}}$.
> (b) The bijection preserves the property of being hermitian.
> (c) It also preserves unitarity because of a "Bottom–layer argument". That is, there is a finite set $\Lambda_{_{L_u}}$ of $(L_u \cap K)$-types satisfying
>
> > (i) they detect *non-unitarity* for any representation of $L_u$;
> > (ii) they are in one-to-one correspondence with a set $\Lambda_G$ of $K$-*dominant* weights of $K$. And this correspondence preserves signature of hermitian forms.

## 2. Notation and preliminary results

2.1. **Structure theory and a unitary test.** As usual, let us denote by $\Delta(\mathfrak{k}, \mathfrak{t})$ the set of compact roots of $\mathfrak{t}$ in $\mathfrak{k}$. Let $\mathfrak{h}$ be a maximally compact Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. That is, put $\mathfrak{h} = \mathfrak{g}^{\mathfrak{t}} = \mathfrak{t} + \mathfrak{p}^{\mathfrak{t}}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $W_{\mathfrak{g}}$ the Weyl group $W(\mathfrak{g}, \mathfrak{h})$, of $\mathfrak{h}$ in $\mathfrak{g}$ and by $W_{\mathfrak{k}}$, the Weyl group $W(\mathfrak{k}, \mathfrak{t})$, of $\mathfrak{t}$ in $\mathfrak{k}$.

Also, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ be the set of non-zero roots of $\mathfrak{h}$ in $\mathfrak{g}$. Then every root has support on $\mathfrak{t}$. Let $\Delta(\mathfrak{g}, \mathfrak{t})$ be the set of non-zero roots of $\mathfrak{t}$ in $\mathfrak{g}$. More generally, if $V$ is any $\mathfrak{t}$-invariant subspace of $\mathfrak{g}$, we denote by $\Delta(V,\mathfrak{t})$ the set of $\mathfrak{t}$- roots with multiplicities (see [6]).

**Definition 2.1.** Let us fix once and for all a positive root system for $\Delta(\mathfrak{k},\mathfrak{t})$ denoted $\Delta^+(\mathfrak{k},\mathfrak{t})$. We say that a set $\Delta^+(\mathfrak{g},\mathfrak{h})$ of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$, is compatible with $\Delta^+(\mathfrak{k},\mathfrak{t})$ if

1. for all $\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})$, there is $\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})$ such that $\alpha = \beta\big|_{\mathfrak{t}}$,
2. for all $\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})$, $\theta\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})$.

$\Delta^+(\mathfrak{g},\mathfrak{h})$ induces a positive root system $\Delta^+(\mathfrak{g},\mathfrak{t})$. Let $\Delta(\mathfrak{p},\mathfrak{t})$ be the set of non-compact roots for $\mathfrak{t}$ and $\Delta^+(\mathfrak{p},\mathfrak{t}) = \Delta^+(\mathfrak{g},\mathfrak{t}) - \Delta^+(\mathfrak{k},\mathfrak{t})$. Set

$$\begin{aligned}
\rho_c &= \rho_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}\alpha,\\
\rho &= \rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}\alpha,\\
\rho_n &= \rho-\rho_c = \frac{1}{2}\sum_{\beta\in\Delta^+(\mathfrak{p},\mathfrak{t})}\beta.
\end{aligned}$$

More generally, for any $\theta$–stable parabolic subalgebra $\mathfrak{q}$ (see Definition 2.2 below), if $\mathfrak{m}$ is any subspace of $\mathfrak{q}$ stable under $ad(\mathfrak{t})$, then there is a subset $\Phi = \{\alpha_1,\dots,\alpha_f\}$ of $\mathfrak{t}^*$ and subspaces $\mathfrak{m}_{\alpha_i}$ of $\mathfrak{m}$, such that if $y \in \mathfrak{t}$ and v is in $\mathfrak{m}_{\alpha_i}$, then

$$ad(y)v = \alpha_i(y)v.$$

We set

$$\Delta(\mathfrak{m},\mathfrak{t}) = \Phi \quad\text{and}\quad \rho(\mathfrak{m}) = \frac{1}{2}\sum_{\alpha\in\Phi} m_\alpha\alpha.$$

Here $m_\alpha$ is the multiplicity of $\alpha$. Then

$$\rho(\mathfrak{m})(y) = \frac{1}{2}tr\big(ad(y)\big)\big|_{\mathfrak{m}}.$$

If $\mathfrak{m}$ is a reductive subalgebra then we can choose a subset $\Phi^+ \subset \Phi$ which is a system of positive roots. We write

$$\rho_{\mathfrak{m}} = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

For example, if $G$ is $Sp(p,q)$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$\Delta(\mathfrak{k},\mathfrak{t}) = \pm\left\{2e_i;\ e_j\pm e_k\big|1\le i\le p+q; 1\le j<k\le p,\ \text{or } p<j<k\le p+q\right\},$$

