

A DIMENSION RESULT ARISING FROM THE L^q -SPECTRUM OF A MEASURE

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ABSTRACT. We give a rigorous proof of the following heuristic result: Let μ be a Borel probability measure and let $\tau(q)$ be the L^q -spectrum of μ . If $\tau(q)$ is differentiable at $q = 1$, then the Hausdorff dimension and the entropy dimension of μ equal $\tau'(1)$. Our result improves significantly some recent results of a similar nature; it is also of particular interest for computing the Hausdorff and entropy dimensions of the class of self-similar measures defined by maps which do not satisfy the open set condition.

1. INTRODUCTION

Let μ be a Borel probability measure on \mathbb{R}^d with bounded support and let $\text{supp}(\mu)$ denote the support of μ . For a finite Borel partition \mathcal{P} of $\text{supp}(\mu)$, we let $|\mathcal{P}|$ be the maximum of the diameters of elements of \mathcal{P} . Define

$$h(\mu, \mathcal{P}) = - \sum_{A \in \mathcal{P}} \mu(A) \ln \mu(A).$$

For $\delta > 0$, let

$$h(\mu, \delta) = \inf \{ h(\mu, \mathcal{P}) : \mathcal{P} \text{ is a finite Borel partition of } \text{supp}(\mu), |\mathcal{P}| \leq \delta \}.$$

The *entropy dimension* (or *Rényi dimension* [Re]) of μ is defined as

$$\dim_e(\mu) = \lim_{\delta \rightarrow 0^+} \frac{h(\mu, \delta)}{-\ln \delta}.$$

Also, we let $\dim_H(E)$ denote the Hausdorff dimension of a set E and define the *Hausdorff dimension* of μ as

$$\dim_H(\mu) = \inf \{ \dim_H(E) : \mu(\mathbb{R}^d \setminus E) = 0 \}.$$

Young [Y] proved that if

$$(1.1) \quad \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \quad \text{for } \mu \text{ a.e. } x \in \text{supp}(\mu),$$

then

$$(1.2) \quad \dim_H(\mu) = \dim_e(\mu) = \alpha.$$

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An important sufficient condition for (1.1) to hold is when μ is a self-similar measure defined by

$$\mu = \sum_{i=1}^m p_i \mu \circ S_i^{-1},$$

where $\{S_i\}_{i=1}^m$ is a family of contractive similitudes satisfying the *open set condition* ([Hut], [F]), and the p_i 's are the probability weights satisfying $p_i > 0$ and $\sum_{i=1}^m p_i = 1$. In this case (1.1) holds for

$$(1.3) \quad \alpha = \sum_{i=1}^m p_i \ln p_i / \sum_{i=1}^m p_i \ln \rho_i,$$

where ρ_i is the contraction ratio of S_i . If we let

$$G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$

then $\dim_H(G) = \alpha$ also. This theorem was proved by Geronimo and Hardin [GH] for $\{S_i\}_{i=1}^m$ satisfying the *strong open set condition* (and also implicitly by Cawley and Mauldin [CM]). It was also proved by Strichartz [S] by using the law of iterated algorithm for $\{S_i\}_{i=1}^m$ satisfying the open set condition.

Another sufficient condition to obtain (1.1) comes from the L^q -spectrum. For $\delta > 0$ and $q \in \mathbb{R}$, the L^q -(*moment*) *spectrum* of μ is defined as

$$(1.4) \quad \tau(q) = \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $\{B_\delta(x_i)\}_i$ is a family of disjoint closed δ -balls with center $x_i \in \text{supp}(\mu)$ and the supremum is taken over all such families. The function $\tau(q)$ is an important function in multifractal theory; under suitable conditions, its Legendre transform equals the *dimension spectrum* of the measure μ ([H], [F]). Moreover, it is suggested in the physics literature that $\tau'(1)$ is equal to the entropy dimension of the measure ([HP], [H], [F]). Falconer [F] gives a heuristic argument for such equality. The purpose of this note is to give a rigorous proof of such a folklore theorem. Specifically, we prove

