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INTEGRATION OF THE INTERTWINING OPERATOR FOR h-HARMONIC POLYNOMIALS ASSOCIATED TO REFLECTION GROUPS

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Abstract. Let V be the intertwining operator with respect to the reflection invariant measure $h_\alpha^2 d\omega$ on the unit sphere S^{d-1} in Dunkl's theory on spherical h-harmonics associated with reflection groups. Although a closed form of V is unknown in general, we prove that

$$\int_{S^{d-1}} Vf(\mathbf{y})h_\alpha^2(\mathbf{y})d\omega = A_\alpha \int_{B^d} f(\mathbf{x})(1-|\mathbf{x}|^2)^{|\alpha|_1-1}d\mathbf{x},$$

where B^d is the unit ball of $\mathbb{R}^d$ and A_α is a constant. The result is used to show that the expansion of a continuous function as Fourier series in h-harmonics with respect to $h_\alpha^2 d\omega$ is uniformly Cesáro (C,δ) summable on the sphere if $\delta > |\alpha|_1 + (d-2)/2$, provided that the intertwining operator is positive.

1. Introduction and preliminaries

The main result of this paper is a formula for the integration of the intertwining operator in Dunkl's theory on h-harmonics associated with finite reflection groups. We start with background on Dunkl's theory (cf. [2]-[5]).

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we let $\langle \mathbf{x}, \mathbf{y}\rangle$ denote the usual inner product of $\mathbb{R}^d$ and $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$ the Euclidean norm. Let $S^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^d$. For a nonzero vector $\mathbf{v} \in \mathbb{R}^d$ define the reflection $\sigma_\mathbf{v}$ by

$$\mathbf{x}\sigma_\mathbf{v} := \mathbf{x} - 2(\langle \mathbf{x}, \mathbf{v}\rangle/|\mathbf{v}|^2)\mathbf{v}, \qquad \mathbf{x} \in \mathbb{R}^d.$$

Suppose that G is a finite reflection group on $\mathbb{R}^d$ with the set $\{\mathbf{v}_i : i = 1, 2, ..., m\}$ of positive roots; assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where we write $\sigma_i = \sigma_{\mathbf{v}_i}$, $1 \le i \le m$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_i : 1 \le i \le m\}$.

Let $d\omega$ be the normalized surface measure on S^{d-1}. We consider weight functions of the form $h_\alpha^2 d\omega$ on S^{d-1}, where

$$h_\alpha(\mathbf{x}) := \prod_{i=1}^m |\langle \mathbf{x}, \mathbf{v}_i\rangle|^{\alpha_i}, \qquad \alpha_i \ge 0, \tag{1.1}$$

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with $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G. Then h_α is a positively homogeneous G-invaraint function of degree $|\alpha|_1 = \alpha_1 + \ldots + \alpha_m$.

The h-harmonics are orthogonal polynomials on S^{d-1} with respect to $h_\alpha^2 d\omega$. The key ingredient of the theory is a family of commuting first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i + \sum_{j=1}^{m} \alpha_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x}, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{e}_i \rangle, \qquad 1 \le i \le d, \tag{1.2}$$

where ∂_i is the ordinary partial derivative with respect to x_i and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-Laplacian, which plays the role similar to that of the ordinary Laplacian, is defined by

$$\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2. \tag{1.3}$$

We keep the notation Δ for the ordinary Laplacian. Let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in $\mathbf{x} = (x_1, \ldots, x_d)$. Then $\mathcal{D}_i \mathcal{P}_n^d \subset \mathcal{P}_{n-1}^d$, $\Delta_h \mathcal{P}_n^d \subset \mathcal{P}_{n-2}^d$; moreover, if $P \in \mathcal{P}_n$, then

$$\int_{S^{d-1}} PQ h_\alpha^2 d\omega = 0, \qquad \forall Q \in \bigcup_{k=0}^{n-1} \mathcal{P}_k^d,$$

if and only if $\Delta_h P = 0$. The space $\mathcal{H}_n^h = \mathcal{H}_n^{h,d} := \mathcal{P}_n^d \cap \ker \Delta_h$ is called the space of *h-harmonic* polynomials of degree n. We denote by $\mathcal{H}_n$ the space of ordinary harmonic polynomials, which corresponds to $h_\alpha = 1$. The dimension of $\mathcal{H}_n^h$ is the same as that of $\mathcal{H}_n$, which we denote by $N(n,d)$; thus,

$$N(n,d) = \dim \mathcal{H}_n^h = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

The intertwining operator V is a linear operator uniquely defined by ([4]):

$$V\mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V = V \partial_i, \qquad 1 \le i \le d.$$

