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ON THE PRODUCT PROPERTY OF THE PLURICOMPLEX GREEN FUNCTION

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Abstract. We prove that the pluricomplex Green function has the product property $g_{D_1\times D_2}=\max\{g_{D_1},g_{D_2}\}$ for any domains $D_1\subset\mathbb{C}^n$ and $D_2\subset\mathbb{C}^m$.

Let E denote the unit disc in $\mathbb{C}$. For any domain $G\subset\mathbb{C}^n$ define

$$g_D(a,z):=\inf_{\substack{\varphi\in\mathcal{O}(E,D)\\ \varphi(0)=z\\ a\in\varphi(E)}}\Big\{\prod_{\lambda\in\varphi^{-1}(a)}|\lambda|^{\mathrm{ord}_\lambda(\varphi-a)}\Big\},\quad a,z\in D,$$

where $\mathcal{O}(E,D)$ denotes the set of all holomorphic mappings $E\to D$ and $\mathrm{ord}_\lambda(\varphi-a)$ denotes multiplicity of $\varphi-a$ at λ.

The function g_D is proposed by Poletsky (cf. [Pol]) and is called the *pluricomplex Green function for D*. We have that (see [Jar-Pfl1], Chapter IV)

(a) $$g_D(a,z)=\inf_{\substack{\varphi\in\mathcal{O}(\overline{E},D)\\ \varphi(0)=z\\ a\in\varphi(E)}}\Big\{\prod_{\lambda\in\varphi^{-1}(a)}|\lambda|^{\mathrm{ord}_\lambda(\varphi-a)}\Big\},\quad a,z\in D.$$

Note that in the formula (a) we take only $\lambda\in\varphi^{-1}(a)$ such that $\lambda\in E$.

(b) For any domains D_1, D_2 and any holomorphic mapping $f:D_1\to D_2$ we have the following contractible property: $g_{D_2}\big(f(z),f(w)\big)\le g_{D_1}(z,w)$, $z,w\in D_1$.

The main result of the paper is the following product property.

Theorem. *Let $D_1\subset\mathbb{C}^n$ and $D_2\subset\mathbb{C}^m$ be domains. Then*

$$g_{D_1\times D_2}\big((z_1,w_1),(z_2,w_2)\big)=\max\{g_{D_1}(z_1,z_2),g_{D_2}(w_1,w_2)\},\qquad (z_1,w_1),(z_2,w_2)\in D_1\times D_2.$$

Remark. The product property for $D_1\times D_2$ for the pluricomplex Green function in the case when D_1 or D_2 is pseudoconvex was proved in [Jar-Pfl2]. Note that in [Jar-Pfl2] the authors used the description of the pluricomplex Green function given by M. Klimek.

Proof. The inequality "$\ge$" follows from the property (b). So, we have to prove "$\le$".

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Let $(a_1,b_1),(a_2,b_2)\in D_1\times D_2$. If $a_1=a_2$ or $b_1=b_2$, then the required inequality follows from the property (b). So, we may assume that $a_1\neq a_2$ and $b_1\neq b_2$.

Suppose that $N\in(0,1)$ is such that

$$\max\{g_{D_1}(a_1,a_2),g_{D_2}(b_1,b_2)\}<N. \tag{1}$$

It is sufficient to prove that

$$g_{D_1\times D_2}\big((a_1,b_1),(a_2,b_2)\big)<N.$$

There are holomorphic mappings $\varphi_1:\bar{E}\to D_1$ and $\varphi_2:\bar{E}\to D_2$ such that $\varphi_1(0)=a_2$, $\varphi_2(0)=b_2$,

$$\prod_{\lambda\in\varphi_1^{-1}(a_1)}|\lambda|^{\operatorname{ord}_\lambda(\varphi_1-a_1)}<N \quad\text{and}\quad \prod_{\lambda\in\varphi_2^{-1}(b_1)}|\lambda|^{\operatorname{ord}_\lambda(\varphi_1-b_1)}<N.$$

Note that $\nu:=\#\big(\varphi_1^{-1}(a_1)\cap E\big)<\infty$ and $\mu:=\#\big(\varphi_2^{-1}(b_1)\cap E\big)<\infty$. We may assume that φ_1 and φ_2 are such that ν and μ are minimal.

Let $\varphi_1^{-1}(a_1)\cap E=\{\zeta_1,\dots,\zeta_\nu\}$ and $\varphi_2^{-1}(b_1)\cap E=\{\xi_1,\dots,\xi_\mu\}$, where each point counts with its multiplicity.[1] Since $\varphi_1(E)\Subset D_1$ and $\varphi_2(E)\Subset D_2$, we may assume that $|\zeta_1|<|\zeta_2|<\dots<|\zeta_\nu|$ and $|\xi_1|<|\xi_2|<\dots<|\xi_\nu|$, i.e. each point ζ_j and ξ_j is with multiplicity one.[2] Then

$$|\zeta_1\dots\zeta_\nu|\geq N|\zeta_\nu|^\nu \quad\text{and}\quad |\xi_1\dots\xi_\mu|\geq N|\xi_\mu|^\mu. \tag{2}$$

For, if $|\zeta_1\dots\zeta_\nu|<N|\zeta_\nu|^\nu$, then we may consider the mapping $\varphi_1(\zeta_\nu\lambda)$, and it contradicts the minimality of ν.