Theorem 1.1. *Let μ be a Borel probability measure on \mathbb{R}^d with bounded support. Then*

(a) *for μ a.e. $x \in \text{supp}(\mu)$, we have*

$$\tau'_+(1) \leq \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \tau'_-(1).$$

(b) *If $\tau(q)$ is differentiable at $q = 1$, then*

$$\lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \quad \text{for } \mu \text{ a.e. } x \in \text{supp}(\mu).$$

Consequently, μ is concentrated on $G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \right\}$, and

$$\dim_H(G) = \dim_H(\mu) = \dim_e(\mu) = \tau'(1).$$

We will prove Theorem 1.1 in Section 3. The main idea is to show that the set of points $x \in \text{supp}(\mu)$ such that

$$\varliminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \tau'_+(1) \quad \text{or} \quad \tau'_-(1) < \varlimsup_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}$$

has μ measure zero. The proof of this relies on estimations of some counting functions (Lemma 2.2) together with a standard covering lemma. For the special case of self-similar measures defined by contractive similitudes satisfying the open set condition, $\tau(q)$ is given by

$$\sum_{i=1}^m p_i^q \rho_i^{-\tau(q)} = 1.$$

Moreover, $\tau(q)$ is differentiable and $\tau'(1) = \alpha$, where α is given by (1.3) (see [CM]). Such results have also been proved for some extensions of the self-similar measures [AP], [R] (with the open set condition), and for equilibrium measures of Hölder continuous conformal expanding maps [PW]. The equality of $\dim_H(\mu)$ and $\tau'(1)$, under the assumption that $\tau(q)$ is differentiable at $q = 1$, was recently studied by Fan for a certain class of infinite product measures [Fa]. An additional example is the infinitely convolved Bernoulli measure associated with the golden ratio. This is a good illustration and the main motivation for our result because the open set condition fails. This will be discussed in Section 4.

2. PRELIMINARIES

Let $\tau : \mathbb{R} \rightarrow [-\infty, \infty)$ be a concave function. We define the *effective domain* of τ as

$$\text{Dom } \tau = \{x : -\infty < \tau(x) < \infty\}.$$

The *concave conjugate* (or the *Legendre transform*) of τ is the function $\tau^* : \mathbb{R} \rightarrow [-\infty, \infty)$ defined by

$$\tau^*(\alpha) = \inf\{\alpha x - \tau(x) : x \in \mathbb{R}\}.$$

For $x \in \text{Dom } \tau$, we let $\partial\tau(x) \subseteq \mathbb{R}$ be the *subdifferential* of τ at x , i.e.,

$$\partial\tau(x) = \{\alpha : \tau(y) \leq \tau(x) + \alpha(y - x) \quad \text{for all } y \in \mathbb{R}\}.$$

Then $\tau^*(\alpha) + \tau(x) = \alpha x$ for $\alpha \in \partial\tau(x)$ [Ro]. If $\tau(x)$ is differentiable at x , then $\partial\tau(x)$ is the singleton $\tau'(x)$. Otherwise, $\partial\tau(x)$ is a closed interval. We will denote the special subdifferentials $\partial\tau(0)$ and $\partial\tau(1)$ respectively by $[\alpha_0^-, \alpha_0^+]$ and $[\alpha_1^-, \alpha_1^+]$.