Note that $V\mathcal{H}_n \subset \mathcal{H}_n^h$; the intertwining operator allows the transfer of certain results about $\mathcal{H}_n$ to $\mathcal{H}_n^h$. In particular, the Poisson kernel for h-harmonics in the unit ball is given by [4, Theorem 4.2, p. 1225]

$$P^h(\mathbf{x}, \mathbf{y}) = V\Big((1 - |\mathbf{x}|^2)(1 - 2\langle \mathbf{x}, \cdot \rangle + |\mathbf{x}|^2)^{-|\alpha|_1 - \frac{d}{2}}\Big)(\mathbf{y}), \qquad |\mathbf{x}| < |\mathbf{y}| = 1. \tag{1.4}$$

A closed form of V is known only for $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$, associated with the group $\mathbb{Z}^2 \times \cdots \times \mathbb{Z}^2$ ([13], the case $d = 1$ appeared early in [4]), and for $h_\alpha(\mathbf{x}) = |(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)|^\alpha$, associated with the symmetric group S_3 ([5]). We state the first case as

Example 1.1. For $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$,

$$Vf(\mathbf{x}) = \int_{[-1,1]^d} f(x_1 t_1, \ldots, x_d t_d) \prod_{i=1}^{d} (1 + t_i) \prod_{i=1}^{d} c_{\alpha_i} (1 - t_i^2)^{\alpha_i - 1} d\mathbf{t}, \tag{1.5}$$

where the constant c_λ is defined by

$$c_\lambda^{-1} = \int_{-1}^{1} (1 - t^2)^{\lambda - 1} dt = \frac{\pi^{\frac{1}{2}} \Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})}. \tag{1.6}$$

The closed form of V in the case of S_3 in [5] is rather complicated. The problem of finding a closed form of V, however, is very important but also very difficult. The main result in the present paper is to show that integration of Vf with respect to $h_\alpha^2 d\omega$ on S^{d-1} has a simple closed form. We state and prove the main result in the following section. In Section 3, we use the result to study the summability of Fourier series in terms of h-harmonics, where we shall prove that the expansion of a continuous function as Fourier series in h-harmonics is uniformly (C,δ) summable on S^{d-1} if $\delta > |\alpha|_1 + (d-2)/2$, provided V is a positive operator.

2. Integration of the intertwining operator

Let B^d be the unit ball in $\mathbb{R}^d$. On B^d we define the weight function

$$W_\gamma(\mathbf{x}) = c_{\gamma,d}(1-|\mathbf{x}|^2)^{\gamma-1}, \qquad c_{\gamma,d}^{-1} = \int_{B^d}(1-|\mathbf{x}|^2)^{\gamma-1}d\mathbf{x} = \frac{\pi^{d/2}\Gamma(\gamma)}{\Gamma(\gamma+d/2)},$$

where $c_{\gamma,d}$ is the normalization constant so that the integral of W_γ over B^d takes the value 1. We denote by H_α the normalization constant defined by $H_\alpha^{-1} = \int_{S^{d-1}} h_\alpha^2 d\omega$. There is a closed formula for H_α due to the work of Macdonald, Heckman, Opdam, and others (cf. [7], [8], [9]). We shall not write it down since it is not used in this paper.

Theorem 2.1. *Let h_α be defined as in* (1.1). *Let V be the intertwining operator. Then*

$$\int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = A_\alpha \int_{B^d} f(x)(1-|\mathbf{x}|^2)^{\alpha-1}d\mathbf{x}, \tag{2.1}$$

for $f \in \Pi^d$, where $A_\alpha = H_\alpha^{-1}c_{\gamma,d}$.

Before we turn to the proof of the theorem, some remarks are in order. First, the condition $f \in \Pi^d$ is not necessary. The formula may hold for any function as long as the integrals on both sides are finite. In fact, the first indication that such a formula holds true lies in the fact that the integral of P^h as given in (1.4) is equal to 1. Secondly, we emphasize that the real strength of the theorem lies in the fact that the formula of Vf for a general reflection group is unknown. Nevertheless, let us point out that in the case of Example 1.1, the equation (2.1) takes the form

$$\begin{aligned} &\int_{S^{d-1}}\int_{[-1,1]^d} f(x_1t_1,\ldots,x_dt_d)\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}d\mathbf{t}\prod_{i=1}^d|x_i|^{\alpha_i}d\omega \\ &\quad = A_\alpha\int_{B^d} f(x)(1-|\mathbf{x}|^2)^{|\alpha|_1-1}d\mathbf{x}. \end{aligned} \tag{2.2}$$

This identity can be proved by changing variables and using integration by parts, but such a proof is not a trivial matter.

The following special case of the theorem is of interest in itself.

Corollary 2.2. *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a polynomial. Then*

$$\int_{S^{d-1}} Vg(\langle \mathbf{x},\cdot\rangle)(\mathbf{y})h_\alpha^2(\mathbf{y})d\omega = B_\alpha\int_{-1}^1 g(t|\mathbf{x}|)(1-t^2)^{|\alpha|_1+\frac{d-3}{2}}dt, \tag{2.3}$$

where $B_\alpha = H_\alpha^{-1}c_{|\alpha|_1+(d-1)/2}$.

