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ON THE INTERSECTION PROPERTY OF DUBROVIN VALUATION RINGS

ZHAO YICAI

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ABSTRACT. It is shown that of the three axioms Gräter specified for his intersection property of Dubrovin valuation rings in central-simple algebras, the second and third axioms actually follow from the first.

INTRODUCTION

In [G1] Gräter introduced the intersection property of Dubrovin valuation rings which proved to be a crucial property in studying extensions of valuation rings and the structure of intersections of a finite number of Dubrovin valuation rings. In that paper a numerical invariant was obtained which has a close relation to the intersection property and this invariant was equal to that of the Ostrowski Theorem for Dubrovin valuation rings in [W], and so a new Defektsatz was stated. On the other hand, the intersection property is equivalent to Morandi's condition which is necessary and sufficient to get the general approximation theorem in [M1]. Furthermore, any semilocal Bézout order in a central simple algebra is precisely an intersection of a finite number of Dubrovin valuation rings having the intersection property [G2]. Thus Dubrovin valuation rings with the intersection property form a useful tool in studying prime PI-Bézout rings.

The purpose of this paper is to show that the intersection property can be replaced by a simpler property, i.e., property P. It turns out to be convenient in studying Dubrovin valuation rings by using property P.

This paper makes strong use of the results of [G1] and [M3].

1. PRELIMINARIES

Throughout this paper Q will denote a central simple algebra with finite dimension over its center F. If R is any ring, then $Z(R)$ is the center of R, $J(R)$ the Jacobson radical of R, R^* the group of units of R and $\mathrm{Spec}(R)$ the set of all prime ideals of R. A subring B of Q is called a Dubrovin valuation ring of Q if B has an ideal I such that B/I is simple Artinian and for any $q \in Q \setminus B$ there exist $b, b' \in B$ with $bq, qb' \in B \setminus I$. It is shown in [D1] that $I = J(B)$, the only maximal ideal of B, and that $Z(B) = B \cap F$, a valuation ring of F. Other properties of Dubrovin valuation rings can be found in [D1], [D2], [W] and [G1]. Let B be a Dubrovin

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valuation ring of Q, then $\mathcal{B}(B)$ denotes the set of all overrings of B in Q. In this paper, $\subseteq$ stands for inclusion and $\subset$ for proper inclusion.

Following Gräter [G1] the intersection property is defined as follows.

Definition. Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q and let $R = B_1 \cap \cdots \cap B_n$. Then $B_1, \ldots, B_n$ have the *intersection property* if

$$\begin{aligned} f : \mathcal{B}(B_1) \cup \cdots \cup \mathcal{B}(B_n) &\to \operatorname{Spec}(R) \\ B &\mapsto J(B) \cap R \end{aligned}$$

is a well-defined anti-order-isomorphism.

In other words, we say that $B_1, \ldots, B_n$ have the *intersection property* if the following conditions are satisfied:

(1) For any overring B of B_i in Q $(i = 1, \ldots, n)$, $J(B) \cap R$ is a prime ideal of R.

(2) For any prime ideal P of R, there exists a unique $B \in \mathcal{B}(B_i)$ for some $i = 1, \ldots, n$ such that $P = J(B) \cap R$.

(3) For any $B,\ B' \in \mathcal{B}(B_1) \cup \cdots \cup \mathcal{B}(B_n)$, $B \subseteq B'$ if and only if $J(B') \cap R \subseteq J(B) \cap R$.

Definition. Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q and let $R = B_1 \cap \cdots \cap B_n$. Then $B_1, \ldots, B_n$ have *property* P if for any overring B of B_i in Q $(i = 1, \ldots, n)$, $J(B) \cap R$ is a prime ideal of R.

We will show that the intersection property is equivalent to property P. This means that conditions (2) and (3) follow from condition (1).

2. Proof of the main theorem

First of all, we need some lemmas.

Lemma 1. *Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q and let $R = B_1 \cap \cdots \cap B_n$ and $D = R \cap F$.*

(i) *Let $S \neq \varnothing$ be a multiplicative subset of D and I an ideal of R_s. Then I is a prime ideal of R_s if and only if $I \cap R$ is a prime ideal of R, where $R_s = RS^{-1}$, the localization of R at S.*

In particular, if $B_1, \ldots, B_n$ have property P*, then $B_{1s}, \ldots, B_{ns}$ have property* P.