(2.1)

$$\Delta(\mathfrak{p},\mathfrak{t}) = \pm\left\{e_i\pm e_j\big|1\le i\le p<j\le p+q\right\},$$

and we can choose

$$\Delta^+(\mathfrak{k},\mathfrak{t}) = \left\{2e_i;\ e_j\pm e_k\big|1\le i\le p+q; 1\le j<k\le p,\ \text{or } p<j<k\le p+q\right\},$$

(2.2)

$$\Delta^+(\mathfrak{p},\mathfrak{t}) = \left\{e_i\pm e_j\big|1\le i\le p<j\le p+q\right\}$$

so that

$$\rho_c = (p, p-1, \dots, 1 \big| q, q-1, \dots, 1), \tag{2.3}$$

$$\rho = (p+q, p+q-1, \dots, q+1 \big| q, q-1, \dots, 1), \tag{2.4}$$

$$\rho_n = (q, q, \dots, q \big| 0, 0, \dots, 0). \tag{2.5}$$

**Proposition 2.1** ([3, 9, 1, Parthasarathy's Dirac operator inequality])**.** *Let $\pi$ be a unitary representation of $G$ and $X$ its Harish–Chandra module. Fix a representation of $\mathfrak{k}$ occurring in $X$, of highest weight $\chi \in \mathfrak{t}^*$ and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$. Write $\rho_c$, $\rho_n$ as in Definition 2.1; fix an element $w \in W_{\mathfrak{k}}$ such that $w(\chi - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Let $c_o$ denote the eigenvalue of the Casimir operator of $\mathfrak{g}$ in $X$. Then*

$$(w(\chi - \rho_n) + \rho_c\ , w(\chi - \rho_n) + \rho_c) \geq c_o + (\rho, \rho).$$

2.2. **Construction of $\theta$–stable parabolic subalgebras.** Let us recall from [7] the construction of certain subalgebras of $\mathfrak{g}$.

**Definition 2.2** (See [7], section 5.2)**.** Suppose $x \in i(\mathfrak{t}_o)^*$ is a $\Delta^+(\mathfrak{k}, \mathfrak{t})$–dominant weight. We will denote by $\mathfrak{q}(x) = \mathfrak{l}(x) + \mathfrak{u}(x)$ the $\theta$–stable parabolic subalgebra obtained as follows:

$$\mathfrak{l}(x) = \bigoplus_{\alpha \in \Delta(\mathfrak{l}, x)} \mathbb{C} X_\alpha,$$

$$\mathfrak{u}(x) = \bigoplus_{\alpha \in \Delta(\mathfrak{u}, x)} \mathbb{C} X_\alpha,$$

where

$$\Delta(\mathfrak{l}, x) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) | (x, \alpha) = 0\},$$

$$\Delta(\mathfrak{u}, x) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) | (x, \alpha) > 0\}.$$

Define also $L$= normalizer of $\mathfrak{q}$ in $G$. Then $L$ is called the Levi subgroup of $\mathfrak{q}$.

In our case, when $G$ is $Sp(p, q)$, then

$$x = (\underbrace{x_1, \dots, x_1}_{p_1\ times}, \underbrace{x_2, \dots, x_2}_{p_2\ times}, \dots, \underbrace{x_t, \dots, x_t}_{p_t\ times}, \underbrace{0, \dots, 0}_{r\ times} \big| \underbrace{x_1, \dots, x_1}_{q_1\ times}, \underbrace{x_2, \dots, x_2}_{q_2\ times}, \dots, \underbrace{x_t, \dots, x_t}_{q_t\ times}, \underbrace{0, \dots, 0}_{s\ times}) \tag{2.6}$$

is a dominant weight for $\Delta^+(\mathfrak{k}, \mathfrak{t})$, where $p = \sum_{i=1}^t p_i + r$, $q = \sum_{i=1}^t q_i + s$. Then if $n_i = p_i + q_i$

$$\Delta(\mathfrak{l}, x) \cong \left( \bigoplus_{i=1}^t A_{n_i - 1} \right) \oplus C_{r+s}$$

and

$$L \cong \left( \prod_{i=1}^t U(p_i, q_i) \right) \times Sp(r, s).$$

Using Definition 2.2 Vogan associates ([7], Section 5.3) in a canonical way, a $\theta$–stable parabolic subalgebra $\mathfrak{q}_V = \mathfrak{l}_V + \mathfrak{u}_V$ of $\mathfrak{g}$ to any highest weight $\mu$ of any irreducible finite dimensional representation of $K$. This is not necessarily $\mathfrak{q}(\mu)$ (Definition 2.2) but rather, he first attaches to $\mu$ ([7], Proposition 5.3.3), in a unique way, a weight $\lambda_V \in i(\mathfrak{t}_0)^*$ which in turn gives $\mathfrak{q}_V$. In other words,

$$\mathfrak{q}_V = \mathfrak{q}(\lambda_V). \tag{2.7}$$

To reduce our problem to an appropriate subgroup $L$ we need to use $L_V$ as our starting point. Then keeping in mind the algorithm that attaches $\lambda_V$ to $\mu$, we need the following data.

