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# EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS WITH PSEUDO MONOTONE OPERATORS

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Abstract. In this paper, we study the existence of $T$-periodic solutions for the problem

$$u'(t) + A(t)u(t) = 0, \quad t \in \mathbb{R},$$

where $A(t)$ is a $T$-periodic, pseudo monotone mapping from a reflexive Banach space into its dual.

## 1. Introduction

Let $V$ be a reflexive Banach space, let $V'$ be its topological dual and let $H$ be a Hilbert space such that $V$ is densely and continuously imbedded in $H$. Let $A(t) : V \to V'$ be a $T$-periodic, pseudo monotone operator, let $A : V \to V'$ be a maximal monotone operator and let $G : \mathbb{R} \times H \to H$ be a Carathéodory mapping which is $T$-periodic in its first variable. In this paper, we study the existence of $T$-periodic solutions to a class of a nonlinear evolution equations of the form

$$u'(t) + A(t)u(t) = 0, \quad t \in \mathbb{R}. \tag{1.1}$$

This problem is a generalized problem of

$$u'(t) + Au(t) = G(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

since if $V$ is also compactly imbedded in $H$ and $A$ is also hemicontinuous then $v \mapsto Av - G(t, v)$ is pseudo monotone for almost every $t \in \mathbb{R}$.

Problems of this kind have been studied by many authors. When $A$ is linear, the problem (1.2) was studied by Amann [1], Becker [3], Prüss [13] and others. Vrabie [15] considered the case that $A$ is a fully nonlinear operator. He assumed that $(X, \|\cdot\|)$ is a Banach space, $A : D(A) \subset X \to X$ is an $m$-accretive operator and $G : \mathbb{R} \times \overline{D(A)} \to X$ is a Carathéodory mapping such that

(i) $\overline{D(A)}$ is convex, $-A$ generates a compact semigroup and there exists $a > 0$ such that $A - aI$ is $m$-accretive,
(ii) $G$ is $T$-periodic in its first variable and satisfies

$$\lim_{r\to\infty} (1/r) \sup\{\|G(t, v)\| : t \in \mathbb{R}, v \in \overline{D(A)}, \|v\| \le r\} < a,$$

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and he showed that (1.2) has a $T$-periodic, mild solution. Hirano [11] considered the case that $(H, \|\cdot\|)$ is a Hilbert space, $A$ is a subdifferential of a lower semicontinuous, proper, convex function from $H$ into $(-\infty, \infty]$ and $G : \mathbb{R}\times H \to H$ is a Carathéodory mapping such that

(iii) the resolvents of $A$ are compact,

(iv) $G$ is $T$-periodic in its first variable and there exist positive numbers $M_1$ and $M_2$ such that

$$\|G(t, v)\| \leq M_1\|v\| + M_2 \quad \text{for a.e. } t \in \mathbb{R} \text{ and for every } v \in H,$$

(v) there exist positive constants $a$ and $b$ such that

$$(z - G(t, v), v) \geq a\|v\|^2 - b \quad \text{for all } v \in D(A) \text{ and for all } z \in Av.$$

Caşcaval and Vrabie [5] extended Hirano's result to the case that $A$ is $m$-accretive and $-A$ generates a compact semigroup. On the other hand, Hirano [10] investigated the existence of solutions of initial value problems under the conditions (A1)–(A4) in our Theorem 1. In this paper, we show that these conditions guarantee the existence of $T$-periodic solutions of (1.1). We do not need either $A$ is $m$-accretive or $A$ is a subdifferential of a lower semicontinuous, proper, convex function. In order to prove our result, we use the method employed in [9, 10], the Galerkin method [16, 17], and Gossez and Mawhin's result [7, 12] which ensures the existence of periodic solutions for the finite dimensional case. Our method gives simple proofs for Hirano's theorems in [9, 10].

The next section is devoted to some preliminaries and notations. In section 3, we state our main result and we prove it in section 4. In the final section, we study an example.