It is known (e.g. [LN1, Proposition 2.3]) that $\text{Dom } \tau^*$ is an interval and $(\text{Dom } \tau^*)^o = (\alpha_{\min}, \alpha_{\max})$, where

$$\begin{aligned} \alpha_{\min} &:= \inf\{\alpha : \alpha \in \partial\tau(x), x \in \text{Dom } \tau\}, \\ \alpha_{\max} &:= \sup\{\alpha : \alpha \in \partial\tau(x), x \in \text{Dom } \tau\}. \end{aligned}$$

For the rest of this note, we assume that $\tau(q)$ is the L^q -spectrum of a Borel probability measure μ defined by (1.4). It is known that $\tau(q)$ is increasing, concave and $\tau(1) = 0$ (see Figure 1). Moreover, it is proved in [LN1] that

$$(2.1) \quad \alpha_{\min} = \varliminf_{\delta \rightarrow 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta} \quad \text{and} \quad \alpha_{\max} = \varlimsup_{\delta \rightarrow 0^+} \frac{\ln(\inf_x \mu(B_\delta(x)))}{\ln \delta},$$

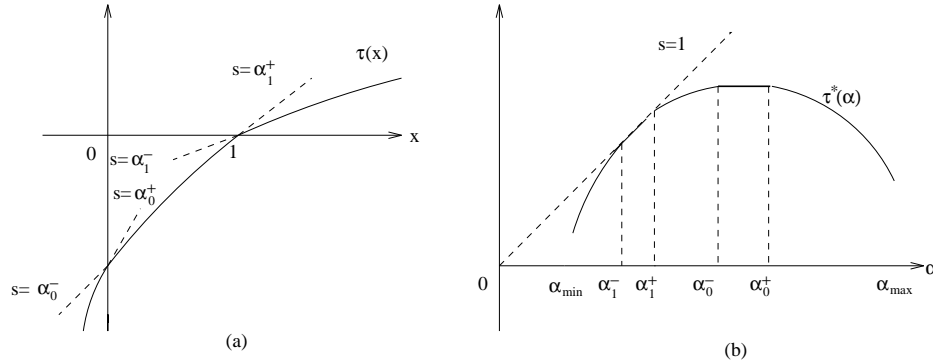


FIGURE 1. A concave function τ and its concave conjugate τ^* (s means slope)

where the supremum and infimum are taken over all $x \in \text{supp}(\mu)$. Define

$$\tau^*(\alpha_{\min}) := \lim_{q \rightarrow \infty} \tau^*(\alpha),$$

where $\alpha \in \partial\tau(q)$. The following proposition will be used in the proof of Lemma 3.1.

Proposition 2.1. *Assume that $\alpha_{\min} < \alpha_1^-$. Then $\alpha_{\min} > \tau^*(\alpha_{\min})$.*

Proof. Let $\alpha_{\min} < \tilde{\alpha} < \alpha_1^-$ and $q \in \partial\tau^*(\tilde{\alpha})$ (i.e., $\tilde{\alpha} \in \partial\tau(q)$). Consider the line with slope $\tilde{\alpha}$ passing through the point $(q, \tau(q))$. This line intersects the vertical line $q = 1$ at $(1, \tau(q) - (q - 1)\tilde{\alpha})$. By using the identity $\tau(q) + \tau^*(\tilde{\alpha}) = q\tilde{\alpha}$ together with the facts that τ is concave with $\tau(1) = 0$ and $\tilde{\alpha} < \alpha_1^-$, we have

$$\tilde{\alpha} - \tau^*(\tilde{\alpha}) = \tau(q) - (q - 1)\tilde{\alpha} > 0.$$

The same argument shows that $\alpha - \tau^*(\alpha)$ is an increasing function of q and hence

$$\alpha - \tau^*(\alpha) \geq \tilde{\alpha} - \tau^*(\tilde{\alpha}) \quad \text{for all } \alpha \leq \tilde{\alpha}.$$

The result follows by letting $q \rightarrow \infty$. □

Let \mathcal{B}_δ denote a disjoint family of closed balls of radii δ centered at points in $\text{supp}(\mu)$. For $\alpha \in (\text{Dom } \tau^*)^\circ$, we define the counting functions

$$N_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta, \mu(B) \geq \delta^\alpha\},$$

$$\tilde{N}_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta, \mu(B) < \delta^\alpha\}.$$

The following lemma is proved in [LN1, Lemma 4.2].

