

## AN IMPROVED ESTIMATE FOR THE HIGHEST LYAPUNOV EXPONENT IN THE METHOD OF FREEZING

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ABSTRACT. Let  $\dot{x} = A(t)x$  and  $\lambda_k(t)$  be the eigenvalues of the matrix  $A(t)$ . The main result of the Method of Freezing states that if  $\sup_J \|A(t)\| \leq M$ ,  $\sup_J \max_{1 \leq k \leq n} \operatorname{Re} \lambda_k(t) \leq \rho$  and  $\sup_J (\|A(t) - A(s)\|/|t - s|) \leq \delta$ , then

$$x_{\max} \leq \rho + 2M\lambda_\delta,$$

for the highest exponent  $x_{\max}$  of the system, where

$$\lambda_\delta = \left( \frac{C_n \delta}{4M^2} \right)^{\frac{1}{n+1}}.$$

The previous best known value  $C_n = \frac{n(n+1)}{2}$  and the substantially smaller values of  $C_n$  are reduced to the still smaller value.

1.

Let us consider an  $n$ -dimensional system

$$(1) \quad \dot{x} = A(t)x, \quad t \in J = [t_0, \infty),$$

with (real or complex) bounded and continuous on  $J$  matrix function  $A(t)$  and let  $\lambda_k(t)$ ,  $k = 1, \dots, n$ , be the eigenvalues of  $A(t)$ .

The Lyapunov exponent  $\chi[x(t)]$  of a solution  $x(t)$  of (1) and the highest Lyapunov exponent  $\chi_{\max}$  of the system (1) are given by [3]:

$$(2) \quad \chi[x(t)] = \overline{\lim}_{t \rightarrow \infty} \frac{\log \|x(t)\|}{t}, \quad \chi_{\max} = \sup_{x \neq 0} \chi[x(t)] = \max_{x \neq 0} \chi[x(t)].$$

(Throughout this paper  $\|\cdot\|$  denotes the Euclidean norm for a vector or matrix argument:

$$\|x\|_E = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_E = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2},$$

if  $x = \operatorname{colon}(x_1, \dots, x_n)$  and  $A = ((a_{ij}))$ , respectively. These norms are compatible:  $\|Ax\| \leq \|A\| \|x\|$ . Moreover, the Euclidean matrix norm is the ring norm:  $\|AB\| \leq \|A\| \|B\|$ , but it does not preserve the unity:  $\|I\| = \sqrt{n} \neq 1$  if  $n > 1$ .)

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In a series of papers the estimate of the highest Lyapunov exponent  $\chi_{\max}$  of the system (1) with “slowly changing” coefficients (with  $\delta > 0$  small enough) was given by values:

$$(3) \quad \sup_J \|A(t)\| \leq M,$$

$$(4) \quad \sup_J \frac{\|A(t) - A(s)\|}{|t - s|} \leq \delta,$$

$$(5) \quad \sup_J \max_{1 \leq k \leq n} \operatorname{Re} \lambda_k[A(t)] \leq \rho.$$

In these works the estimate has the following form:

$$(6) \quad \chi_{\max} \leq \rho + C_0 \delta^\gamma,$$

where  $C_0 \geq 0$ ,  $\gamma \geq 0$  are suitable constants.

So, for example, from [9] we can take (6) with  $\gamma = 1/(2n + 2)$ ; in [1] a better estimate with  $\gamma = 1/2n$  was obtained. Later in [2] (see also [3]) these estimates were strengthened to the following:

$$(7) \quad \chi_{\max} \leq \rho + C_0 \delta^{\frac{1}{n+1}},$$

where  $C_0 = C_n = 2M \left(\frac{n(n+1)}{8M^2}\right)^{\frac{1}{n+1}}$ .

Finally, in [7] for  $n = 2$  and in [5] for arbitrary  $n$ , it was proved that the index  $\frac{1}{n+1}$  in (7) gives the best value of the constant  $\gamma$  in (6).

On the other hand, in [8], [10] we have an improvement of (6) in another direction: a decrease of the constant  $C_0$  in (6) by the better (or the best) “point of freezing” (see [3], [10]).

It goes without saying, that every such improvement of (6) depends upon the choice of the norm, while the definitions (2) are independent of the norm.

Moreover, for every fixed norm we can formulate the question about the best or the exact value of the constant  $C_n$ : this value of  $C_n$  with the attainability of the index  $\gamma = 1/(n + 1)$  will give the best estimate of type (6).