The corollary shows, in particular, that the integration of $Vg(\langle \mathbf{x}, \cdot\rangle)$ against h_α^2 yields a radial function. For the Lebesgue measure, this is evident since $V = id$ and the integral in the left side of (2.3) with $h_\alpha = 1$ is clearly invariant under orthogonal transformations. For $h_\alpha \neq 1$ this is far from being obvious since h_α is invariant under a subgroup of the group of orthogonal transformations. Even in the case of h_α in Example 1.1 this is not obvious in view of (2.2).

In order to prove the theorem we explore the orthogonal polynomials with respect to the weight function W_γ on B^d. The following formula is very useful: If g is a continuous function on B^d, then

$$\int_{B^d} g(\mathbf{x})(1-|\mathbf{x}|^2)^{\gamma-1}d\mathbf{x} = \int_0^1 \left(\int_{S^{d-1}} g(r\mathbf{x}')d\omega\right) r^{d-1}(1-r^2)^{\gamma-1}dr, \tag{2.4}$$

under the standard change of variables $\mathbf{x} = r\mathbf{x}'$, $\mathbf{x}' \in S^{d-1}$. An orthogonal basis of $L^2(B^d, W_\gamma d\mathbf{x})$ can be given in terms of Jacobi polynomials and the ordinary spherical harmonics. We use the standard notation $P_n^{(\alpha,\beta)}$ for the Jacobi polynomials orthogonal with respect to $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ (cf. [10, Chapter IV]). Let $\{S_{n,i}, 1 \le i \le N(n,d)\}$ be an orthonormal basis for $\mathcal{H}_n$. Let

$$\mathbb{P}_n = \{b_{k,n}P_k^{(\gamma-1,n-2k+\frac{d-2}{2})}(2|\mathbf{x}|^2-1)S_{n-2k,i}(\mathbf{x}),$$
$$1 \le i \le N(n-2k,d),\ 0 \le 2k \le n\},$$

where $b_{k,n}$ are the normalization constants so that the square of each element has integral equal to 1 with respect to $W_\gamma d\mathbf{x}$ over B^d. Using formula (2.4) and the fact that S_m is homogeneous of degree m, it follows from the orthogonality of the Jacobi polynomials and the orthogonality of the (ordinary) spherical harmonics that $\mathbb{P}_n$ is orthogonal to $\mathbb{P}_m$ for $n \neq m$ and the elements in $\mathbb{P}_n$ are mutually orthogonal to each other. Since the polynomials in $\mathbb{P}_n$ are of degree exactly n and there are exactly

$$\sum_{0\le 2k\le n} \dim \mathcal{H}_{n-2k,d} = \sum_{0\le 2k\le n} (\dim \mathcal{P}_{n-2k} - \dim \mathcal{P}_{n-2k-2}) = \dim \mathcal{P}_n^d$$

many polynomials in $\mathbb{P}_n$, it follows that $L^2(B^d, W_\gamma d\mathbf{x}) = \bigoplus_{n=0}^\infty \operatorname{span} \mathbb{P}_n$. Therefore, the elements of $\mathbb{P}_n$ form an orthonormal basis of $L^2(W_\gamma d\mathbf{x}, B^d)$. We choose a total order for the elements of $\mathbb{P}_n$ and treat it as a polynomial vector. Such a vector notation has been used to study orthogonal polynomials in several variables quite generally (cf. [11]). In [12], a different orthonormal basis of $\mathbb{P}_n$ with respect to $W_\gamma d\mathbf{x}$, given in terms of Gegenbauer polynomials, has been used to study the summability of Fourier series in terms of $\mathbb{P}_n$.

For $f \in L^2(B^d, W_\gamma d\mathbf{x})$, we can write its Fourier expansion with respect to $\{\mathbb{P}_n\}_{n=0}^\infty$ as (see [11])

$$f(\mathbf{x}) = \sum_{n=0}^\infty \mathbf{a}_n^T(f)\mathbb{P}_n(\mathbf{x}), \qquad \mathbf{a}_n(f) = \int_{B^d} f(\mathbf{y})\mathbb{P}_n(\mathbf{y})W_\gamma(\mathbf{y})d\mathbf{y}. \tag{2.5}$$

We want to study the effect of V applied to $P_n \in \mathbb{P}_n$. Because of the formulae [10, p. 62, (4.21.2)] and [10, p. 59, (4.1.4)],

(2.6)

$$P_k^{(\alpha,\beta)}(2t^2-1) = \frac{(-1)^k}{k!}\sum_{j=0}^k \binom{k}{j}\frac{\Gamma(k+\alpha+\beta+j+1)}{\Gamma(k+\alpha+\beta+1)}\frac{\Gamma(k+\beta+1)}{\Gamma(j+\beta+1)}(-1)^j t^{2j},$$

we first consider the effect of V applied to $|\mathbf{x}|^{2j}S_{n-2k,i}(\mathbf{x})$. We have