If $|\zeta_1\dots\zeta_\nu|<|\xi_1\dots\xi_\mu|$, then we replace φ_1 with the mapping $\widetilde{\varphi}_1(\lambda)=\varphi_1(t\lambda)$, where $t:=\left(\frac{|\zeta_1\dots\zeta_\nu|}{|\xi_1\dots\xi_\mu|}\right)^{\frac{1}{\nu}}$. Then $\left|\frac{1}{t}\zeta_j\right|<1$, $j=1,\dots,\nu$ (use (2)), and

$$\left|\left(\frac{\zeta_1}{t}\right)\dots\left(\frac{\zeta_\nu}{t}\right)\right|=|\xi_1\dots\xi_\mu|.$$

Hence, we may assume that

$$|\zeta_1\dots\zeta_\nu|=|\xi_1\dots\xi_\mu|=C<N.$$

Moreover, replacing $\varphi_1(\lambda)$ with $\varphi_1(e^{-i\theta_1}\lambda)$ and $\varphi_2(\lambda)$ with $\varphi_2(e^{-i\theta_2}\lambda)$, where θ_1,θ_2 are chosen such that $e^{i\theta_1}\zeta_1\dots e^{i\theta_1}\zeta_\nu=C$ and $e^{i\theta_2}\xi_1\dots e^{i\theta_2}\xi_\mu=C$, we may assume that $\zeta_1\dots\zeta_\nu=\xi_1\dots\xi_\mu=C$.

We consider Blaschke products

$$B_1(\lambda):=\prod_{j=1}^{\nu}\frac{\zeta_j-\lambda}{1-\bar{\zeta}_j\lambda}$$

and

$$\widetilde{B}_1(\lambda)=\frac{B_1(\lambda)-B_1(0)}{1-\overline{B_1}(0)B_1(\lambda)}=e^{i\theta}\prod_{j=1}^{\nu}\frac{\lambda-w_j}{1-\bar{w}_j\lambda},\quad \lambda\in\bar{E}.$$

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[1] Note that mappings φ_1 and φ_2 are holomorphic in some neighborhood of $\overline{E}$, and the sets $\varphi_1^{-1}(a_1)$ and $\varphi_2^{-1}(b_1)$ may contain points outside of E.

[2] For example, it is enough to change very little the mappings φ_1 and φ_2 by the formula (3) given below.

We choose different w'_j, $1 \leq j \leq \nu$, as close to w_j as we want such that $0 \in \{w'_1, \dots, w'_\nu\}$. Define

$$G_1(\lambda) = e^{i\theta} \prod_{j=1}^{\nu} \frac{\lambda - w'_j}{1 - \bar{w}'_j \lambda}.$$

Note that $\widetilde{B}_1^{-1}(-C) = \{\zeta_1, \dots, \zeta_\nu\}$. We can find $w'_1, \dots, w'_\nu$ such that $G_1^{-1}(-C)$ consists of ν different points ζ'_j, $1 \leq j \leq \nu$, as close to points ζ_j as we want. Let us replace the mapping φ_1 with the mapping

$$\widetilde{\varphi}_1(\lambda) := (\varphi(\lambda) - a_1) \frac{\prod_{j=1}^{\nu} \zeta_j (\lambda - \zeta'_j)}{\prod_{j=1}^{\nu} \zeta'_j (\lambda - \zeta_j)} + a_1. \tag{3}$$

Clearly, when ζ'_j, $1 \leq j \leq \nu$, are sufficiently close to ζ_j, $\widetilde{\varphi}_1$ maps E into D_1 (recall that φ_1 maps some neighborhood of $\bar{E}$ into D_1, hence $\varphi_1(E) \Subset D_1$), and $\widetilde{\varphi}_1(0) = \varphi_1(0)$, $\widetilde{\varphi}_1(\zeta'_j) = \varphi_1(\zeta_j)$.

Repeating this process for φ_2, we may assume that for Blaschke products B_1 and B_2 derivatives are not equal to 0 either on preimages of C or at points ζ_j or ξ_j respectively.

Let A be the union of images of singular points under mappings B_1 and B_2. Note that neither 0 nor C is in A. Let π be a holomorphic universal covering of $E \setminus A$ by E with $\pi(0) = C$. There are liftings ψ_1 and ψ_2 mapping E into E such that $\pi = B_1 \circ \psi_1 = B_2 \circ \psi_2$ and $\psi_1(0) = \psi_2(0) = 0$. If $\pi^{-1}(0) = \{\eta_1, \eta_2, \dots\}$, then mappings $\varphi_1 \circ \psi_1$ and $\varphi_2 \circ \psi_2$ map 0 into a_2 and b_2, and all points η_j into a_1 and b_1 respectively.

Note that π has all radial limits either in ∂E or in A. Since A is finite, π is an inner function. By Theorem 2 of Ch. III in [Nos], every inner function which has no zero radial limits is a Blaschke product. Thus

$$\pi(\lambda) = e^{i\alpha} \prod_{j=1}^{\infty} \frac{\bar{\eta}_j}{|\eta_j|} \frac{\eta_j - \lambda}{1 - \bar{\eta}_j \lambda}$$

and

$$|\prod_{j=1}^{\infty} \eta_j| = \pi(0) = C < N.$$

Since $(\varphi_1 \circ \psi_1, \varphi_2 \circ \psi_2)$ maps E into $D_1 \times D_2$,

$$g_{D_1 \times D_2}\big((a_1, b_1), (a_2, b_2)\big) \leq \prod_{j=1}^{\infty} |\eta_j| < N.$$

$\square$

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References

[Jar-Pfl1] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, 1993. MR **94k:**32039

[Jar-Pfl2] M. Jarnicki and P. Pflug, *Remarks on the pluricomplex Green function*, Indiana Univ. Math. Journal **44 (2)** (1995), 535-543. MR **96k:**32026

[Nos] K. Noshiro, *Cluster sets*, Springer-Verlag, 1960. MR **24:**A3295

[Pol] E. Poletsky, *Holomorphic currents*, Indiana Univ. Math. Journal **42 (1)** (1993), 85-144. MR **94c:**32007

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