## 2. Preliminaries and notations

Throughout this paper, all vector spaces are real and if $E$ is a Banach space then $E'$ denotes its topological dual. Let $E$ be a Banach space. We write $(y, x)$ in place of $y(x)$ for $x \in E$ and $y \in E'$. Let $T > 0$. $C(0, T; E)$ denotes the space of all continuous $E$-valued functions defined on $[0, T]$. For $1 \leq p < \infty$, $L^p(0, T; E)$ denotes the space of all strongly measurable, $p$-integrable, $E$-valued functions defined almost everywhere on $[0, T]$. We know [6] that if $E$ is reflexive, $1 < p < \infty$ and $q$ satisfy $1/p + 1/q = 1$ then the dual of $L^p(0, T; E)$ is $L^q(0, T; E')$.

Let $V$ be a reflexive Banach space which is densely and continuously imbedded in a Hilbert space $H$ and let $p$, $q$ and $T$ be positive constants such that $1/p + 1/q = 1$. Since we identify $H$ with its dual, we have $V \subset H \subset V'$. For each $u \in L^p(0, T; V)$ and $v \in L^q(0, T; V')$, $\langle v, u\rangle$ is defined by $\int_0^T (v(t), u(t))\, dt$. We denote by $W_p^1(0, T; V, H)$ the Banach space

$$W_p^1(0, T; V, H) = \{u \in L^p(0, T; V) : u' \in L^q(0, T; V')\}$$

with the norm $\|u\| + \|u'\|_*$, where $u'$ is the generalized derivative [2, 16] of $u$ and $\|\cdot\|$ and $\|\cdot\|_*$ are the norms of $L^p(0, T; V)$ and $L^q(0, T; V')$, respectively. It is well known [16] that $W_p^1(0, T; V, H)$ is a reflexive Banach space and that $W_p^1(0, T; V, H)$ is continuously imbedded in $C(0, T; H)$.

Let $V$ be a reflexive Banach space and let $A$ be a mapping from $V$ into $V'$. $A$ is said to be monotone if $(Ax - Ay, x - y) \geq 0$ for each $x, y \in V$. $A$ is said to be hemicontinuous if for each one dimensional subspace $L$ of $V$, $A$ is continuous from $L$ to $V'$ with $V'$ given its weak topology. $A$ is said to be finitely continuous if for each finite dimensional subspace $F$ of $V$, $A$ is continuous from $F$ to $V'$ with $V'$ given its weak topology. $A$ is said to be pseudo monotone if for each sequence $\{x_n\}$ in $V$ such that it converges weakly to $x_0 \in V$ and $\overline{\lim}(Ax_n, x_n - x_0) \leq 0$,

$$(Ax_0, x_0 - y) \leq \varliminf_{n\to\infty} (Ax_n, x_n - y) \quad \text{for all } y \in V.$$

It is well known [4] that if $A$ is monotone and hemicontinuous then $A$ is pseudo monotone. The following is Proposition 7.2 in [4]. Since this proposition holds, we don't need to use nets in the definition of pseudo monotone operators.

**Proposition 1** (Browder)**.** *Let $X$ be a reflexive Banach space, let $C$ be a bounded subset of $X$ and let $x_0$ be a point in the weak closure of $C$. Then there exists a sequence $\{x_n\}$ in $C$ converging weakly to $x_0$.*

To find solutions of Galerkin equations in the proof of our main result, we need the following. For its proof, see [7] or [12, Corollary VI.4].

**Proposition 2** (Gossez and Mawhin)**.** *Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory mapping (i.e., for every $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable and for almost every $t \in [0,T]$, $f(t,\cdot)$ is continuous) such that for each $\rho > 0$, there exists $\alpha_\rho \in L^1(0,T;\mathbb{R})$ such that $|f(t,x)| \leq \alpha_\rho(t)$ for almost every $t \in [0,T]$ and for every $x \in \mathbb{R}^n$ with $|x| \leq \rho$. Assume that there exist a nonnegative function $a \in L^1(0,T;\mathbb{R})$ and a positive number $r$ such that*

$$(x, f(t,x)) \leq a(t)(|x|^2 + 1)$$

*for almost every $t \in [0,T]$ and for every $x \in \mathbb{R}^n$ and*

$$\int_0^T \big(x(t), f(t,x(t))\big)\, dt \leq 0$$

*for every absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ with $x(0) = x(T)$ and $\min\limits_{0\leq t\leq T} |x(t)| \geq r$. Then there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $x'(t) = f(t,x(t))$ for almost every $t \in [0,T]$ and $x(0) = x(T)$.*

## 3. Main result

In the rest of this paper, $T$, $p$ and $q$ are positive constants such that $1/p+1/q = 1$. Now we state our main result which implies the existence of periodic solutions.