**Definition 2.3.** Let $\Delta^+(\mathfrak{k},\mathfrak{t})$ and $2\rho_c$ be as in Definition 2.1. Since $\mu$ is a highest weight, $\mu + 2\rho_c$ is $\Delta^+(\mathfrak{k},\mathfrak{t})$-dominant. Choose a positive set $\Delta^+_\mu(\mathfrak{g},\mathfrak{t})$, of roots that are positive on $\mu + 2\rho_c$ and denote by $\Pi_\mu$ its subset of simple roots. Set

$$\rho_\mu = \frac{1}{2}\left(\sum_{\alpha\in\Delta^+_\mu(\mathfrak{g},\mathfrak{t})} \alpha\right).$$

The following lemma is true in a more general setting but this version is enough for the purposes of this paper.

**Lemma 2.2.** *Suppose $G$ is $Sp(p,q)$ and $\mu \in i(\mathfrak{t}_o)^*$ is a dominant weight for $\mathfrak{k}$ satisfying*

$$(\mu + 2\rho_c, \check{\alpha}) \le (2\rho_\mu\ , \check{\alpha}) = 2 \tag{2.8}$$

*for all $\alpha \in \Pi_\mu$. Then*

$$\mu = \sum_{\substack{\beta\in\Delta^+(\mathfrak{p},\mathfrak{t})\\ 0\le c_\beta\le 1}} c_\beta\beta. \tag{2.9}$$

*Proof.* The proof is by induction on the dimension of $G$. Suppose

$$\mu + 2\rho_c = \left(r_1, r_2, \dots, r_p \middle| s_1, s_2, \dots, s_q\right).$$

Without loss of generality we may assume that $r_1 \ge s_1$. Let $\zeta = e_1$; then, since $\zeta$ is a positive sum of simple roots, we have

$$(\mu + 2\rho_c, \zeta) \le (2\rho_\mu, \zeta) = 2p + 2q.$$

So $\mu_1 = (\mu, \zeta) \le 2q$. Now choose $L = U(1) \times Sp(p-1, q)$ which is built from

$$x = (1, 0, 0, \dots, 0|0, 0, \dots, 0) \in i(\mathfrak{t}_o)^*$$

using Definition 2.2. Then

$$2\rho(\mathfrak{u}\cap\mathfrak{p}) = (2q, 0, 0, \dots, 0|0, 0, \dots, 0)$$

(see Definition 2.1).

Set $y = \frac{\mu_1}{2q}$; then $0 \le y \le 1$ and

$$\mu^L = \mu - y2\rho(\mathfrak{u}\cap\mathfrak{p}) = \left(0, \mu_2, \mu_3, \dots, \mu_p \middle| \mu_{p+1}, \mu_{p+2}, \dots, \mu_{p+q}\right)$$

is a weight in $L \cap K$ and $\mu^L + 2\rho_{\mathfrak{l}\cap\mathfrak{k}}$ satisfies our induction hypothesis. In fact

$$\Pi_{\mu^L} = \Pi_\mu \cap \Delta(\mathfrak{l},\mathfrak{t})$$

and

$$(\mu^L + 2\rho_{\mathfrak{l}\cap\mathfrak{k}}\ , \check{\alpha}) \le (2\rho_{\mathfrak{l}}\ , \check{\alpha})$$

for all $\alpha \in \Pi_{\mu^L}$. Then

$$\mu^L = \sum_{\substack{\beta\in\Delta^+(\mathfrak{l}\cap\mathfrak{p},\mathfrak{t})\\ 0\le c_\beta\le 1}} c_\beta \beta.$$

Now

$$\begin{aligned}\mu &= \mu^L + y2\rho(\mathfrak{u}\cap\mathfrak{p})\\ &= \sum_{\substack{\beta\in\Delta^+(\mathfrak{l}\cap\mathfrak{p},\mathfrak{t})\\ 0\le c_\beta\le 1}} c_\beta\beta + \sum_{\substack{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})-\Delta^+(\mathfrak{l}\cap\mathfrak{p},\mathfrak{t})\\ 0\le y\le 1}} y\alpha.\end{aligned}$$

□

2.3. $A_{\mathfrak{q}}\ (\lambda)$ **modules.** In order to prove Theorem 1.2 we need to find some characterizations of the $A_{\mathfrak{q}}\ (\lambda)$ modules. In this subsection we record some results needed in the next section for establishing such parameters together with some lemmas used in the proof of Theorem 1.2 .

**Definition 2.4.** Fix a $\theta$–stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as in Definition 2.2 and let $L \subseteq G$ be the centralizer of $\mathfrak{q}$ in $G$. Let $\lambda : \mathfrak{l} \to \mathbb{C}$ be a one–dimensional representation of $\mathfrak{l}$. $\lambda$ is called admissible if the following are satisfied:

1. $\lambda$ is the differential of a unitary character of $L$;
2. $(\alpha, \lambda\big|_{\mathfrak{t}}\ ) \geq 0$, for all $\alpha \in \Delta(\mathfrak{u})$.