(ii) *$B_1, \ldots, B_n$ have property* P *if and only if $B_{1m}, \ldots, B_{nm}$ have property* P *for all maximal ideals m of D, where $B_{im} = B_i(D \setminus m)^{-1}$.*

Proof. (i) Suppose I is a prime ideal of R_s. For any $a, b \in R$, if $aRb \subseteq I \cap R$, then $aR_sb \subseteq (I \cap R)_s = I$. Hence a or $b \in I$ and then a or $b \in I \cap R$. It follows that $I \cap R$ is a prime ideal of R.

Conversely, suppose $I \cap R$ is a prime ideal of R. For any $x, y \in R_s$, we have $x = us^{-1}$ and $y = vt^{-1}$ for some $u, v \in R$ and some $s, t \in S$. If $xR_sy \subseteq I$, then $uR_sv \subseteq I$. So $uRv \subseteq I \cap R$. Since $I \cap R$ is a prime ideal of R, u or $v \in I \cap R$. Thus x or $y \in (I \cap R)_s = I$. Therefore I is a prime ideal of R_s.

Finally, assume that $B_1, \ldots, B_n$ have property P. For any given B_{is}, if $B \in \mathcal{B}(B_{is})$ then $B \in \mathcal{B}(B_i)$. By assumption, $J(B) \cap R$ is a prime ideal of R. It follows that $J(B) \cap R_s$ is a prime ideal of R_s, as $(J(B) \cap R_s) \cap R = J(B) \cap R$. Hence $B_{1s}, \ldots, B_{ns}$ have property P.

(ii) Suppose $B_{1m}, \ldots, B_{nm}$ have property P for all maximal ideals m of D. For any B_i, write $V_i = Z(B_i)$; then $J(V_i) \cap D$ is a prime ideal of D. There exists a maximal ideal m of D containing $J(V_i) \cap D$. We have $B_{im} = B_i(D \setminus m)^{-1} \subseteq$

$B_i(D \setminus J(V_i))^{-1} \subseteq B_i(V_i \setminus J(V_i))^{-1} = B_i$. If $B \in \mathcal{B}(B_i)$, then $B \in \mathcal{B}(B_{im})$. By assumption, $B_{1m}, \ldots, B_{nm}$ have property P. Hence $J(B) \cap R_m$ is a prime ideal of R_m. By (i) $J(B) \cap R$ is a prime ideal of R as $(J(B) \cap R_m) \cap R = J(B) \cap R$. Therefore $B_1, \ldots, B_n$ have property P.

The converse follows from (i).

Lemma 2. *Let $B_1, \ldots, B_n, B$ be Dubrovin valuation rings of Q such that $B_1, \ldots, B_n \subseteq B$. Then $B_1, \ldots, B_n$ have property* P *if and only if $B_1/J(B), \ldots, B_n/J(B)$ have property* P.

Proof. Let $R = B_1 \cap \cdots \cap B_n$. Assume that $B_1, \ldots, B_n$ have property P. $B_1/J(B), \ldots, B_n/J(B)$ are Dubrovin valuation rings of $B/J(B)$. For any given B_i, if $C' \in \mathcal{B}(B_i/J(B))$, then there exists a Dubrovin valuation ring C of Q with $B_i \subseteq C \subseteq B$ such that $C' = C/J(B)$. By assumption, $J(C) \cap R$ is a prime ideal of R. Then $J(C)/J(B) \cap R/J(B)$ is a prime ideal of $R/J(B)$. Hence $B_1/J(B), \ldots, B_n/J(B)$ have property P.