Lemma 2.2. *Let $\alpha_{\min} < \alpha < \alpha_0^+$, $q \in \partial\tau^*(\alpha)$ and $\xi > 0$. Then for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $0 < \delta < \delta_\epsilon$,*

$$N_\delta(\alpha \pm \epsilon) \leq \delta^{-\tau^*(\alpha) - (\xi \pm q)\epsilon}.$$

For $\alpha_0^+ \leq \alpha < \alpha_{\max}$, the above holds with \tilde{N}_δ replacing N_δ .

Lemma 2.2 and the counting functions play a key role in the proof of the main theorem.

3. PROOF OF THE MAIN THEOREM

We need two lemmas.

Lemma 3.1. *Let μ be a Borel probability measure on \mathbb{R}^d with bounded support. Then*

$$\mu\left\{x \in \text{supp}(\mu) : \alpha_{\min} \leq \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0.$$

Proof. Part 1. We claim that $\mu\left\{x \in \text{supp}(\mu) : \alpha_{\min} < \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0$. Let $\alpha_{\min} < \alpha < \alpha_1^-$ and $q \in \partial\tau^*(\alpha)$. Then $q > 1$. Since τ is increasing, concave, $\alpha < \alpha_1^-$, and since $\tau(1) = 0$, we have $(\tau(q) - \tau(1))/(q - 1) \geq \tau'_-(q) \geq \alpha > 0$. We choose $\epsilon > 0$ small enough so that

$$(3.1) \quad \sigma := (\tau(q) - (q - 1)\alpha)/2 \leq \tau(q) - (q - 1)\alpha - (2 + q)\epsilon.$$

(This implies that $\alpha + \epsilon < \alpha_1^-$.) Define

$$L_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \liminf_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3}\right\}.$$

We will show that $\mu(L_\epsilon(\alpha)) = 0$. Putting $\xi = 1$ in Lemma 2.2, then there exists $\delta_\epsilon > 0$ such that for all $0 < \delta \leq \delta_\epsilon$,

$$(3.2) \quad N_\delta(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha) - (1+q)\epsilon}.$$

Fix $m \in \mathbb{N}$ satisfying

$$(3.3) \quad 2^{-m} < \delta_\epsilon \quad \text{and} \quad m \geq 3\alpha/\epsilon + 2.$$

For each $x \in L_\epsilon(\alpha)$, we let n_x be the smallest integer satisfying the following conditions:

- (i) $n_x \geq m$;
- (ii) $\mu(B_\delta(x)) < \delta^{\alpha - \epsilon}$ for all $0 < \delta \leq 2^{-(n_x - 2)}$;
- (iii) there exists $\delta_x > 0$ such that

$$2^{-(n_x + 1)} < \delta_x \leq 2^{-n_x} \quad \text{and} \quad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha + 2\epsilon/3}.$$

Note that n_x is uniquely determined by x . Partition $L_\epsilon(\alpha)$ into a countable disjoint union of subsets $L_\epsilon^n(\alpha)$ where $L_\epsilon^n(\alpha) = \{x \in L_\epsilon(\alpha) : n_x = n\}$. Then

$$(3.4) \quad L_\epsilon(\alpha) = \bigcup_{n=m}^\infty L_\epsilon^n(\alpha).$$

Clearly for each $n \geq m$,

$$L_\epsilon^n(\alpha) \subseteq \bigcup_{x \in L_\epsilon^n(\alpha)} B_{2^{-n}}(x).$$

By a standard covering lemma (see [F, Lemma 4.8]), there exists a finite sequence $\{x_i\}_{i=1}^\ell$ in $L_\epsilon^n(\alpha)$ such that $\{B_{2^{-n}}(x_i)\}_{i=1}^\ell$ is a disjoint family and

$$(3.5) \quad L_\epsilon^n(\alpha) \subseteq \bigcup_{i=1}^\ell B_{2^{-(n-2)}}(x_i).$$