Given $\mathfrak{q}$ and an admissible $\lambda$ define, as in [9], Section 5, $\mu(\mathfrak{q}, \lambda)$ to be the representation of $K$ of highest weight $\lambda\big|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p})$.

**Proposition 2.3.** *Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a $\theta$–stable parabolic subalgebra and $\lambda \in \mathfrak{t}^*$ satisfies 2 of Definition 2.4. Let $X$ be an irreducible unitary Harish–Chandra module with infinitesimal character $\gamma$ satisfying Condition* (1.1) *of Conjecture 1.1; and $\mu \in \mathfrak{t}^*$, the highest weight of a lowest $K$-type of $X$. Suppose in addition that $\lambda$ is zero on $[\mathfrak{l}, \mathfrak{l}]$ and assume that $\mu = \mu(\mathfrak{q}, \lambda)$. Then*

$$X \cong A_{\mathfrak{q}}\ (\lambda).$$

This is a variant of Proposition 6.1 in [9].

**Proposition 2.4** ([2])**.** *Suppose $X$ is an irreducible unitary Harish–Chandra module of a real semisimple Lie group $G$ with infinitesimal character $\gamma$ satisfying Condition* (1.1)*. Let $\Delta^+$ be a $\theta$–stable positive root system compatible with $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Let $(\Delta^+)'$ be another $\theta$–stable positive root system. Set $\rho_n = \rho_n(\Delta^+)$ and ${\rho_n}' = \rho_n(\Delta^+)'$ and assume that $\chi = \rho_n + \rho_n'$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$–dominant and that $Hom_{\mathfrak{k}}\ (V_\chi, X)$ $\neq 0$.*

*Then there is a $\theta$–stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ of $\mathfrak{g}$, such that $\chi = 2\rho(\mathfrak{u}\cap\mathfrak{p})$.*

This proposition is stated in [2] in a weaker form. Namely the infinitesimal character is assumed to be trivial. However in the proof of that statement the author is only using Condition (1.1). A statement for general $\lambda$ is given in [9], Proposition 5.16 and the extension to the case of Condition (1.1) is immediate.

**Lemma 2.5.** *Suppose $X$ is an irreducible Harish–Chandra module of $G$, with infinitesimal character satisfying* (1.1) *and $\mu \in i(\mathfrak{t}_o)^*$ is a lowest $K$ type of $X$ satisfying Equation* (2.9)*. Then, either Parthasarathy's Dirac operator inequality (Proposition 2.1) fails on $\mu$, or there is a $\theta$–stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $X$ is isomorphic to $A_{\mathfrak{q}}\ (0)$. In that case $\mu = 2\rho(\mathfrak{u}\cap\mathfrak{p})$ and $(\gamma, \gamma) = (\rho, \rho)$.*

*Proof.* For any choice of $\Delta^+(\mathfrak{g},\mathfrak{t}) = \Delta^+(\mathfrak{p},\mathfrak{t}) \cup \Delta^+(\mathfrak{k},\mathfrak{t})$, if $\mu$ satisfies Equation (2.9) and $w \in W_{\mathfrak{k}}$ makes $\mu - \rho_n$ a $w\Delta^+(\mathfrak{k},\mathfrak{t})$–dominant weight, then

$$\begin{aligned} \mu - \rho_n + w\rho_c &= \sum_{\substack{\beta\in\Delta^+(\mathfrak{p},\mathfrak{t})\\ 0\le c_\beta\le 1}} c_\beta\beta - \frac{1}{2}\sum_{\beta\in\Delta^+(\mathfrak{p},\mathfrak{t})}\beta + \sum_{\substack{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})\\ -\frac12\le c_\alpha\le\frac12}} c_\alpha\alpha \\ &= \sum_{\substack{\beta\in\Delta^+(\mathfrak{p},\mathfrak{t})\\ -\frac12\le c_\beta{}'\le\frac12}} c_\beta{}'\beta + \sum_{\substack{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})\\ -\frac12\le c_\alpha{}'\le\frac12}} c_\alpha{}'\alpha \end{aligned}$$

and hence $\mu - \rho_n + w\rho_c$ lies in the convex hull of $W\rho$. Then

$$(\mu - \rho_n + w\rho_c, \mu - \rho_n + w\rho_c) \le (\rho,\rho)$$

and equality holds if and only if $\mu - \rho_n + w\rho_c = \sigma(\rho)$, for some $\sigma \in W_{\mathfrak{k}}$. Then if $X$ is unitary, by Proposition 2.1, the left-hand side of the last inequality is at least $(\gamma,\gamma)$, forcing equality to hold.