**Theorem 1.** *Let $(V, \|\cdot\|)$ be a reflexive Banach space which is densely and continuously imbedded in a Hilbert space $(H, |\cdot|)$ and let $\{A(t) : 0 \leq t \leq T\}$ be a family of mappings from $V$ into $V'$ such that*

(A1) *$A(t) : V \to V'$ is pseudo monotone for almost every $t \in [0,T]$;*
(A2) *for each $u \in L^p(0,T;V)$, $A(t)u(t)$ is $V'$-measurable on $[0,T]$;*
(A3) *there exist a positive constant $C_1$ and a nonnegative function $C_2 \in L^1(0,T;\mathbb{R})$ such that*

$$(A(t)x, x) \geq C_1\|x\|^p - C_2(t)$$

*for almost every $t \in [0,T]$ and for every $x \in V$;*

(A4) *there exist a positive constant* $C_3$ *and a nonnegative function* $C_4 \in L^q(0,T;\mathbb{R})$ *such that*

$$\|A(t)x\|_* \le C_3\|x\|^{p-1} + C_4(t)$$

*for almost every* $t \in [0,T]$ *and for every* $x \in V$*, where* $\|\cdot\|_*$ *is the norm of* $V'$.

*Then there exists* $u \in W_p^1(0,T;V,H)$ *such that*

$$u'(t) + A(t)u(t) = 0 \quad \text{for almost every } t \in [0,T]$$

*and*

$$u(0) = u(T).$$

## 4. Proof of Theorem 1

We denote by $\mathcal{V}$ and $\mathcal{V}'$ the spaces $L^p(0,T;V)$ and $L^q(0,T;V')$, respectively and the norms of these spaces are also denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. By $\mathcal{W}$, we mean $W_p^1(0,T;V,H)$. For $u \in \mathcal{V}$ and $t \in [0,T]$, we write $\mathcal{A}u(t)$ instead of $A(t)u(t)$.

The following is essentially due to Hirano [9, 10].

**Lemma 1** (Hirano)**.** *Let* $\{v_n\}$ *be a sequence in* $\mathcal{W}$ *such that* $\{v_n\}$ *converges to* $v_0$ *weakly in* $\mathcal{W}$ *and*

$$\varlimsup_{n\to\infty} \langle \mathcal{A}v_n, v_n - v_0\rangle \le 0.$$

*Then for each* $z \in \mathcal{V}$,

$$\langle \mathcal{A}v_0, v_0 - z\rangle \le \varliminf_{n\to\infty} \langle \mathcal{A}v_n, v_n - z\rangle. \tag{4.1}$$

*Especially,* $\{\mathcal{A}v_n\}$ *converges to* $\mathcal{A}v_0$ *weakly in* $\mathcal{V}'$.

*Proof.* We can show easily by (A3) and (A4) that there exist positive numbers $K_1$, $K_2$ and a nonnegative function $K_3 \in L^1(0,T;\mathbb{R})$ such that

$$(A(t)v(t), v(t) - z(t)) \ge K_1\|v(t)\|^p - K_2\|z(t)\|^p - K_3(t) \tag{4.2}$$

for almost every $t \in [0,T]$ and for every $v, z \in \mathcal{V}$. Since $\mathcal{W}$ is continuously imbedded in $C(0,T;H)$, we remark that $\{v_n(t)\}$ converges to $v_0(t)$ weakly in $H$ for all $t \in [0,T]$. We shall show that

$$\varliminf_{n\to\infty} (A(t)v_n(t), v_n(t) - v_0(t)) \ge 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.3}$$