In this work we show that

1) the well-known inequality of Gelfand-Schilov (G.-S.) (see [3], p. 131) for the matrix exponent  $\exp(A(t)s)$  can give an essential improvement in the Euclidean norm;

2) using this improved inequality of G.-S, in the standard way of [3] we can have a better estimate of  $\chi_{\max}$  than in [3], [8], [10];

3) using this improved inequality of G.-S. and the best “point of freezing” of [10] we have a still better estimate of the highest Lyapunov exponent  $\chi_{\max}$  in the Method of Freezing.

*Remark 1.* If  $A(t)$  is differentiable, then (4) is equivalent to  $\|\dot{A}(t)\| \leq \delta$ .

*Remark 2.* (7) is true but trivial, when  $n(n + 1)\delta/(8M^2) \leq 1$ . So, the method is of interest just for  $\delta > 0$  small.

## 2. THE “FROZEN” EQUATION

For simplicity we let  $t_0 = 0$  in (1); the general case can be treated quite similarly—just replace  $(0, t)$  with  $(t_0, t_0 + t)$ .

The fundamental role in the Method of Freezing is played by the following integral equation for every solution  $x(t)$  of (1) obtained by the Variation Constants

Formula (see [3] or [10])

$$x(t) = e^{A(t_1)t}x(0) + \int_0^t e^{A(t_1)(t-s)}[A(s) - A(t)]x(s) ds,$$

where  $t_1 \in J$  and can be chosen arbitrarily. We notice also that, for example, in [3] the point  $t_1$  was chosen as  $t$ , but in [10] a proper choice of  $t_1$  was playing the crucial role.

### 3.

To obtain our main results we need a good estimate of the norm of the following matrix function:  $\exp(A(t_1)t)$ . One such estimate (for an arbitrary ring norm) is the well-known inequality of G.-S. (see [3], p. 131, or [4], p. 92),

$$(8) \quad \|e^{A(t_1)t}\| \leq e^{\rho t} \left( \|\mathbf{I}\| + \sum_{k=1}^{n-1} \frac{(2\|A(t_1)\|t)^k}{k!} \right),$$

where  $\mathbf{I}$  is the  $n \times n$  unit matrix.

In the following Theorem 1 we shall prove a better estimate for the matrix function  $\|\exp(A(t_1)t)\|$  in the Euclidean norm.

**Theorem 1.** *If (3) and (5) hold, then*

$$(9) \quad \begin{aligned} \|e^{A(t_1)t}\| &\leq e^{\rho t} \left( \|\mathbf{I}\| + \sum_{k=1}^{n-1} \frac{(\|A(t_1)\|t)^k}{k!} \right) \\ &\leq e^{\rho t} \left( \|\mathbf{I}\| + \sum_{k=1}^{n-1} \frac{(Mt)^k}{k!} \right), \end{aligned}$$

where  $\|\cdot\|$  is the Euclidean matrix norm.

To prove this Theorem 1 for an arbitrary matrix  $A$  we need some preliminary notions and results.

We introduce the notion of the absolute value (modulus) of a matrix  $A = ((a_{ij}))$ , supposing  $|A| = \text{mod } A = ((|a_{ij}|))$  and the notion of the inequality  $A \leq B$ , if all corresponding elements  $a_{ij}$  and  $b_{ij}$  of these matrices satisfy the inequalities:  $a_{ij} \leq b_{ij}$  (in particular,  $A \leq 0$  if  $\forall a_{ij} \leq 0$ ).

*Remark 3.* Notice that the Euclidean matrix norm is

- $\alpha$ ) absolute:  $\|A\| = \|\text{mod } A\|$  for all matrices  $A$ ;
- $\beta$ ) invariant with respect to unitary matrices:  $\|UA\| = \|AU\| = \|A\|$  for all matrices  $A$  and all unitary matrices  $U$ ;
- $\gamma$ ) monotone:  $\|A\| \leq \|B\|$  for all matrices  $A, B$  such that  $\text{mod } A \leq \text{mod } B$ .

*Proof of Theorem 1.* Indeed, according to the Theorem of Schur [6] there exists a unitary matrix  $U$  such that  $A = U^*(D + \Psi)U$ , where  $D + \Psi$  is the upper triangular matrix with diagonal part  $D = \text{diag}[\lambda_1, \dots, \lambda_n]$  and  $\lambda_k = \lambda_k[A]$ ,  $k = 1, \dots, n$ , are the eigenvalues of the matrix  $A$ .