Now, because of the choice of $w$, $\mu - \rho_n + w\rho_c$ is dominant for $w\Delta^+(\mathfrak{k},\mathfrak{t})$ and therefore $\sigma(\rho)$ is as well. This implies that $w\Delta^+(\mathfrak{k},\mathfrak{t}) \subset \sigma\Delta^+(\mathfrak{g},\mathfrak{t})$ and hence $\sigma(\rho) = \rho_n' + w\rho_c$ and $\mu = \rho_n + \rho_n'$.

By Proposition 2.4, if $X$ is unitary, there is a $\theta$–stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that

$$\mu = 2\rho(\mathfrak{u}\cap\mathfrak{p})$$

making $\mu$ an $A_{\mathfrak{q}}$ (0) lowest $K$ type. Now, by Proposition 2.3, $X \cong A_{\mathfrak{q}}$ (0)) and the rest of the lemma follows. □

## 3. Proof of Theorem 1.2

Now to prove Theorem 1.2 suppose $X$ is an irreducible Harish–Chandra module satisfying the hypotheses of that theorem, and suppose $\mu$ is a lowest $K$ type of $X$. Having in mind Vogan's algorithm for attaching $\mathfrak{q}_V$ to a weight in $K$, we write the coordinates of $\mu$ as follows:

$$(3.1)\quad \begin{aligned} \mu = (&\underbrace{x_1,\dots,x_1}_{r_1\ times},\underbrace{x_2,\dots,x_2}_{r_2\ times},\dots,\underbrace{x_h,\dots,x_h}_{r_h\ times},\underbrace{0,\dots,0}_{r\ times}\,| \\ &\underbrace{y_1,\dots,y_1}_{s_1\ times},\underbrace{y_2,\dots,y_2}_{s_2\ times},\dots,\underbrace{y_k,\dots,y_k}_{s_k\ times},\underbrace{0,\dots,0}_{s\ times}). \end{aligned}$$

Set $\Pi_\mu$ , $\rho_\mu$ as in Definition 2.3 and suppose we label the simple roots in $\Pi_\mu$ as

$$\alpha_1,\alpha_2,\dots,\alpha_l$$

according to their place in the Dynkin diagram. Define $d$ by

$$(3.2)\qquad \begin{aligned} (\mu + 2\rho_c\ ,\check{\alpha}_i) &\le (2\rho_\mu\ ,\check{\alpha}_i) = 2, \quad \text{for}\ \ i > d, \\ (\mu + 2\rho_c,\check{\alpha}_d) &> 2. \end{aligned}$$

Recall that $\Pi_\mu$ is a choice of simple roots that are positive on $\mu + 2\rho_c$. So if $w\{\alpha_1,\alpha_2,\dots,\alpha_l\} = \{e_1 - e_2; e_2 - e_3; \dots e_{l-1} - e_l; 2e_l\}$, the coordinates of $w(\mu + 2\rho_c)$ are in decreasing order. Then the inequalities in (3.2) say that $d$ is the first place (from right to left) where the jump in the coordinates of $w(\mu + 2\rho_c)$ is strictly greater than 2.

Note that the last $l-d$ roots of $\Pi_\mu$ form a Dynkin diagram of type $C$ and the rest of the roots one of type $A$. Define a $\theta$-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ by

$$\begin{aligned}\Delta(\mathfrak{l},\mathfrak{t}) &\cong A_{d-1}\oplus C_{l-d},\\ \Delta(\mathfrak{u}) &= \Delta(\mathfrak{u}_V)\setminus\Delta(\mathfrak{l}).\end{aligned}$$

Hence, there are $a$, $b$ in $\mathbb{Z}$ such that $a+b=l-d$ and

$$L\cong U(p-a,q-b)\times Sp(a,b). \tag{3.3}$$

We claim that $\mathfrak{q}\supset\mathfrak{q}_V$. In fact, since by definition $\mathfrak{u}\subseteq\mathfrak{u}_V$ we only need to establish that $\mathfrak{l}_V\subseteq\mathfrak{l}$.

Note that if $\tilde{\lambda}=\mu+2\rho_c-\rho_\mu$ as in Proposition 5.3.3 in [7], then

$$(\tilde{\lambda},\check{\alpha_d})>1,$$

which implies that $\alpha_d\notin\Delta(\mathfrak{l}_V,\mathfrak{t})$. In fact the weight

$$\lambda=\tilde{\lambda}+\frac{1}{2}v$$

of Proposition 5.3.3 in [7] defines $\mathfrak{l}_V$. So we only need to see that $(\lambda,\check{\alpha_d})>0$. But from the inductive construction of $v$ and Proposition 5.3.1 in [7], we can conclude that the coordinates of $v$ are between $-1$ and 1, which gives the inequality for $\lambda$ we want. Hence $\Delta(\mathfrak{l}_V,\mathfrak{t})\subset\Delta(\mathfrak{l},\mathfrak{t})$ and $\mathfrak{l}_V\subset\mathfrak{l}$.