Lemma 2.3. *Let $S_m \in \mathcal{H}_m$, $m \geq 0$. Then*

$$\int_{S^{d-1}} V(|\cdot|^{2j}S_m)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha \frac{\Gamma(|\alpha|_1+d/2)\Gamma(j+d/2)}{\Gamma(|\alpha|_1+d/2+j)\Gamma(d/2)}\delta_{m,0}. \tag{2.7}$$

Proof. Since $|\mathbf{x}|^{2j}S_m$ is homogeneous of degree $m+2j$, so is $V(|\cdot|^{2j}S_m)$. Using the canonical decomposition [2, Theorem 1.7, p. 37]

$$\mathcal{P}_n^d = \bigoplus_{j=0}^{[n/2]} |\mathbf{x}|^{2j}\mathcal{H}_{n-2j}^h$$

and the harmonic projection operator $\mathrm{proj}_{\mathcal{H}_k^h} : \mathcal{P}_k \mapsto \mathcal{H}_k^h$ [2, Theorem 1.11, p. 38], we have

$$\begin{aligned} &V(|\cdot|^{2j}S_m) \\ &\quad= \sum_{i=0}^{[m/2]+j} |\mathbf{x}|^{2i}\frac{1}{4^i i!}\frac{\Gamma(|\alpha|_1+m+2j-i+d/2)}{\Gamma(|\alpha|_1+m+2j-2i+d/2)}\,\mathrm{proj}_{\mathcal{H}^h_{m+2j-2i}}(\Delta_h^i V(|\cdot|^{2j}S_m)).\end{aligned}$$

Since $\mathcal{H}_k^h \perp 1$, $k > 0$, with respect to $h_\alpha^2 d\omega$, it follows that the integral of the function $\mathrm{proj}_{\mathcal{H}^h_{m+2j-2i}}\Delta_h^i V(|\cdot|^{2j}S_m)$ is zero unless $m+2j-2i = 0$. In case $2i = m+2j$, we have

$$\begin{aligned}&\int_{S^{d-1}} V(|\cdot|^{2j}S_m)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega \\ =&\frac{1}{4^{j+m/2}(j+m/2)!}\frac{\Gamma(|\alpha|_1+j+(d+m)/2)}{\Gamma(|\alpha|_1+d/2)}\int_{S^{d-1}}\Delta_h^{j+m/2}V(|\cdot|^{2j}S_m)h_\alpha^2(\mathbf{x})d\omega,\end{aligned}$$

since $\mathrm{proj}_{\mathcal{H}_0^h} = id$. By the intertwining property of V, it follows that

$$\Delta_h^{j+m/2}V(|\cdot|^{2j}S_m) = V\Delta^{j+m/2}(|\cdot|^{2j}S_m).$$

Using the identity

$$\Delta(|\mathbf{x}|^{2j}g_m) = 4j(m+j-1+d/2)|\mathbf{x}|^{2j-2}g_m + |\mathbf{x}|^{2j}\Delta g_m,$$

for $g_m \in \mathcal{P}_m$, $m = 0,1,2,...$, and the fact that $\Delta S_m = 0$, we see that

$$\Delta_h^{j+m/2}(|\cdot|^{2j}S_m)(\mathbf{x}) = 4^j j!\frac{\Gamma(j+d/2)}{\Gamma(d/2)}\Delta_h^{m/2}S_m(\mathbf{x}),$$

which is zero if $m > 0$. For $m = 0$, we use the fact that $S_0(\mathbf{x}) = 1$ and put the constants together to finish the proof. □

Lemma 2.4. *For $P_n \in \mathbb{P}_n$,*

$$\int_{S^{d-1}} V(P_n)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1}\delta_{0,n}. \tag{2.8}$$

Proof. By the definition of $\mathbb{P}_n$, we need to show that

$$\int_{S^{d-1}} V(P_k^{(|\alpha|_1-1,n-2k+(d-2)/2)}(2|\cdot|^2-1)S_{n-2k,i})(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1}\delta_{0,n}$$

for $0 \le 2k \le n$ and $S_{n-2k,i} \in \mathcal{H}_{n-2k}$. From Lemma 2.3 and (2.6) this follows readily if $2k < n$. For the remaining case $2k = n$, we use (2.6) and (2.7) to derive that