For $1 \leq i \leq \ell$, condition (iii) and (3.3) imply that

$$\mu(B_{2^{-n}}(x_i)) > 2^{-(n+1)(\alpha + 2\epsilon/3)} \geq 2^{-n(\alpha + \epsilon)}.$$

Hence by (3.2),

$$(3.6) \quad \ell \leq 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)}.$$

Combining condition (ii), (3.5), (3.6) and (3.1), we have

$$\begin{aligned} \mu(L_\epsilon^n(\alpha)) &\leq \sum_{i=1}^{\ell} \mu(B_{2^{-(n-2)}}(x_i)) \leq 2^{-(n-2)(\alpha-\epsilon)} \cdot 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)} \\ &\leq C \cdot 2^{-n(\tau(q)-(q-1)\alpha-(2+q)\epsilon)} \leq C \cdot 2^{-n\sigma}. \end{aligned}$$

(C is a constant independent of n .) Using this and (3.4), we have

$$\mu(L_\epsilon(\alpha)) \leq \sum_{n=m}^{\infty} \mu(L_\epsilon^n(\alpha)) \leq C \sum_{n=m}^{\infty} 2^{-n\sigma} = C \frac{2^{-\sigma m}}{1-2^{-\sigma}}.$$

Letting $m \rightarrow \infty$, we get $\mu(L_\epsilon(\alpha)) = 0$. The claim follows easily by taking a countable cover for $(\alpha_{\min}, \alpha_1^-)$ by sets of the form $L_\epsilon(\alpha)$.

Part 2. We will show that if $\alpha_{\min} < \alpha_1^-$, then

$$\mu\left\{x \in \text{supp}(\mu) : \lim_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha_{\min}\right\} = 0.$$

By Proposition 2.1, we may choose $\epsilon > 0$ sufficiently small and $\alpha \in (\text{Dom } \tau^*)^\circ$ sufficiently close to α_{\min} such that

$$0 < \sigma := (\alpha_{\min} - \tau^*(\alpha))/2 \leq \alpha_{\min} - \tau^*(\alpha) - (2+q)\epsilon,$$

where $q \in \partial\tau^*(\alpha)$. By Lemma 2.2, there exists $\delta_\epsilon > 0$ such that for all $0 < \delta \leq \delta_\epsilon$,

$$N_\delta(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha)-(1+q)\epsilon}.$$

Now choose m and n_x as in the proof of Part 1 but replace conditions (ii) and (iii) respectively by

- (ii)' $\mu(B_\delta(x)) < \delta^{\alpha_{\min}-\epsilon}$ for all $0 < \delta \leq 2^{-(n_x-2)}$;
- (iii)' there exists $\delta_x > 0$ such that

$$2^{-(n_x+1)} < \delta_x \leq 2^{-n_x} \quad \text{and} \quad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha_{\min}+\epsilon/2}.$$

The same proof yields the result and the lemma follows by combining the above two parts. \square

Lemma 3.2. *Under the same hypotheses of Lemma 3.1, then*

$$\mu\left\{x \in \text{supp}(\mu) : \alpha_1^+ < \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\right\} = 0.$$

Proof. Again we divide the proof into two parts.