Then by [7], 6.5.9 and 6.5.12 and Corollary 5.3 in [4], $X$ is the Langlands subrepresentation of some module $\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}(X^L)$, with $X^L$ a Harish–Chandra module of $L$. Moreover, the hermitian dual of $X^L$ has a hermitian form, say $\langle\ ,\ \rangle^L$.

Now $\mu^L=\mu-2\rho(\mathfrak{u}\cap\mathfrak{p})$ is a lowest $K$–type of $X^L$. Also, $2\rho(\mathfrak{u}\cap\mathfrak{p})$ is constant on $(Sp(a,b))\cap K$ and

$$2\rho_c|_{Sp(a,b)}=2\rho_{\mathfrak{l}\cap\mathfrak{k}}\ |_{Sp(a,b)}.$$

Then $\mu_1=\mu^L|_{Sp(a,b)}$ satisfies the hypothesis of Lemma 2.2, so by Lemma 2.5, either Parthasarathy's Dirac operator inequality (see Proposition 2.1) fails on $\mu_1$, or there is a $\theta$–stable parabolic subalgebra $\mathfrak{q}'$ of Sp(a,b) such that

$$X^L\big|_{Sp(a,b)}\cong A_{\mathfrak{q}'}(0). \tag{3.4}$$

In the first case, if Dirac inequality fails on $\mu_1$, by Lemma 6.3 in [4] there is another representation $(V_{\delta_1},\delta_1)$ of $Sp(a,b)\cap K$, occurring in $V_{\mu_1}\otimes(\mathfrak{sp}\ (a,b)\cap\mathfrak{p})$ such that the hermitian form $\langle\ ,\ \rangle^L$ is indefinite on $V_{\mu_1}\oplus V_{\delta_1}$.

We claim that any possible highest weight $\delta_1$ as above is such that

$$\delta=\delta_1+2\rho(\mathfrak{u}\cap\mathfrak{p})$$

is dominant for $\mathfrak{k}$ and hence, by Theorem 5.8 in [4], $V_\delta$ is a $K$ type of $X$ and the hermitian form $<\ ,\ >$ on $X$ is indefinite on $V_\mu\oplus V_\delta$.

To prove the claim let $Q=\Delta(\mathfrak{sp}\ (a,b)\cap\mathfrak{p})$. If $\delta_1$ is a highest weight as above, then $\delta_1=\mu^L+\beta$, for some $\beta\in Q$ and $\delta=\mu+\beta$. Note that

$$Q=\{\underline{+}(e_i\underline{+}e_j)\big|p-a<i\le p;\quad p+q-b<j\le p+q\}.$$

Therefore, if $\delta_1$ is dominant for $\Delta^+(\mathfrak{sp}\ (a,b)\cap\mathfrak{k},\mathfrak{t})$, then $\delta$ will be dominant for $\Delta^+(\mathfrak{k},\mathfrak{t})$ provided that

$$(\delta,e_{p-a}-e_{p-a+1})\ \ge\ 0, \tag{3.5}$$

$$(\delta,e_{p+q-b}-e_{p+q-b+1})\ \ge\ 0. \tag{3.6}$$

In case (3.5)

$$\begin{aligned}(\delta, e_{p-a} - e_{p-a+1}) &= (\mu, e_{p-a} - e_{p-a+1}) + (\beta, e_{p-a} - e_{p-a+1}) \\ &\geq 0 + (\beta, e_{p-a} - e_{p-a+1}).\end{aligned}$$

The right-hand side of the inequality is either 0, 1 or $-1$. So the only bad cases are when $(\mu, e_{p-a} - e_{p-a+1}) = 0$. But after rearranging the coordinates of $\mu + 2\rho_c$ in decreasing order, there will be at most two $q$-coordinates in between the $p-a$ and the $p-a+1$ coordinates and all the jumps down those – at most four – coordinates must be at most 2.

This contradicts our assumption that the simple root that joins the simple roots of $Sp(a,b)$ with those of $U(p-a, q-b)$ is the first one on which the jump is greater than two. This proves (3.5). (3.6) can be proved in a similar way and this proves our claim.

Now assume (3.4). If $X^L\big|_{U(p-a,q-b)}$ is not an $A_{\mathfrak{q}}\ (\lambda)$ module we need to see that $X$ will not be unitary. Set $\mu_2 = \mu^L\ |_{U(p-a,q-b)}$. By Theorem 1.2 in [4] and its proof, there is a representation $(V_{\tau_1}\ , \tau_1)$ of $U(p-a, q-b) \cap K$, occurring in $V_{\mu_2} \otimes \mathfrak{u}(p-a, q-b) \cap \mathfrak{p}$ such that the hermitian form $\langle\ ,\ \rangle^L$ is indefinite on $V_{\mu_2} \oplus V_{\tau_1}$. As we did above, we need to show that any weight of the form $\tau = \tau_1 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$, with $\tau_1$ a highest weight in

$$V_{\mu_2} \otimes \mathfrak{u}(p-a, q-b) \cap \mathfrak{p},$$

is dominant for $\mathfrak{k}$. But the roots in $\mathfrak{u}(p-a, q-b) \cap \mathfrak{p}$ are the set

$$R = \{\underline{+}(e_i - e_j) \mid 1 \leq i \leq p-a;\ p < j \leq p+q-b\}.$$

Then it all reduces to checking that inequalities (3.5) and (3.6) hold for $\tau$ instead of $\delta$, where $\tau = \mu + \eta$, $\eta \in R$. This in turn reduces to making sure that $(\mu, e_{p-a} - e_{p-a+1}) > 0$ and $(\mu, e_{p+q-b} - e_{p+q-b+1}) > 0$, which was done above.