$$\begin{aligned}
&\int_{S^{d-1}} V(P_k^{(|\alpha|_1-1,n-2k+\frac{d-2}{2})}(2|\cdot|^2-1)S_{n-2k,i})(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega \\
&\quad= \frac{(-1)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{\Gamma(k+|\alpha|_1+j-1+d/2)\Gamma(k+d/2)}{\Gamma(k+|\alpha|_1-1+d/2)\Gamma(j+d/2)}(-1)^j\int_{S^{d-1}} V(|\cdot|^{2j})h_\alpha^2 d\omega \\
&\quad= H_\alpha^{-1}\frac{\Gamma(|\alpha|_1+d/2)}{\Gamma(d/2)}\frac{(-1)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{\Gamma(k+|\alpha|_1+j-1+d/2)\Gamma(k+d/2)}{\Gamma(k+|\alpha|_1-1+d/2)\Gamma(|\alpha|_1+j+d/2)}(-1)^j \\
&\quad= H_\alpha\frac{\Gamma(|\alpha|_1+d/2)}{\Gamma(d/2)}\frac{\Gamma(k+d/2)}{\Gamma(k+|\alpha|_1+d/2)}P_k^{(-1,|\alpha|_1-1+d/2)}(1),
\end{aligned}$$

where we note that the definition of $P_k^{(\alpha,\beta)}$ can be extended to $\alpha = -1$ by using (2.6). From [10, (4.22.2), p. 64],

$$P_k^{(-1,\beta)}(x) = \frac{k+\beta}{k}\frac{x-1}{2}P_{k-1}^{(1,\beta)}(x),$$

it follows that $P_k^{(-1,|\alpha|_1-1+d/2)}(1) = 0$ for $k > 0$ or $n = 2k > 0$. For $n = 0$ we use the fact that $P_0^{(\alpha,\beta)}(x) = 1$ and $V1 = 1$. □

Proof of Theorem 2.1. We expand f into the Fourier series with respect to the orthonormal basis $\mathbb{P}_n$ as in (2.5). Since V is a linear operator, we have that

$$\int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = \sum_{n=0}^{\infty}\mathbf{a}_n^T(f)\int_{S^{d-1}} V(\mathbb{P}_n)(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega.$$

Thus, Lemma 2.4 implies that

$$\int_{S^{d-1}} Vf(\mathbf{x})h_\alpha^2(\mathbf{x})d\omega = H_\alpha^{-1}\mathbf{a}_0(f) = A_\alpha\int_{B^d} f(\mathbf{x})(1-|\mathbf{x}|^2)^{|\alpha|_1-1}d\mathbf{x},$$

which is the desired result. □

Proof of Corollary 2.2. The following formula is known to hold for $f : \mathbb{R}^d \mapsto \mathbb{R}^d$:

$$\int_{S^{d-1}} f(\langle\mathbf{x},\mathbf{y}\rangle)d\omega(\mathbf{y}) = \omega_{d-2}\int_{-1}^{1} f(s|\mathbf{x}|)(1-s^2)^{\frac{d-3}{2}}ds, \quad \mathbf{x}\in\mathbb{R}^d.$$

Using this formula and Theorem 2.1 we have

$$\begin{aligned}
I(\mathbf{x}) := \int_{S^{d-1}} Vg(\langle\mathbf{x},\cdot\rangle)(\mathbf{y})h_\alpha^2(\mathbf{y})d\omega &= A_\alpha\int_{B^d} g(\langle\mathbf{x},\mathbf{y}\rangle)(1-|\mathbf{y}|^2)^{|\alpha|_1-1}d\mathbf{y} \\
&= c\int_0^1 r^{d-1}\int_{S^{d-1}} g(r\langle\mathbf{x},\mathbf{y}'\rangle)d\omega(\mathbf{y}')(1-r^2)^{|\alpha|_1-1}dr \\
&= c\int_0^1\int_{-1}^{1} g(sr|\mathbf{x}|)(1-s^2)^{\frac{d-3}{2}}ds(1-r^2)^{|\alpha|_1-1}dr,
\end{aligned}$$

where $c = A_\alpha \omega_{d-2}$. Changing the variable $s \mapsto t/r$ in the last formula and interchanging the order of the integrations we have

$$\begin{aligned} I(\mathbf{x}) =& c\int_{-1}^{1} g(t|\mathbf{x}|) \int_{|t|}^{1} (r^2 - t^2)^{\frac{d-3}{2}} r(1-r^2)^{|\alpha|_1 - 1} dr dt \\ =& c\frac{1}{2}\int_0^1 u^{\frac{d-3}{2}}(1-u)^{|\alpha|_1-1} du \int_{-1}^{1} g(t|\mathbf{x}|)(1-t^2)^{|\alpha|_1 + \frac{d-3}{2}} dt, \end{aligned}$$

where the last step follows from another change of variable. This is the desired formula. Instead of keeping track of the constant we can determine it by setting $g = 1$ and using the fact that $V1 = 1$. □

We end this section with the following remark: Formula (2.8) in Lemma 2.4 shows that $V(\mathbb{P}_n)$, $n > 0$, is orthogonal to $\mathcal{H}_0^h$ with respect to $h_\alpha^2 d\omega$. In fact, with little more effort, the same approach can be used to show that $V(\mathbb{P}_n)$ is orthogonal to $\mathcal{H}_k^h$ for $k \neq n$. This may suggest a possible way of studying the intertwining operator V. However, we do not know how to compute the integral of $V(P)Q$ for $P \in \mathbb{P}_n$ and $Q \in \mathcal{H}_n^h$.