Suppose that the following set

$$\begin{aligned}\{t \in [0,T] :\ &\varliminf_{n\to\infty} (A(t)v_n(t), v_n(t) - v_0(t)) < 0,\\ &(A(t)v_n(t), v_n(t) - v_0(t)) \ge K_1\|v_n(t)\|^p - K_2\|v_0(t)\|^p - K_3(t) \quad \text{for all } n \in \mathbb{N}\}\end{aligned}$$

has positive measure. Let $t$ be an element of the set. Since $\{v_n(t)\}$ is bounded in $V$, $\{v_n(t)\}$ converges to $v_0(t)$ weakly in $V$. By (A1), we have

$$\varliminf(A(t)v_n(t), v_n(t) - v_0(t)) = 0,$$

which contradicts that $t$ is an element of the above set. Hence we have (4.3). By (4.2) and Fatou's lemma, we have

$$\begin{aligned} 0 &\leq \int_0^T \varliminf_{n\to\infty} (A(t)v_n(t), v_n(t) - v_0(t))\, dt \\ &\leq \varliminf_{n\to\infty} \langle \mathcal{A}v_n, v_n - v_0\rangle \leq \varlimsup_{n\to\infty} \langle \mathcal{A}v_n, v_n - v_0\rangle \leq 0. \end{aligned}$$

Hence we obtain $\lim\langle \mathcal{A}v_n, v_n - v_0\rangle = 0$. Next we shall show that there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i\to\infty} (A(t)v_{n_i}(t), v_{n_i}(t) - v_0(t)) = 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.4}$$

Put $h_n(t) = (A(t)v_n(t), v_n(t) - v_0(t))$ for $t \in [0,T]$. We know that $\varliminf h_n(t) \geq 0$ for almost every $t \in [0,T]$ and $\lim \int_0^T h_n(t)\, dt = 0$. By (4.2) and Lebesgue's dominated convergence theorem, we get $\lim \int_0^T h_n^-(t)\, dt = 0$, where $h_n^-(t) = -\min\{h_n(t), 0\}$. Hence we obtain $\lim \int_0^T |h_n(t)|\, dt = 0$. Then we can choose a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ which satisfies (4.4).

Let $z \in \mathcal{V}$. By the preceding, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i\to\infty} \langle \mathcal{A}v_{n_i}, v_{n_i} - z\rangle = \varliminf_{n\to\infty} \langle \mathcal{A}v_n, v_n - z\rangle$$

and

$$\lim_{i\to\infty} (A(t)v_{n_i}(t), v_{n_i}(t) - v_0(t)) = 0 \quad \text{for a.e. } t \in [0,T].$$

Since $\{v_{n_i}(t)\}$ is bounded in $V$ by (4.2), $\{v_{n_i}(t)\}$ converges to $v_0(t)$ weakly in $V$ for almost every $t \in [0,T]$. So (A1) yields

$$(A(t)v_0(t), v_0(t) - z(t)) \leq \varliminf_{i\to\infty} (A(t)v_{n_i}(t), v_{n_i}(t) - z(t)) \quad \text{for a.e. } t \in [0,T].$$

Then, from (4.2) and Fatou's lemma, we find that

$$\begin{aligned} \langle \mathcal{A}v_0, v_0 - z\rangle &\leq \int_0^T \varliminf_{i\to\infty} (A(t)v_{n_i}(t), v_{n_i}(t) - z(t))\, dt \\ &\leq \varliminf_{i\to\infty} \langle \mathcal{A}v_{n_i}, v_{n_i} - z\rangle \\ &= \varliminf_{n\to\infty} \langle \mathcal{A}v_n, v_n - z\rangle. \end{aligned}$$

So we have $\lim\langle \mathcal{A}v_n, v_n - v_0\rangle = 0$ and hence we get (4.1). □

By using Proposition 2, we find solutions of Galerkin equations [16, 17].