Conversely, assume that $B_1/J(B), \ldots, B_n/J(B)$ have property P. For any $C \in \mathcal{B}(B_i)$, if $C \subseteq B$, then $C/J(B) \in \mathcal{B}(B_i/J(B))$. By assumption, $J(C)/J(B) \cap R/J(B)$ is a prime ideal of $R/J(B)$. Hence $J(C) \cap R$ is a prime ideal of R. If $C \nsubseteq B$, then $B \subset C$ as $\mathcal{B}(B_i)$ is totally ordered by inclusion. Now $B_1/J(C), \ldots, B_n/J(C)$ are Dubrovin valuation rings of $C/J(C)$. By [K, Theorem 107], $\bigcap Z(B_i/J(C))$ has the same quotient field as the $Z(B_i/J(C))$'s, so $R/J(C) = B_1/J(C) \cap \cdots \cap B_n/J(C)$ is an order in $C/J(C)$, hence $J(C) = J(C) \cap R$ is a prime ideal of R. Therefore $B_1, \ldots, B_n$ have property P.

Lemma 3. *Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q having property* P *such that $V = B_1 \cap F = \cdots = B_n \cap F$. If each B_i is integral over V, then $B_1 = \cdots = B_n$.*

Proof. Suppose $B_i \neq B_j$ for some B_i and B_j. We can assume that $B_1, \ldots, B_n$ are incomparable. For any B_i, $i = 1, \ldots, n$, B_1 and B_i are conjugate by [W, Theorem A], that is, there exist $q_i \in Q^*$ such that $B_i = q_i B_1 q_i^{-1}$ for $i = 1, \ldots, n$. We first note that for any $q \in Q^*$, there exists $v \in \mathrm{st}(B_1) = \{x \in Q^*, xB_1 = B_1 x\}$ such that $qv \in B_1 \setminus J(B_1)$. In fact, since B_1 is integral over V, by [W, Theorem F] there exists $a \in \mathrm{st}(B_1)$ such that $B_1 q B_1 = aB_1$, i.e., $B_1 q a^{-1} B_1 = B_1$. Let $v = a^{-1}$; then $qv \in B_1 \setminus J(B_1)$ and $v \in \mathrm{st}(B_1)$. Now, for each q_i, there exists a $v_i \in \mathrm{st}(B_1)$ such that $q_i v_i \in B_1 \setminus J(B_1)$. By replacing $q_i v_i$ by q_i, we may assume $q_i \in B_1 \setminus J(B_1)$. For q_i^{-1}, there exists a $u_i \in \mathrm{st}(B_1)$ such that $q_i^{-1} u_i \in B_1 \setminus J(B_1)$. Hence $q_i^{-1} u_i B_1 q_i \subseteq B_1$ and then we have that $u_i B_1 \subseteq q_i B_1 q_i^{-1} = B_i$ for $i = 1, \ldots, n$. Set $I_1 = \bigcap u_i B_1$ and $R = B_1 \cap \cdots \cap B_n$; then I_1 is an ideal of R. Since $q_i^{-1} u_i, q_i \in B_1 \setminus J(B_1)$, $u_i B_1 = B_1 u_i$ is a two-sided ideal of B_1. By [D1, Theorem 4] the $u_i B_1$ are totally ordered by inclusion. Thus there exists some u_{i_1} in $\{u_1, \ldots, u_n\}$ such that $I_1 = u_{i_1} B_1$ and $B_{i_1} = q_{i_1} B_1 q_{i_1}^{-1}$. Similarly, we have that for each $B_k \in \{B_1, \ldots, B_n\}$ there exists a $u_{i_k} \in \mathrm{st}(B_k), q_{i_k} \in Q^*$ and $B_{i_k} \in \{B_1, \ldots, B_n\}$ such that $B_{i_k} = q_{i_k} B_k q_{i_k}^{-1}$ and $u_{i_k} B_k$ is a two-sided ideal of $R = B_1 \cap \cdots \cap B_n$ and $q_{i_k}, q_{i_k}^{-1} u_{i_k} \in B_k \setminus J(B_k)$. Set $P_i = J(B_i) \cap R$ for $i = 1, \ldots, n$. Then each P_i is a prime ideal of R by assumption. Suppose that for each $B_k \in \{B_1, \ldots, B_n\}$, $u_{i_k} B_k \nsubseteq P_{i_k}$. Because B_k and R have a common ideal $u_{i_k} B_k$ which does not lie in P_{i_k}, there is by [AS, Theorem 2.5] a prime ideal P'_k of B_k such that $P_{i_k} = P'_k \cap R$. Thus $P_{i_k} \subseteq J(B_k) \cap R = P_k$. It follows that $P_{i_k} \subset P_k$ as $u_{i_k} B_k \subseteq J(B_k) \cap R = P_k$. Noting that