Part 1. $\mu\left\{x \in \text{supp}(\mu) : \alpha_1^+ < \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_0^+\right\} = 0$. Let $\alpha_1^+ < \alpha < \alpha_0^+$ and $q \in \partial\tau^*(\alpha)$. The condition $\tau(q) - (q-1)\alpha > 0$ still holds by the assumption $\alpha > \alpha_1^+$ and by the fact that τ is increasing and concave. Instead of $L_\epsilon(\alpha)$, we define

$$U_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3}\right\}.$$

Let $\delta_\epsilon > 0$ and $m \in \mathbb{N}$ be as in the proof of Lemma 3.1. For each $x \in U_\epsilon(\alpha)$, we let n_x be chosen as in Lemma 3.1 but replace conditions (ii) and (iii) by (ii)' and (iii)' respectively as follows:

- (ii)' For all $0 < \delta \leq 2^{-(n_x-1)}$, $\mu(B_\delta(x)) \geq \delta^{\alpha+\epsilon}$;
- (iii)' there exists $\delta_x > 0$ such that

$$2^{-n_x} < \delta_x \leq 2^{-(n_x-1)} \quad \text{and} \quad \mu(B_{\delta_x}(x)) \leq \delta_x^{\alpha-\epsilon}.$$

Then apply the same technique.

Part 2. We need to show that if $\alpha > \alpha_1^+$ and $\alpha_0^+ \leq \alpha \leq \alpha_{\max}$, then

$$\mu\left\{x \in \text{supp}(\mu) : \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha\right\} = 0.$$

Choose $\epsilon > 0$ as in the proof of Part 1 of Lemma 3.1 and define

$$U'_\epsilon(\alpha) = \left\{x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\right\}.$$

Using Lemma 2.2, we can replace inequality (3.2) by $\tilde{N}_\epsilon(\alpha - \epsilon) \leq \delta^{-\tau^*(\alpha) - (1-q)\epsilon}$. A similar argument yields $\mu(U'_\epsilon(\alpha)) = 0$ and the result follows. \square

We now proof the main theorem by combining Lemmas 3.1 and 3.2.

Proof of Theorem 1.1. (a) It follows easily from (2.1) that for each $x \in \text{supp}(\mu)$,

$$\alpha_{\min} \leq \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha_{\max}.$$

Consequently, Lemma 3.1 implies that

$$\mu\left\{x \in \text{supp}(\mu) : \underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0.$$

By Lemma 3.2,

$$\mu\left\{x \in \text{supp}(\mu) : \overline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha_1^+\right\} = 0.$$

Part (a) now follows.

(b) The assumption that $\tau(q)$ is differentiable at $q = 1$ implies that $\partial\tau(1)$ is a singleton, i.e., $\alpha_1^- = \alpha_1^+ = \tau'(1)$. Part (a) now implies that for μ a.e. $x \in \text{supp}(\mu)$,

$$\underline{\lim}_{\delta \rightarrow 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1).$$

The result follows from Theorem 4.4 in [Y]. \square

4. INFINITE BERNOULLI CONVOLUTIONS

Let $0 < \rho < 1$, $S_1(x) = \rho x$, $S_2(x) = \rho x + (1 - \rho)$, and let μ_ρ be the self-similar measure defined by S_1, S_2 , i.e.,

$$\mu_\rho = \frac{1}{2}\mu_\rho \circ S_1^{-1} + \frac{1}{2}\mu_\rho \circ S_2^{-1}.$$

μ_ρ is known as an *infinitely convolved Bernoulli measure* (ICBM) because it can be identified with the distribution of the random variable $(1 - \rho) \sum_{n=0}^\infty \rho^n \epsilon_n$ where $\{\epsilon_n\}$ are i.i.d. random variables each taking values 0 or 1 with probability 1/2. Such measures have been studied extensively since the 30's. For $1/2 < \rho < 1$, $\{S_1, S_2\}$ does not satisfy the open set condition and hence the dimension result stated in (1.2) (with $\alpha = \tau'(1)$) does not cover such measures. An important result of Erdős says that if ρ^{-1} is a P.V. number, then μ is singular [E]. (Recall that an algebraic integer $\beta > 1$ is a *P.V. number* if all of its conjugates have moduli strictly less than 1.)