Let  $\lambda_k = p_k + iq_k$ ,  $k = 1, \dots, n$ , and  $\psi_{km}$  be the elements of the matrices  $D$  and  $\Psi$ , respectively, where  $p_k$  and  $q_k$  are the real and imaginary parts of  $\lambda_k$ . Then using the Re-transformation ([3], p. 249)  $L = L(t) = \text{diag}[\exp(iq_1t), \dots, \exp(iq_nt)]$  we shall have

$$(10) \quad e^{At} = U^*LYU,$$

where  $Y = Y(t)$  ( $Y(0) = \mathbf{I}$ ) is the fundamental matrix of the solutions of the system  $\dot{y} = B(t)y$  and

$$B = \begin{pmatrix} p_1 & \psi_{12}e^{i(q_2-q_1)t} & \psi_{13}e^{i(q_3-q_1)t} & \dots & \psi_{1n}e^{i(q_n-q_1)t} \\ 0 & p_2 & \psi_{23}e^{i(q_3-q_2)t} & \dots & \psi_{2n}e^{i(q_n-q_2)t} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p_n \end{pmatrix}.$$

Further, using the  $\lambda$ -transformation ([3], p. 249) with  $\lambda \geq -\min\{0, \min_{1 \leq k \leq n} p_k\}$ , let us consider the system  $\dot{z} = B_1(t)z$ , where  $B_1 = B + \lambda I$  and  $z = (\exp(\lambda t)y)$ . Let  $Z(t)$  be a fundamental matrix of its solutions such that  $Z(0) = \mathbf{I}$ ; then we have for all  $t \in J$

$$|Z(t)| \leq \mathbf{I} + \int_0^t |B_1(\xi)| |Z(\xi)| d\xi \quad (|Z| = \text{mod } Z).$$

Since  $|B_1(\xi)| \leq (\lambda + \rho)\mathbf{I} + |\Psi|$  for all  $\xi \in J$ , then

$$(11) \quad |Z(t)| \leq W(t), \quad Z(t) = e^{\lambda t}Y(t)$$

and

$$W(t) = \mathbf{I} + [(\lambda + \rho)\mathbf{I} + |\Psi|] \int_0^t W(\xi) d\xi$$

or

$$(12) \quad \dot{W} = [(\lambda + \rho)\mathbf{I} + |\Psi|]W, \quad W(0) = \mathbf{I}$$

(see, for example, [3], pp. 506–507).

The solution of problem (12) is given by

$$(13) \quad W(t) = e^{(\lambda+\rho)t} e^{|\Psi|t}.$$

Consequently, from (10)–(13) and Remark 3 we shall have statement (9) of our Theorem 1.  $\square$

*Remark 4.* As follows from (13) this proof of Theorem 1 gives more than (9) for the Euclidean matrix norm:

$$(14) \quad \|e^{At}\| \leq e^{\rho t} \left[ \|\mathbf{I}\| + \|\Psi\|t + \dots + \|\Psi\|^{n-1} \frac{t^{n-1}}{(n-1)!} \right].$$

Also, there exist examples when such an estimate has the following form:

$$(15) \quad \|e^{At}\| \leq e^{\rho t} \left[ \|\mathbf{I}\| + \sum_{k=1}^{n-1} \frac{\|A^k\|t^k}{k!} \right]$$

or even

$$(16) \quad \|e^{At}\| \leq e^{\rho t} \left[ \|\mathbf{I}\|^2 + \left( \sum_{k=1}^{n-1} \frac{\|A^k\|t^k}{k!} \right)^2 \right]^{1/2},$$

but, seemingly, it is not known: is it possible to lower the estimate (9) to the forms (15) or (16)?

Moreover, notice that for a sparse matrix  $A$  (see [12]) often the norm  $\|A^k\|$  or  $\| |A|^k \|$  is essentially smaller than  $\|A\|^k$ . This fact can be important in applications.

*Remark 5.* Another matrix norm is known,

$$\|A\|_S = \sup_{x \neq 0} \frac{\|Ax\|_E}{\|x\|_E} = \max_{1 \leq k \leq n} \sqrt{\lambda_k[A^*A]},$$

where  $\|\cdot\|$  is the Euclidean vector norm and  $s_k = \sqrt{\lambda_k[A^*A]}$  are the singular values of the matrix  $A$ . This matrix norm (the Spectral matrix norm; see, for example, [11] p. 65 or [6], p. 236) is also the ring norm, invariant with respect to unitary matrices and preserves the unity, but it is not monotone. Therefore, using the Variation of Constant Formula we can prove only (14) for the Spectral matrix norm.

4.