$\{P_{i_1}, \ldots, P_{i_n}\} \subseteq \{P_1, \ldots, P_n\}$, we conclude that each $P_k \in \{P_1, \ldots, P_n\}$ properly contains some one of $\{P_1, \ldots, P_n\}$. This is impossible. Indeed, since $\{P_1, \ldots, P_n\}$ is finite, we may choose P_k minimal in $\{P_1, \ldots, P_n\}$. But P_k strictly contains P_{i_k}, a contradiction. Hence there is B_k in $\{B_1, \ldots, B_n\}$ such that $u_{i_k} B_k \subseteq P_{i_k}$. Then $q_{i_k}^{-1} u_{i_k} B_k q_{i_k} \subseteq q_{i_k}^{-1} P_{i_k} q_{i_k} \subseteq q_{i_k}^{-1} J(B_{i_k}) q_{i_k} = J(B_k)$. Since $q_{i_k}^{-1} u_{i_k}, q_{i_k} \in B_k \setminus J(B_k)$, it follows that $B_k = B_k q_{i_k}^{-1} u_{i_k} B_k q_{i_k} B_k \subseteq B_k J(B_k) B_k = J(B_k)$, a contradiction. Therefore, $B_1 = \cdots = B_n$.

By [G1, Corollary 5.3], we know that each proper Dubrovin valuation ring is contained in a proper valuation ring which is integral over its center.

Lemma 4. *Let $B_1, \ldots, B_n$ be incomparable Dubrovin valuation rings of Q having property* P *such that $D = F \cap B_1 \cap \cdots \cap B_n$ is a valuation ring of F. Let R_i be the minimal Dubrovin valuation ring of Q containing B_i such that R_i is integral over its center $D_i = R_i \cap F$. Then $R_1 = \cdots = R_n$.*

Proof. Since D is a valuation ring of F, overrings of D in F are totally ordered by inclusion. Now $D \subseteq D_i$ for $i = 1, \ldots, n$, so $D_1, \ldots, D_n$ are totally ordered, say, $D_1 \subseteq \cdots \subseteq D_n$. Suppose that $D_1 \neq D_n$. Write $V_i = F \cap B_i$. Then either $D_1 \subseteq V_n$ or $V_n \subset D_1$ as V_n and D_1 contain D. If $V_n \subset D_1$, then $Z(D_1 B_n) = Z(R_1) = D_1$. So $D_1 B_n$ and R_1 are conjugate and then $D_1 B_n$ is integral over D_1. However, $Z(D_1 B_n) = D_1 \subset D_n = Z(R_n)$. It follows that $B_n \subset D_1 B_n \subset D_n B_n = R_n$. This is contrary to the minimality of R_n. Thus we have $D_1 \subseteq V_n$. Since R_1 is integral over D_1, $R_1 V_n$ is integral over V_n. Since $Z(R_1 V_n) = Z(B_n) = V_n$, B_n and $R_1 V_n$ are conjugate by [W, Theorem A]. Hence B_n is integral over V_n. It follows that $B_n = R_n$. Let $P = J(D_n)$; then $D_n = D_p$ and $(B_1)_p \cap F = \cdots = (B_n)_p \cap F = D_n$ as $D \subseteq B_i \cap F \subseteq D_n$. Since $B_{n_p} = B_n$ is integral over D_n, each B_{i_p} is integral over D_n by [W, Theorem A]. Furthermore, by Lemma 1(i) $B_{1_p}, \ldots, B_{n_p}$ have property P. Then by Lemma 3, $B_{1_p} = \cdots = B_{n_p} = B_n$. It follows that each $B_i \subseteq B_n$. This is a contradiction. Hence $D_1 = \cdots = D_n$ and then $R_i = B_{i_p}$ for all i. By Lemma 1(i) $R_1, \ldots, R_n$ have property P and by Lemma 3, $R_1 = \cdots = R_n$.

Lemma 5. *Let $B_1, \ldots, B_n, B$ be Dubrovin valuation rings of Q such that $B_i \subseteq B$ for some i. Then $B_1, \ldots, B_n, B$ have property* P *if and only if $B_1, \ldots, B_n$ have property* P.

