ON THE INTERSECTION PROPERTY
OF DUBROVIN VALUATION RINGS

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(Communicated by Ken Goodearl)

Abstract. It is shown that of the three axioms Gräter specified for his intersection property of Dubrovin valuation rings in central-simple algebras, the second and third axioms actually follow from the first.

Introduction

In [G1] Gräter introduced the intersection property of Dubrovin valuation rings which proved to be a crucial property in studying extensions of valuation rings and the structure of intersections of a finite number of Dubrovin valuation rings. In that paper a numerical invariant was obtained which has a close relation to the intersection property and this invariant was equal to that of the Ostrowski Theorem for Dubrovin valuation rings in [W], and so a new Defektsatz was stated. On the other hand, the intersection property is equivalent to Morandi’s condition which is necessary and sufficient to get the general approximation theorem in [M1]. Furthermore, any semilocal Bézout order in a central simple algebra is precisely an intersection of a finite number of Dubrovin valuation rings having the intersection property [G2]. Thus Dubrovin valuation rings with the intersection property form a useful tool in studying prime PI-Bézout rings.

The purpose of this paper is to show that the intersection property can be replaced by a simpler property, i.e., property P. It turns out to be convenient in studying Dubrovin valuation rings by using property P.

This paper makes strong use of the results of [G1] and [M3].

1. Preliminaries

Throughout this paper Q will denote a central simple algebra with finite dimension over its center F. If R is any ring, then Z(R) is the center of R, J(R) the Jacobson radical of R, R* the group of units of R and Spec(R) the set of all prime ideals of R. A subring B of Q is called a Dubrovin valuation ring of Q if B has an ideal I such that B/I is simple Artinian and for any q ∈ Q \ B there exist b, b′ ∈ B with bq, qb′ ∈ B \ I. It is shown in [D1] that I = J(B), the only maximal ideal of B, and that Z(B) = B ∩ F, a valuation ring of F. Other properties of Dubrovin valuation rings can be found in [D1], [D2], [W] and [G1]. Let B be a Dubrovin valuation ring of Q.
valuation ring of \( Q \), then \( \mathcal{B}(B) \) denotes the set of all overrings of \( B \) in \( Q \). In this paper, \( \subseteq \) stands for inclusion and \( \subset \) for proper inclusion.

Following Gräter \cite{G1} the intersection property is defined as follows.

**Definition.** Let \( B_1, \ldots, B_n \) be Dubrovin valuation rings of \( Q \) and let \( R = B_1 \cap \ldots \cap B_n \). Then \( B_1, \ldots, B_n \) have the **intersection property** if

\[
f : \mathcal{B}(B_1) \cup \ldots \cup \mathcal{B}(B_n) \to \text{Spec}(R)
\]

\[
B \mapsto J(B) \cap R
\]

is a well-defined anti-order-isomorphism.

In other words, we say that \( B_1, \ldots, B_n \) have the **intersection property** if the following conditions are satisfied:

1. For any overring \( B \) of \( B_i \) in \( Q \) (\( i = 1, \ldots, n \)), \( J(B) \cap R \) is a prime ideal of \( R \).
2. For any prime ideal \( P \) of \( R \), there exists a unique \( B \in \mathcal{B}(B_i) \) for some \( i = 1, \ldots, n \) such that \( P = J(B) \cap R \).
3. For any \( B, B' \in \mathcal{B}(B_1) \cup \cdots \cup \mathcal{B}(B_n) \), \( B \subseteq B' \) if and only if \( J(B') \cap R \subseteq J(B) \cap R \).

**Definition.** Let \( B_1, \ldots, B_n \) be Dubrovin valuation rings of \( Q \) and let \( R = B_1 \cap \cdots \cap B_n \). Then \( B_1, \ldots, B_n \) have **property** \( P \) if for any overring \( B \) of \( B_i \) in \( Q \) (\( i = 1, \ldots, n \)), \( J(B) \cap R \) is a prime ideal of \( R \).

We will show that the intersection property is equivalent to property \( P \). This means that conditions (2) and (3) follow from condition (1).

2. **Proof of the main theorem**

First of all, we need some lemmas.

**Lemma 1.** Let \( B_1, \ldots, B_n \) be Dubrovin valuation rings of \( Q \) and let \( R = B_1 \cap \cdots \cap B_n \) and \( D = R \cap F \).

(i) Let \( S \neq \emptyset \) be a multiplicative subset of \( D \) and \( I \) an ideal of \( R_s \). Then \( I \) is a prime ideal of \( R_s \) if and only if \( I \cap R \) is a prime ideal of \( R \), where \( R_s = RS^{-1} \), the localization of \( R \) at \( S \).

In particular, if \( B_1, \ldots, B_n \) have property \( P \), then \( B_{1s}, \ldots, B_{ns} \) have property \( P \).

(ii) \( B_1, \ldots, B_n \) have property \( P \) if and only if \( B_{1m}, \ldots, B_{nm} \) have property \( P \) for all maximal ideals \( m \) of \( D \), where \( B_{im} = B_i(D \setminus m)^{-1} \).

**Proof.** (i) Suppose \( I \) is a prime ideal of \( R_s \). For any \( a, b \in R \), if \( aRb \subseteq I \cap R \), then \( aR_s b \subseteq (I \cap R)_s = I \). Hence \( a \) or \( b \in I \) and then \( a \) or \( b \in I \cap R \). It follows that \( I \cap R \) is a prime ideal of \( R \).