**Lemma 2.** *For each finite dimensional subspace $F$ of $V$, there exists an absolutely continuous function $u : [0,T] \to F$ such that $u' \in L^q(0,T;F)$, $u(0) = u(T)$ and*

$$(u'(t) + A(t)u(t), v) = 0 \quad \textit{for a.e. } t \in [0,T] \textit{ and for every } v \in F. \tag{4.5}$$

*Proof.* Let $F$ be an $n$-dimensional subspace of $V$. Since $F$ is finite dimensional, there exist positive numbers $M_1, M_2$ such that $M_1|v| \leq \|v\| \leq M_2|v|$ for all $v \in F$. Let $\{w_1, \cdots, w_n\}$ be a basis of $V$ such that

$$(w_i, w_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Define $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(t,x) = -\begin{pmatrix} \left(A(t)\left(\sum_{i=1}^{n} x_i w_i\right), w_1\right) \\ \vdots \\ \left(A(t)\left(\sum_{i=1}^{n} x_i w_i\right), w_n\right) \end{pmatrix}, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Since $A(t)$ is finitely continuous by Lemma 3 in [8], $f$ is a Carathéodory mapping. Let $t \in [0,T]$ and $x \in \mathbb{R}^n$. By (A4), we have

$$\begin{aligned} |f(t,x)| &\le K \sum_{j=1}^{n} \left\| A(t)\left(\sum_{i=1}^{n} x_i w_i\right)\right\|_* \|w_j\| \\ &\le K M_2 n \left( C_3 \left\| \sum_{i=1}^{n} x_i w_i \right\|^{p-1} + C_4(t)\right) \\ &\le K M_2 n \left( C_3 M_2^{p-1} |x|^{p-1} + C_4(t)\right), \end{aligned}$$

where $K$ is a positive constant such that $|y| \le K \sum_{i=1}^{n} |y_i|$ for all $y \in \mathbb{R}^n$. Let $t \in [0,T]$ and $x \in \mathbb{R}^n$. By (A3), we have

$$\begin{aligned} (x, f(t,x)) &= -\sum_{j=1}^{n} x_j \left( A(t)\left(\sum_{i=1}^{n} x_i w_i\right), w_j\right) \\ &= -\left( A(t)\left(\sum_{i=1}^{n} x_i w_i\right), \sum_{j=1}^{n} x_j w_j\right) \\ &\le -C_1 \left\| \sum_{i=1}^{n} x_i w_i \right\|^p + C_2(t) \\ &\le C_2(t). \end{aligned}$$

Let $r > 0$ such that $C_1 M_1^p r^p T \ge \int_0^T C_2(t)\, dt$. Let $x : [0,T] \to \mathbb{R}^n$ be an absolutely continuous function with $x(0) = x(T)$ and $\min_{0 \le t \le T} |x(t)| \ge r$. Then we have

$$\begin{aligned} \int_0^T \big(x(t), f(t,x(t))\big)\, dt &\le \int_0^T \left( -C_1 \left\| \sum_{i=1}^{n} x_i(t) w_i \right\|^p + C_2(t)\right) dt \\ &\le -C_1 M_1^p r^p T + \int_0^T C_2(t)\, dt \le 0. \end{aligned}$$

Hence, by Proposition 2, there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $x'(t) = f(t,x(t))$ for almost every $t \in [0,T]$ and $x(0) = x(T)$. Let $u : [0,T] \to F$ be the absolutely continuous function defined by

$$u(t) = \sum_{i=1}^{n} x_i(t) w_i, \quad t \in [0,T].$$

It is easy to see that $u$ satisfies (4.5) and $u(0) = u(T)$. By (A4), $(A(\cdot)u(\cdot), w_i) \in L^q(0,T;\mathbb{R})$ for $i = 1, \cdots, n$. So, by (4.5), we get $x'_i(\cdot) \in L^q(0,T;\mathbb{R})$ for $i = 1, \cdots, n$. Hence we have $u' \in L^q(0,T;F)$. $\square$

Let $\mathcal{F}$ be the set of all finite dimensional subspaces of $V$. For $F, G \in \mathcal{F}$, we define $F \leq G$ when $F \subset G$. For each $F \in \mathcal{F}$, let $u_F$ be one of the functions which are obtained by Lemma 2.

We denote by $J$ the duality mapping from $L^q(0,T;V')$ onto $L^p(0,T;V)$, i.e.,

$$Jv = \{u \in L^p(0,T;V) : \langle v, u\rangle = \|u\|^2 = \|v\|_*^2\}$$

for each $v \in L^q(0,T;V')$.