Then $X^L \cong A_{\mathfrak{q}_0}\ (\lambda_0) = \left[\mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}_0}\ \right]^{s_0}\ (\mathbb{C}_{\lambda_0})$ where $s_0 = dim(\mathfrak{u}_0 \cap \mathfrak{k})$. Then, by induction by stages again,

$$\left[\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}\ \right]^{s}\ (A_{\mathfrak{q}_0}\ (\lambda_0)) \cong \left[\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_1}\ \right]^{s_1}\ (\mathbb{C}_{\lambda_0}),$$

where we set

$$\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{u}_1, \quad \mathfrak{u}_1 = \mathfrak{u}_0 + \mathfrak{u}, \quad \mathfrak{l}_1 = \mathfrak{l}_0,$$

$$s_1 = dim(\mathfrak{u}_1 \cap \mathfrak{k}), \text{ and } s = dim(\mathfrak{u} \cap \mathfrak{k}).$$

We want to use Proposition 2.3. The lowest K type of the module on the left-hand side has highest weight

$$\mu = \lambda_0 + 2\rho(\mathfrak{u}_1 \cap \mathfrak{p}).$$

On the other hand, $\gamma$ is the infinitesimal character of

$$\left[\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_1}\ \right]^{s_1}\ (\mathbb{C}_{\lambda_0}),$$

so

$$\gamma = \lambda_0 + \rho(\mathfrak{u}_1) + \rho_{\mathfrak{l}_1}, \tag{3.7}$$

for some choice of positive roots for $\mathfrak{l}_1$. Also, $\lambda_0$ is orthogonal to the roots of $\mathfrak{l}_1$ and $(\lambda_0, \alpha) \geq 0$ for all roots $\alpha \in \Delta(\mathfrak{u}_0)$. To see that $X \cong A_{\mathfrak{q}_1}\ (\lambda_0)$, using Proposition 2.3, we only need to see that

$$(\lambda_0, \alpha) \geq 0, \text{ for all } \alpha \in \Delta(\mathfrak{u}). \tag{3.8}$$

We will use the following:

**Lemma 3.1** (Vogan, unpublished). *Assume the above notation and suppose $X$ is the Langlands submodule of the Harish–Chandra module*

$$Y = \left[\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}\right]^{s} (A_{\mathfrak{q}_0} (\lambda_0)).$$

*Then, if the infinitesimal character $\gamma$ of $X$ satisfies condition* (1.1)*, $Y$ is irreducible and*

$$X \cong A_{\mathfrak{q}_1} (\lambda_0).$$

*Proof.* Set $\Delta^+(\gamma) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \big| (\gamma, \alpha) > 0\}$ and $\rho_\gamma = \rho(\Delta^+(\gamma))$. Recall that for $\alpha$ in this set,

$$(\gamma, \check{\alpha}) \geq (\rho_\gamma, \check{\alpha}) \geq 1.$$

By (3.7), if we can show that

$$\rho_\gamma = \rho(\mathfrak{u}_1) + \rho_{\mathfrak{l}_1} ,$$

then $\gamma - \rho_\gamma = \lambda_0$ and $\lambda_0$ will satisfy (3.8). Therefore it is enough to show that

$$\Delta^+(\gamma) = \Delta(\mathfrak{u}_1) + \Delta^+(\mathfrak{l}_1).$$

We can also write $\gamma$ in terms of its Langlands parameters. Let $M_1 A_1 N_1$ be a minimal real parabolic subgroup and $H_1 = T_1 A_1$ a maximally split Cartan subgroup of $L_1$. Recall that $\mathfrak{q} \supset \mathfrak{q}_{_V}$ and moreover, we can assume that $\mathfrak{q}_0 \supset \mathfrak{q}_{_V}(X^{^L})$ and

$$\lambda_{_V}(Y) = \lambda_0 + \rho(\mathfrak{u}_1) + \rho_{\mathfrak{m}_1} \tag{3.9}$$

(see [7], Section 5.3).