**Theorem 2.** *If (3)-(5) hold in the Euclidean norm, then*

$$(17) \quad \chi_{\max} \leq \rho + \frac{1}{\zeta},$$

where  $\zeta$  is the positive root of the algebraic equation

$$(18) \quad \delta e \zeta^2 + \delta \sum_{k=2}^n k M^{k-1} \zeta^{k+1} = 1 \quad (e = \|\mathbf{I}\| = \sqrt{n}).$$

In particular, there exists a number  $\Delta = \Delta(n, M)$  such that for all  $\delta \leq \Delta$

$$(19) \quad \chi_{\max} \leq \rho + M \left[ \frac{2e + (n-1)(n+2)}{2M^2} \delta \right]^{\frac{1}{n+1}}.$$

The value of  $\Delta$  can be expressed explicitly:

$$(20) \quad \Delta = \frac{2M^2}{2e + (n-1)(n+2)}.$$

Moreover, the estimate (19) is better than the inequality of R. Vinograd (see [10], Theorem 1) for all  $n \geq 4$ .

*Proof.* In fact, using our improved estimate (9) in Theorem 10.2.2 and its Consequences 10.2.2 and 10.2.3 from [3], we shall have (17), where  $\zeta$  is a positive root of (18) (unique, because the left part of (18) is monotone on  $\zeta$ ).

Further, putting  $M\zeta = z$ , we can rewrite (18) in the following form:

$$(21) \quad \frac{\delta e}{M^2} z^2 + \frac{\delta}{M^2} \sum_{k=2}^n k z^k = 1.$$

If, in addition,  $\delta \leq \Delta$ , where  $\Delta$  is defined by (20), then the unique positive root  $z_0$  of (21) is not less than 1 ( $z \geq 1$ ) and thus from (21) we shall have

$$z_0 \geq \left[ \frac{2M^2}{[2e + (n-1)(n+2)]\delta} \right]^{\frac{1}{n+1}}$$

or the required estimate (19).

To prove the last part of Theorem 2 we notice that the inequality of R. Vinograd from [10] for the highest Lyapunov exponent has the following form:

$$(22) \quad X_{\max} \leq \rho + 2M \left[ \frac{C'_n + \varepsilon}{4M^2} \delta \right]^{\frac{1}{n+1}}, \quad C'_n = \frac{2n^n e^{-n}}{(n-1)!}, \quad \varepsilon > 0,$$

for the Euclidean norm (see also Theorem 3 below).

Therefore, according to Stirling's Formula, the last statement of our Theorem 2 for the comparison of the estimates (19) and (22) is equivalent to the following:

$$2e + (n-1)(n+2) < 2^n \left( \sqrt{\frac{2n}{\pi}} e^{-\frac{\Theta_n}{12n}} + \varepsilon \right), \quad 0 < \Theta_n < 1.$$

But this inequality is true for all  $n \geq 4$  (even if  $\varepsilon = 0$  when Theorem 1 in [10] loses its meaning).  $\square$

**Theorem 3.** *Let (3)–(5) hold in the Euclidean norm. Then given  $\varepsilon$ ,  $0 < \varepsilon < (n+1)(n+4\sqrt{n}-2)/2$ , there is a  $\delta(\varepsilon) > 0$  such that for  $\delta < \delta(\varepsilon)$*

$$X_{\max} \leq \rho + M \left[ \frac{C'_n + \varepsilon}{M^2} \delta \right]^{\frac{1}{n+1}}, \quad C'_n = \frac{2n^n e^{-n}}{(n-1)!}.$$

The value of  $\delta(\varepsilon)$  can be expressed explicitly:

$$\delta(\varepsilon) = M^2 \varepsilon^{n+1} \left[ \frac{2}{(n+1)(n+\sqrt{n}-2)} \right]^{n+2}.$$

*Remark 6.* The distinction between this Theorem 3 and Theorem 1 in [10] is the following: in Theorem 3 we use the improved inequality (9) in the Euclidean norm of matrix exponent, whereas in Theorem 1 in [10] the standard inequality (8) of Gelfand-Schilov was used in a matrix norm  $\|\cdot\|$  such that  $\|\mathbf{I}\| = 1$ .

*Remark 7.* In Theorem 1 in [10] and in our Theorem 3 the decrease (increase) of the interval with respect to  $\varepsilon$  does not change the estimate for  $X_{\max}$  of system (1), but increases (decreases) the interval  $[0, \delta(\varepsilon)] \ni \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$  of their applications.

## 5. COMPARISON OF THE RESULTS

$\alpha$ ) Theorem 2 gives a better estimate than (22) (the estimate of R. Vinograd in [10]) for all  $n > 3$ ;

$\beta$ ) Theorem 3 gives a better estimate than (22) for all  $n > 1$  (in the case  $n = 1$  the results coincide);

$\gamma$ ) Theorem 3 gives a better estimate than Theorem 2 for all  $n \geq 1$ .

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