We will consider the special P.V. number $\rho_o^{-1} = (\sqrt{5} + 1)/2$ (the golden ratio), which is so far the best understood case. The Hausdorff and entropy dimensions of this particular measure have been studied by a number of authors ([AY], [AZ], [LP], [La]). It is known that these two dimensions are equal and it is conjectured that they are equal to 0.99571312... [AZ]. In [LN2], a closed formula which defines the corresponding $\tau(q)$ for all $q > 0$ is derived. Moreover, it is proved that $\tau(q)$ is differentiable on $(0, \infty)$ and

$$(4.1) \quad \tau'(1) = \frac{1}{9 \ln \rho_o} \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \ln c_J,$$

where

$$c_J = \frac{1}{8 \cdot 4^k} [1, 1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and $P_J = P_{j_1} \cdots P_{j_k}$, with $j_i = 0$ or 1 . Theorem 1.1 implies that $\tau'(1)$ is equal to the Hausdorff and entropy dimensions of the measure. Numerical calculations using (4.1) suggest that $\tau'(1) \approx 0.9957$, agreeing with the result obtained in [AY], [AZ] and [La]. It is an open question how to obtain the L^q -spectrum $\tau(q)$ for other P.V. numbers.

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REFERENCES

- [AP] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*, Math. Nachr. **181** (1996), 5-42. CMP 97:01
- [AY] J.C. Alexander and J.A. Yorke, *Fat baker's transformations*, Ergodic Theory Dynamical Systems. **4** (1984), 1-23. MR **86c**:58090
- [AZ] J.C. Alexander and D. Zagier, *The entropy of a certain infinitely convolved Bernoulli measure*, J. London Math. Soc. **44** (1991), 121-134. MR **92g**:28035
- [CM] R. Cawley and R.D. Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math. **92** (1992), 196-236. MR **93b**:58085
- [E] P. Erdős, *On a family of symmetric Bernoulli convolutions*, Amer. J. Math. **61** (1939), 974-976. MR **1**:52a
- [F] K.J. Falconer, *Fractal geometry-Mathematical foundations and applications*, John Wiley and Sons, New York, 1990. MR **92j**:28008
- [Fa] A.-H. Fan, *Multifractal analysis of infinite products*, J. Statist. Phys. **86** (1997), 1313-1336.
- [GH] J.S. Geronimo and D.P. Hardin, *An exact formula for the measure dimensions associated with a class of piecewise linear maps*, Constr. Approx. **5** (1989), 89-98. MR **90d**:58076
- [H] T.C. Halsey, M. H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman, *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A **33** (1986), 1141-1151. MR **87h**:58125a
- [HP] H. Hentschel and I. Procaccia, *The infinite number of generalized dimensions of fractals and strange attractors*, Physica **8D** (1983), 435-444. MR **85a**:58064
- [Hut] J.E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713-747. MR **82h**:49026
- [La] S.P. Lalley, *Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution*, preprint.
- [LN1] K.-S. Lau and S.-M. Ngai, *Multifractal measures and a weak separation condition*, Adv. Math. (to appear).

- [LN2] ———, *L^q -spectrum of the Bernoulli convolution associated with the golden ratio*, preprint.
- [LP] F. Ledrappier and A. Porzio, *A dimension formula for Bernoulli convolutions*, J. Statist. Phys. **76** (1994), 1307-1327. MR **95i**:58111
- [PW] Y. Pesin and H. Weiss, *A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions*, J. Statist. Phys. **86** (1997), 233-275.
- [R] D.A. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters*, Ergodic Theory Dynamical Systems. **9** (1989), 527-541. MR **90k**:58115
- [Re] A. Rényi, *Probability Theory*, North-Holland, Amsterdam, 1970. MR **47**:4296
- [Ro] R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, New Jersey, 1970. MR **43**:445
- [S] R.S. Strichartz, *Self-similar measures and their Fourier transforms I*, Indiana Univ. Math. J. **39** (1990), 797-817. MR **92k**:42015
- [Y] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynamical Systems. **2** (1982), 109-124. MR **84h**:58087

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