3. Summability of orthogonal series in h-harmonics

As an application of Theorem 2.1, we consider the Cesáro summability of the Fourier expansion of a function with respect to h-harmonics on the sphere. Let $\{S_{n,i}^h, 1 \le i \le N(n,d)\}$ be an orthonormal basis for $\mathcal{H}_n^h$. For $f \in L^2(S^{d-1}, h_\alpha^2 d\omega)$, its Fourier orthogonal expansion in terms of $\{S_{n,i}^h\}$ is given by

$$f \sim \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} a_i^n(f) S_{n,i}^h, \qquad a_i^n(f) = \int_{S^{d-1}} f S_{n,i}^h h_\alpha^2 d\omega.$$

We denote the n-th partial sum of this expansion by $S_n(h_\alpha^2; f)$; thus,

$$S_n(h_\alpha^2; f, \mathbf{x}) = \sum_{k=0}^{n} \sum_{i=1}^{N(k,d)} a_i^k(f) S_{k,i}^h(\mathbf{x}).$$

Let us recall the definition of Cesàro summability. The sequence $\{s_n\}$ is summable by Cesàro's method of order δ, (C, δ), to s if

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k$$

converges to s as $n \to \infty$. If, for each $n \in \mathbb{N}_0$, s_n is the n-th partial sum of the series $\sum_{k=0}^{\infty} c_k$, the Cesàro means can be rewritten as

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} c_k.$$

For the basic properties of Cesàro summability see [14, Chapter III]. We are interested in the (C, δ) means of $S_n(h_\alpha^2, f)$. Our main result in this section is the following.

Theorem 3.1. *Assume the intertwining operator V with respect to the weight function h_α is positive. Let f be continuous on S^{d-1}. Then the expansion of f as Fourier series with respect to h_α^2 is uniformly (C, δ) summable over S^{d-1} provided $\delta > |\alpha|_1 + (d-2)/2$.*

We note that the intertwining operator is known to be positive in the case of Example 1.1 and for some parameters in the S_3 case ([5]). Dunkl has conjectured that the intertwining operator is positive for all weight functions invariant under reflection groups. When $\alpha = 0$ the theorem reduces to the classical case of Lebesgue measure.

The proof of this theorem depends on Theorem 2.1 and a compact formula of the reproducing kernel for $\mathcal{H}_n^h$. For each $n \in \mathbb{N}_0$, the reproducing kernel, P_n^h, for $\mathcal{H}_n^h$ is defined by the property

$$H_\alpha \int_{S^{d-1}} Q(\mathbf{y}) P_n^h(\mathbf{x}, \mathbf{y}) h_\alpha^2 d\omega(\mathbf{y}) = Q(\mathbf{x}), \qquad Q \in \mathcal{H}_n^h.$$

In terms of an orthonormal basis of $\mathcal{H}_n^h$, the function P_n^h is given by

$$P_n^h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N(n,d)} S_{n,i}^h(\mathbf{x}) S_{n,i}^h(\mathbf{y}).$$

Using the intertwining operator V, we can express P_n^h in a closed form in terms of the Gegenbauer polynomials $C_n^{(\lambda)}$.

Theorem 3.2. *For* $n \in \mathbb{N}_0$,

(3.1)
$$P_n^h(\mathbf{x}, \mathbf{y}) = \frac{n + |\alpha|_1 + (d-2)/2}{|\alpha|_1 + (d-2)/2} [V C_n^{(|\alpha|_1 + (d-2)/2)}(\langle \cdot, \mathbf{y} \rangle)](\mathbf{x}), \qquad |\mathbf{y}| \le |\mathbf{x}| = 1.$$

Proof. Introducing the notation $K_m(\mathbf{x}, \mathbf{y}) = [V(\langle \cdot, \mathbf{y} \rangle^m / m!)](\mathbf{x})$, Dunkl [4, Theorem 4.1, p. 1224] has proved that P_n^h can be written as

$$P_n^h(\mathbf{x}, \mathbf{y}) = \sum_{j \le n/2} \frac{(|\alpha|_1 + d/2)_n 2^{n-2j}}{(2 - n - |\alpha|_1 - d/2)_j j!} |\mathbf{x}|^{2j} |\mathbf{y}|^{2j} K_{n-2j}(\mathbf{x}, \mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Since V is linear and $\langle \cdot, \mathbf{y} \rangle^m$ is homogeneous of degree m in $\mathbf{y}$, for $|\mathbf{x}| = 1$ we can write

$$P_n^h(\mathbf{x}, \mathbf{y}) = V\Big\{ \sum_{j \le n/2} \frac{(|\alpha|_1 + d/2)_n 2^{n-2j}}{(2 - n - |\alpha|_1 - d/2)_j j! (n-2j)!} \langle \cdot, \mathbf{y}/|\mathbf{y}| \rangle^{n-2j} \Big\}(\mathbf{x}) |\mathbf{y}|^n.$$