**Lemma 3.** *$\{u_F : F \in \mathcal{F}\}$ is bounded in $\mathcal{W}$.*

*Proof.* Let $F$ be an element of $\mathcal{F}$. From $u_F(t) \in F$ for every $t \in [0,T]$, (4.5), $u_F(0) = u_F(T)$ and (A3), we have

$$\begin{aligned} 0 &= \int_0^T (A(t)u_F(t), u_F(t))\,dt + \frac{1}{2}|u_F(T)|^2 - \frac{1}{2}|u_F(0)|^2 \\ &\geq C_1 \int_0^T \|u_F(t)\|^p\,dt - \int_0^T C_2(t)\,dt. \end{aligned}$$

Hence $\{u_F : F \in \mathcal{F}\}$ is bounded in $\mathcal{V}$.

Let $G \in \mathcal{F}$. Since $u_G(t) \in G$ for every $t \in [0,T]$, there exists $v_G \in Ju'_G$ such that $v_G(t) \in G$ for almost every $t \in [0,T]$. So, by (4.5) and (A4), we get

$$\|u'_G\|_*^2 = -\int_0^T (A(t)u_G(t), v_G(t))\,dt \leq \left(C_3\|u_G\|^{\frac{p}{q}} + \left(\int_0^T |C_4(t)|^q\,dt\right)^{\frac{1}{q}}\right)\|v_G\|.$$

Since $\|u'_G\|_* = \|v_G\|$ and $\{u_F : F \in \mathcal{F}\}$ is bounded in $\mathcal{V}$, $\{u'_G : G \in \mathcal{F}\}$ is bounded in $\mathcal{V}'$. $\square$

Since $\{u_F : F \in \mathcal{F}\}$ is bounded in $\mathcal{W}$ and $\{\mathcal{A}u_F : F \in \mathcal{F}\}$ is bounded in $\mathcal{V}'$, there exist $u_0 \in \mathcal{W}$, $w_0 \in \mathcal{V}'$ and a subnet $\{u_{F_\alpha} : \alpha \in \mathcal{D}\}$ of $\{u_F : F \in \mathcal{F}\}$ such that $\{u_{F_\alpha}\}$ converges to $u_0$ weakly in $\mathcal{W}$ and $\{\mathcal{A}u_{F_\alpha}\}$ converges to $w_0$ weakly in $\mathcal{V}'$.

**Lemma 4.** *$u'_0 + w_0 = 0$ and $u_0(0) = u_0(T)$.*

*Proof.* First we shall show $u'_0 + w_0 = 0$. Let $\varphi \in C_0^\infty(0,T)$ and let $v \in V$. Let $L$ be the one dimensional subspace of $V$ spanned by $v$. Then there exists $\alpha_0 \in \mathcal{D}$ such that $\alpha \geq \alpha_0$ implies $F_\alpha \geq L$. Let $\alpha$ be an element of $\mathcal{D}$ with $\alpha \geq \alpha_0$. By (4.5), we have

$$\begin{aligned} 0 &= (u_{F_\alpha}(T), \varphi(T)v) - (u_{F_\alpha}(0), \varphi(0)v) \\ &= \int_0^T \big((u'_{F_\alpha}(t), \varphi(t)v) + (\varphi'(t)v, u_{F_\alpha}(t))\big)\,dt \\ &= \int_0^T \big((-A(t)u_{F_\alpha}(t), \varphi(t)v) + (\varphi'(t)v, u_{F_\alpha}(t))\big)\,dt \\ &= \langle -\mathcal{A}u_{F_\alpha}, \varphi v\rangle + \langle \varphi' v, u_{F_\alpha}\rangle. \end{aligned}$$

So we get $0 = \langle -\varphi w_0 + \varphi' u_0, v\rangle$ for all $v \in V$ and $\varphi \in C_0^\infty(0,T)$. Therefore we obtain $u'_0 + w_0 = 0$. Next we shall show $u_0(0) = u_0(T)$. Since $\{u_{F_\alpha}\}$ converges to $u_0$ weakly in $\mathcal{W}$ and $\mathcal{W}$ is continuously imbedded in $C(0,T;H)$, $\{u_{F_\alpha}(0)\}$ and $\{u_{F_\alpha}(T)\}$ converge to $u_0(0)$ and $u_0(T)$ weakly in $H$, respectively. Hence, by $u_{F_\alpha}(0) = u_{F_\alpha}(T)$, $u_0(0) = u_0(T)$. $\square$