Let's first take a root $\beta \in \Delta^+(\gamma)$ which is also simple for $\mathfrak{l}_1$. Then $\beta$ is orthogonal to the roots in $\mathfrak{u}_1$ and also to $\lambda_0$ and

$$\begin{aligned}(\gamma, \check{\beta}) &= (\lambda_0, \check{\beta}) + (\rho(\mathfrak{u}_1), \check{\beta}) + (\rho_{\mathfrak{l}_1} , \check{\beta}) \\ &= 1.\end{aligned} \tag{3.10}$$

Hence, $\beta$ is also simple for $\Delta^+(\gamma)$ and $W(\mathfrak{l}_1, \mathfrak{h}_1)$ permutes the roots in $\Delta^+(\gamma)$ outside $\mathfrak{l}_1$. So $\mathfrak{l}_1$ is a Levi subgroup consistent with $\Delta^+(\gamma)$.

Now let $w \in W(\mathfrak{l}_1, \mathfrak{h}_1)$ be the longest element. Since $w$ is a product of simple reflections of roots in $\mathfrak{l}_1$, and hence orthogonal to $\lambda_0$, then $w$ fixes $\lambda_0$. Also $w$ fixes $\rho(\mathfrak{u}_1)$, since the roots of $\mathfrak{l}_1$ permute the roots in $\mathfrak{u}_1$. Using (3.7), we have that

$$\gamma + w\gamma = 2(\lambda_0 + \rho(\mathfrak{u}_1)), \tag{3.11}$$

and this must be positive on $\Delta^+(\gamma) \setminus \Delta(\mathfrak{l}_1, \mathfrak{h}_1)$.

We claim that $\gamma + w\gamma$ is positive on $\Delta(\mathfrak{u})$. Let $w'$ be the longest element of $W(M_1, T_1)$. Again $w'$ is a product of simple reflections of roots in $\mathfrak{l}_1$. Hence, it fixes $\lambda_0$ and $\rho(\mathfrak{u}_1)$ and from (3.9) we have

$$\begin{aligned}\lambda_{_V}(Y) + w'(\lambda_{_V}(Y)) &= 2(\lambda_0 + \rho(\mathfrak{u}_1)) \\ &= \gamma + w\gamma.\end{aligned} \tag{3.12}$$

Moreover, $w'$ permutes the roots in $\Delta(\mathfrak{u})$ and since $\mathfrak{q} \supset \mathfrak{q}_{_V}$, $(\lambda_{_V}(Y), \alpha) \geq 0$ on $\Delta(\mathfrak{u})$ and

$$(\lambda_{_V}(Y) + w'(\lambda_{_V}(Y)), \alpha) = (\lambda_{_V}(Y), \alpha + w'\alpha) \geq 0. \tag{3.13}$$

Combining (3.11)–(3.13) proves our claim and hence $\Delta(\mathfrak{u}) \subseteq \Delta^+(\gamma)$ and by (3.10) $\Delta^+(\gamma) = \Delta(\mathfrak{u}_1) \cup \Delta^+(\mathfrak{l}_1)$. So,

$$(\lambda_0, \alpha) = (\gamma - \rho_\gamma, \alpha) \geq 0,$$

for all $\alpha \in \Delta(\mathfrak{u})$. Hence by Proposition 2.3, $Y \cong A_{\mathfrak{q}_1}(\lambda_0)$.

This proves the lemma. □

This concludes the proof of Theorem 1.2.

## References

[1] A. Borel and N. Wallach: Continuous Cohomology, discrete subgroups and representations of reductive subgroups, in *Annals of Mathematics Studies* Vol. 94, Princeton University Press, 1980. MR **83c:**22018

[2] S. Kumaresan, The canonical $\mathfrak{k}$–types of the irreducible unitary $\mathfrak{g}$–modules with non-zero relative cohomology. *Invent. Math.* 59 (1980) 1–11. MR **83c:**17011

[3] R. Parthasarathy, Criteria for the unitarizability of some highest weight modules. *Proc. Indian Acad. Sci.* 89 (1980) 1–24. MR **82c:**22020

[4] S. A. Salamanca–Riba, On the unitary dual of some classical Lie groups. *Compositio Mathematica.* 68 (1988) 251–303 MR **89m:**22018

[5] S. A. Salamanca–Riba, On the unitary dual of the classical Lie groups II. Representations of SO(n,m) inside the dominant Weyl chamber. *Compositio Mathematica.* 86 (1993) 127–146. MR **94e:**22025

[6] D. A. Vogan Jr., The algebraic structure of the representations of semisimple Lie groups I. *Annals of Mathematics.* 109 (1979) 1–60. MR **81j:**22020

[7] D. A. Vogan Jr., *Representations of Real Reductive Lie Groups.* Birkhauser, Boston–Basel–Stuttgart (1981). MR **83c:**22022

[8] D. A, Vogan Jr., Unitarizability of certain series of representations. *Annals of Mathematics.* 120 (1984) 141–187. MR **86h:**22028

[9] D. A. Vogan Jr. and Gregg J. Zuckerman, Unitary representations with non-zero cohomology. *Compositio Mathematica.* 53 (1984) 51–90. MR **86k:**22040

Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

*E-mail address*: ssalaman@nmsu.edu