Setting $t = \langle \cdot, \mathbf{y}/|\mathbf{y}| \rangle$ we now derive a closed formula for the expression inside the curly bracket, which we denote by $J(t)$. If n is even, say, $n = 2m$, then it follows from the definition of $(a)_j = a(a+1)\cdots(a+j-1)$ that

$$(2 - 2m - |\alpha|_1 - d/2)_{m-k} = (-1)^{m-k} \frac{\Gamma(2m + |\alpha|_1 + d/2 - 1)}{\Gamma(m + k + |\alpha|_1 + d/2 - 1)},$$

from which and the formula

$$(2k)! = \Gamma(2k+1) = 2^{2k} \Gamma(k + 1/2) \Gamma(k+1) / \Gamma(1/2),$$

we conclude that

$$\begin{aligned} J(t) =& \sum_{k=0}^{m} \frac{(|\alpha|_1 + d/2)_{2m} 2^{2k}}{(2-2m-|\alpha|_1 - d/2)_{m-k} (m-k)!(2k)!} t^{2k} \\ =& \frac{\Gamma(1/2)(-1)^m}{m!\Gamma(m+1/2)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(m+k+|\alpha|_1+d/2-1)\Gamma(m+1/2)}{\Gamma(2m+|\alpha|_1+d/2-1)\Gamma(k+1/2)} (-1)^k t^{2k} \\ =& \frac{\Gamma(m+|\alpha|_1+d/2-1)\Gamma(1/2)}{\Gamma(2m+|\alpha|_1+d/2-1)\Gamma(m+1/2)} P_m^{(|\alpha|_1+\frac{d-3}{2}, -\frac{1}{2})}(2t^2-1) \end{aligned}$$

by formula (2.6). Upon using formula [10, (4.1.5), p. 59]

$$C_{2m}^{(|\alpha|_1)}(t) = \frac{\Gamma(|\alpha|_1+m)\Gamma(1/2)}{\Gamma(|\alpha|_1)\Gamma(m+1/2)} P_m^{(|\alpha|_1-\frac{1}{2}, -\frac{1}{2})}(2t^2-1),$$

we obtain a compact formula for $J(t)$ in terms of Gegenbauer polynomials for $n = 2m$, which leads to

$$P_n^h(\mathbf{x}, \mathbf{y}) = \frac{n+|\alpha|_1+(d-2)/2}{|\alpha|_1+(d-2)/2} [V C_n^{(|\alpha|_1+(d-2)/2)}(\langle \cdot, \mathbf{y}/|\mathbf{y}|\rangle)](\mathbf{x})|\mathbf{y}|^n.$$

Since each $S_{n,i}$ is homogeneous of degree n, P_n^h is homogeneous of degree n in $\mathbf{y}$. Hence, replacing $\mathbf{y}/|\mathbf{y}|$ by $\mathbf{y}$ in the last identity leads to the desired result for $n = 2m$. The case $n = 2m-1$ is proved similarly. □

We note that this formula can be viewed as the addition formula for the h-harmonics; indeed, it reduces to the addition formula for the ordinary spherical harmonics [6, Vol. II, p. 244, (2)]. The formula agrees with Dunkl's formula for the Poisson kernel (1.4) because

$$\sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^{(\lambda)}(t) r^n = \frac{1-r^2}{(1-2tr+r^2)^{\lambda+1}},$$

which follows easily from the expansion of the generating function of Gegenbauer polynomials ([10, (4.7.23), p.82]).

The proof of Theorem 3.1 depends also on the following fact. Let w_λ be the normalized weight function on $[-1,1]$:

$$w_\lambda = c_{\lambda-1/2}(1-x^2)^{\lambda-1/2}, \qquad \int_{-1}^{1} w_\lambda(t)dt = 1,$$

where $c_{\lambda-1/2}$ is the constant defined in (1.6). A function $f \in L^2(w_\lambda, [-1,1])$ can be expanded into Gegenbauer series

$$f \sim \sum_{k=0}^{\infty} b_k(f) \tilde{C}_n^{(\lambda)}, \qquad b_k(f) = \int_{-1}^{1} \tilde{C}_n^{(\lambda)}(x) w_\lambda(x) dx,$$