*Proof of Theorem 1.* By Proposition 1, there exists a sequence $\{v_n\}$ which is contained in the set $\{u_{F_\alpha} : \alpha \in \mathcal{D}\}$ such that $\{v_n\}$ converges to $u_0$ weakly in $\mathcal{W}$ and $\{\mathcal{A}v_n\}$ converges to $w_0$ weakly in $\mathcal{V}'$, respectively. By Lemma 2 and Lemma 4, we have $\langle \mathcal{A}v_n, v_n\rangle = 0$ and $\langle w_0, u_0\rangle = 0$. So we get $\lim\langle \mathcal{A}v_n, v_n - u_0\rangle = 0$. Hence, by Lemma 1, we have $w_0 = \mathcal{A}u_0$. Therefore we obtain $u_0 \in \mathcal{W}$ such that $u_0(0) = u_0(T)$ and $u_0' + \mathcal{A}u_0 = 0$. $\square$

## 5. Example

Throughout this section, $p > 2$ and $\Omega$ is a bounded, open subset of $\mathbb{R}^n$ $(n \geq 2)$ with sufficiently smooth boundary $\Gamma$.

We consider the following nonlinear differential equation [14, Example 9.66]:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) \\ &\quad + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + a(x)|u(x)|^{p-2}u(x) = g(t,x,u(x)) \end{aligned} \qquad \text{on } \mathbb{R}\times\Omega \tag{5.1}$$

with Dirichlet boundary condition

$$u = 0 \qquad \text{on } \mathbb{R}\times\Gamma, \tag{5.2}$$

where $b_i, a : \mathbb{R} \to \mathbb{R}$ are bounded and continuous and $g : \mathbb{R}\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable in $(t,x)$ and continuous in $u$.

We recall that there exists $\lambda > 0$ such that

$$\lambda \int_\Omega |u(x)|^p\,dx \leq \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We improve [5, Theorem 4.1] in the case of $p > 2$:

**Theorem 2.** *Assume that $g$ is $T$-periodic in its first variable and that there exist positive constants $\alpha$, $\beta$, $\gamma$ and $\delta$ such that*

$$|g(t,x,u)| \leq \alpha|u| + \beta;$$

$$u\cdot g(t,x,u) \leq \gamma|u|^p + \delta \tag{5.3}$$

*for almost every $(t,x) \in \mathbb{R}\times\Omega$ and for every $u \in \mathbb{R}$. Assume also*

$$\gamma < \min\{\lambda, \lambda + \operatorname*{ess\,inf}_{x\in\Omega} a(x)\}. \tag{5.4}$$

*Then* (5.1) *and* (5.2) *have a $T$-periodic, weak solution $u$ such that the restriction $u|_{[0,T]}$ of $u$ belongs to $W_p^1(0,T;W_0^{1,p}(\Omega),L^2(\Omega))$.*

*Proof.* For $t \in \mathbb{R}$ and $u \in W_0^{1,p}(\Omega)$, set

$$\begin{aligned} (A(t)u)(x) = &-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) \\ &+ \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + a(x)|u(x)|^{p-2}u(x) - g(t,x,u(x)). \end{aligned}$$

It is easy to see that $A$ is an operator from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$ and that (A2) and (A4) hold. We get (A1) by the Sobolev imbedding theorem and the monotonicity of $u \mapsto -\sum \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i})$. Set $M = \max_{1\le i\le n} \operatorname{ess\,sup}_{x\in\Omega} |b_i(x)|$. For an arbitrary $\varepsilon > 0$, we have

$$\left|\int_\Omega b_i(x)\frac{\partial u}{\partial x_i}u(x)\right| dx \le \frac{\varepsilon^p}{p}\int_\Omega |u(x)|^p\, dx + \frac{\varepsilon^p}{p}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^p dx + \frac{p-2}{p}\left(\frac{M}{\varepsilon^2}\right)^{\frac{p}{p-2}}|\Omega|.$$

The above inequality, (5.3) and (5.4) yield (A3). Hence, by Theorem 1, we get the conclusion. □

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