Proof. It is trivial as $B_1 \cap \cdots \cap B_n \cap B = B_1 \cap \cdots \cap B_n$ and $\mathcal{B}(B) \subseteq \mathcal{B}(B_i)$.

Lemma 6. *Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q having property* P. *If B, B' are Dubrovin valuation rings of Q such that $B_i \subseteq B, B_j \subseteq B'$ for some $i, j = 1, \ldots, n$, then B, B' have property* P.

Proof. Set $D = F \cap B_1 \cap \cdots \cap B_n$ and $W = F \cap B \cap B'$. For any maximal ideal m of W, by Lemma 1(ii), $B_{1_p}, \ldots, B_{n_p}$ have property P, where $P = m \cap D$. By [E, 11.4, 11.12], $D_p = F \cap B_{1_p} \cap \cdots \cap B_{n_p}$ is a valuation ring of F and $B_{i_p} \subseteq B_m$, $B_{j_p} \subseteq B'_m$. By Lemma 1(ii) again, we may assume that $D = F \cap B_1 \cap \cdots \cap B_n$ is a valuation ring of F, to prove that B, B' have property P. By Lemma 5, it is enough to consider the situation that $n \neq 1$ and $B_1, \ldots, B_n$ are incomparable and B, B' are incomparable. We prove the lemma by induction on $[Q : F]$. Let $[Q : F] > 1$. Let C be the minimal Dubrovin valuation ring of Q containing B_1 such that C is integral over its center $V = F \cap C$. Then by Lemma 4, $B_1, \ldots, B_n \subset C$ and by [G1, Corollary 5.6] $Z(C/J(C)) \neq V/J(V)$. Thus $[C/J(C) : Z(C/J(D))] < [Q : F]$. By

Lemma 2, $B_1/J(C), \ldots, B_n/J(C)$ are Dubrovin valuation rings of $C/J(C)$ having property P. If B or $B' \supseteq C$, then $B \subseteq B'$ or $B' \subset B$ as $\mathcal{B}(B_i)$ and $\mathcal{B}(B_j)$ are totally ordered by inclusion. So we may assume that $B, B' \subset C$. Now, by induction, $B/J(C), B'/J(C)$ have property P. Hence B, B' have property P by Lemma 2

Lemma 7. *Let B_1, B_2 be Dubrovin valuation rings of Q having property P. Then B_1 and B_2 are comaximal in Q if and only if $Z(B_1)$ and $Z(B_2)$ are comaximal in F.*

Proof. Assume that B_1 and B_2 are comaximal in Q. If $V \neq F$ is an overring of $Z(B_1)$ and $Z(B_2)$ in F, then

$$D = Z(B_1) \cap Z(B_2) = Z(B_1) \cap Z(B_2) \cap V.$$

By [K, Theorem 107] $V = D_p$ for some prime ideal P of D. By Lemma 1(i) B_{1_p}, B_{2_p} have property P and $Z(B_{1_p}) = Z(B_{2_p}) = V$. By [G1, Corollary 5.6] and Lemma 4, there is a Dubrovin valuation ring $B \neq Q$ containing B_1 and B_2, a contradiction. Hence $Z(B_1)$ and $Z(B_n)$ are comaximal in F. The converse is trivial.

Now, we can give the main theorem of this paper.

Theorem. *Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of Q. If $B_1, \ldots, B_n$ have property* P*, then $B_1, \ldots, B_n$ have the intersection property.*

Proof. Assume that $B_1, \ldots, B_n$ have property P. Then by Lemma 6 any B_i, B_j have property P for $i, j = 1, \ldots, n$, and by Lemma 2, $B_i/J(B_{ij})$, $B_j/J(B_{ij})$ have property P, where B_{ij} are the least overring of B_i and B_j in Q. By Lemma 7, $Z(B_i/J(B_{ij}))$ and $Z(B_j/J(B_{ij}))$ are comaximal and then by [G1, Corollary 6.2] $B_i/J(B_{ij}), B_j/J(B_{ij})$ have the intersection property. Hence any B_i, B_j have the intersection property by [G1, Prop. 6.3] and then by [G1, Theorem 6.8], $B_1, \ldots, B_n$ have the intersection property.

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Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China