where $\tilde{C}_n^{(\lambda)}$ is the normalized Gegenbauer polynomial, which differs from $C_n^{(\lambda)}$ by a normalization constant chosen so that $\{\tilde{C}_n^{(\lambda)}\}$ forms an orthonormal basis with respect to w_λ (cf. [10, (4.7.15), p.81]). The n-th partial sum of this expansion can be written as

$$s_n(w_\lambda; f) = \int_{-1}^{1} f(y) K_n(w_\lambda; x, y) w_\lambda(y) dy,$$

where the n-th reproducing kernel $K_n(w_\lambda)$ is given by

$$K_n(w_\lambda; x, y) = \sum_{k=0}^{n} \tilde{C}_k^{(\lambda)}(x)\tilde{C}_k^{(\lambda)}(y) = \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^{(\lambda)}(x)C_k^{(\lambda)}(y)/C_k^{(\lambda)}(1);$$

here in the second equation we have switched back to the usual Gegenbauer polynomials.

Proof of Theorem 3.1. From the definition of P_n^h it is easy to see that we can write the partial sum $S_n(h_\alpha^2; f)$ as

$$S_n(h_\alpha^2; f, \mathbf{x}) = \int_{S^{d-1}} \sum_{k=0}^{n} P_k^h(\mathbf{x}, \cdot) f h_\alpha^2 d\omega = \int_{S^{d-1}} G_n(\mathbf{x}, \cdot) f h_\alpha^2 d\omega$$

where, according to Theorem 3.2 and using the notation of $K_n(w_\lambda)$, we have

$$G_n^h(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} P_k^h(\mathbf{x}, \mathbf{y}) = V[K_n(w_{|\alpha|_1+(d-2)/2}; 1, \langle \cdot, \mathbf{y}\rangle)](\mathbf{x}). \tag{3.2}$$

Let us denote by $S_n^\delta(h_\alpha^2; f)$ the (C, δ) means of $S_n(h_\alpha^2; f)$. Then we have

$$S_n^\delta(h_\alpha^2; f, \mathbf{x}) = \int_{S^{d-1}} G_n^\delta(\mathbf{x}, \cdot) f h_\alpha^2 d\omega,$$
$$G_n^\delta(\mathbf{x}, \mathbf{y}) = V[K_n^\delta(w_{|\alpha|_1+(d-2)/2}; 1, \langle \cdot, \mathbf{x}\rangle)](\mathbf{y}),$$

where G_n^δ denotes the (C, δ) means of G_n and $K_n^\delta(w_\lambda)$ the means of $K_n(w_\lambda)$. To prove the theorem it's enough to show that

$$F_n(\mathbf{x}) := \int_{S^{d-1}} |G_n^\delta(\mathbf{x}, \mathbf{y})| h_\alpha^2(\mathbf{y}) d\omega(\mathbf{y}) < \infty$$

uniformly for $\mathbf{x} \in S^{d-1}$ if $\delta > \gamma + (d-2)/2$. If V is positive, we have

$$|V[K_n^\delta(w_{|\alpha|_1+(d-2)/2}; 1, \langle \cdot, \mathbf{x}\rangle)](\mathbf{y})| \le V[|K_n^\delta(w_{|\alpha|_1+(d-2)/2}; 1, \langle \cdot, \mathbf{x}\rangle)|](\mathbf{y}).$$

Therefore, we can apply Corollary 2.2 to conclude that

$$\begin{aligned} F_n(\mathbf{x}) &\le \int_{S^{d-1}} V[|K_n^\delta(w_{|\alpha|_1+(d-2)/2}; 1, \langle \cdot, \mathbf{x}\rangle)|](\mathbf{y}) h_\alpha^2(\mathbf{y}) d\omega \\ &\le B_\alpha \int_{-1}^{1} |K_n^\delta(w_{|\alpha|_1+(d-2)/2}; 1, t)|(1-t^2)^{|\alpha|_1+\frac{d-3}{2}} dt. \end{aligned}$$

The last integral is bounded if and only if $\delta > |\alpha|_1 + (d-2)/2$, since it is precisely the condition on the (C, δ) summability of the Gegenbauer series with index $|\alpha|_1 + (d-2)/2$ at the point $x = 1$ and the conclusion follows from [10, Theorem 9.1.3, p. 246]. □

We conclude the paper with the following remark: From Theorem 3.2 and an inequality of Kogbetliantz (see [1, p. 71]) for the Gegenbauer polynomials, it follows that the $(C, 2|\alpha|_1 + d - 1)$ means of the reproducing kernels P_n^h are nonnegative over S^{d-1} provided that V is positive. In particular, the $(C, 2|\alpha|_1 + d - 1)$ means of the Fourier series with respect to $h_\alpha^2 d\omega$ on S^{d-1} define a positive linear operator, provided that V is positive. Other related results on the positivity of the (C, δ) means can be derived from an inequality of Askey and Gasper (see [1, p. 74]).

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