Conversely, suppose \( I \cap R \) is a prime ideal of \( R \). For any \( x, y \in R_s \), we have \( x = us^{-1} \) and \( y = vt^{-1} \) for some \( u, v \in R \) and some \( s, t \in S \). If \( xR_s y \subseteq I \), then \( uR_s v \subseteq I \). So \( uRv \subseteq I \cap R \). Since \( I \cap R \) is a prime ideal of \( R \), \( u \) or \( v \in I \cap R \). Thus \( x \) or \( y \in (I \cap R)_s = I \). Therefore \( I \) is a prime ideal of \( R_s \).

Finally, assume that \( B_1, \ldots, B_n \) have property \( P \). For any given \( B_{is} \), if \( B \in \mathcal{B}(B_{is}) \) then \( B \in \mathcal{B}(B_i) \). By assumption, \( J(B) \cap R \) is a prime ideal of \( R \). It follows that \( J(B) \cap R_s \) is a prime ideal of \( R_s \), as \( (J(B) \cap R_s) \cap R = J(B) \cap R \). Hence \( B_{1s}, \ldots, B_{ns} \) have property \( P \).

(ii) Suppose \( B_{1m}, \ldots, B_{nm} \) have property \( P \) for all maximal ideals \( m \) of \( D \). For any \( B_i \), write \( V_i = Z(B_i) \); then \( J(V_i) \cap D \) is a prime ideal of \( D \). There exists a maximal ideal \( m \) of \( D \) containing \( J(V_i) \cap D \). We have \( B_{im} = B_i(D \setminus m)^{-1} \subseteq \)
Suppose \( A \), that is, there exist \( q \) have property \( P \). In fact, since \( R/J \) be Dubrovin valuation rings of \( C \). By assumption, \( J(C)/J(B) \) and \( J(B) \) have property \( P \). Conversely, assume that \( B_1/J(B), \ldots, B_n/J(B) \) have property \( P \). For any \( C \in B(B_1), \) if \( C \subseteq B \), then \( C/J(B) \in B(B_1/J(B)) \). By assumption, \( J(C)/J(B) \cap R/J(B) \) is a prime ideal of \( R/J(B) \). Hence \( J(C) \cap R \) is a prime ideal of \( R \). If \( C \not\subseteq B \), then \( B \subset C \) as \( B(B_1) \) is totally ordered by inclusion. Now \( B_1/J(C), \ldots, B_n/J(C) \) are Dubrovin valuation rings of \( C/J(C) \). By [K, Theorem 107], \( J(B_1/J(C)) \) has the same quotient field as the \( Z(B_1/J(C)) \)'s, so \( R/J(C) = B_1/J(C) \cap \cdots \cap B_n/J(C) \) is an order in \( C/J(C) \), hence \( J(C) = J(C) \cap R \) is a prime ideal of \( R \). Therefore \( B_1, \ldots, B_n \) have property \( P \).

**Lemma 3.** Let \( B_1, \ldots, B_n \) be Dubrovin valuation rings of \( Q \) having property \( P \) such that \( V = B_1 \cap F = \cdots = B_n \cap F \). If each \( B_i \) is integral over \( V \), then \( B_1 = \cdots = B_n \).

**Proof.** Suppose \( B_i \neq B_j \) for some \( B_i \) and \( B_j \). We can assume that \( B_1, \ldots, B_n \) are incomparable. For any \( B_i, i = 1, \ldots, n \), \( B_1 \) and \( B_i \) are conjugate by [W, Theorem A], that is, there exist \( q_i \in \mathbb{Q}^* \) such that \( B_i = B_1 q_i^{-1} \) for \( i = 1, \ldots, n \). We first note that for any \( q \in \mathbb{Q}^* \), there exists \( v \in \text{st}(B_1) = \{ x \in \mathbb{Q}^*, x B_1 = B_1 x \} \) such that \( qv \in B_1 \setminus J(B_1) \). In fact, since \( B_1 \) is integral over \( V \), by [W, Theorem F] there exists \( a \in \text{st}(B_1) \) such that \( B_1 aB_1 = B_1 a, \) i.e., \( B_1 qa^{-1}B_1 = B_1 \). Let \( v = a^{-1} \); then \( qv \in B_1 \setminus J(B_1) \) and \( v \in \text{st}(B_1) \). Now, for each \( q_i \), there exists a \( v_i \in \text{st}(B_1) \) such that \( q_i v_i \in B_1 \setminus J(B_1) \). By replacing \( q_i v_i \) by \( q_i \), we may assume \( q_i \in B_1 \setminus J(B_1) \). For \( q_i^{-1} \), there exists a \( u_i \in \text{st}(B_1) \) such that \( q_i^{-1} u_i \in B_1 \setminus J(B_1) \). Hence \( q_i^{-1} u_i B_1 q_i \subseteq B_1 \) and then we have that \( u_i B_1 \subseteq q_i B_1 q_i^{-1} = B_i \) for \( i = 1, \ldots, n \). Set \( I_1 = \bigcap u_i B_1 \) and \( R = B_1 \cap \cdots \cap B_n \); then \( I_1 \) is an ideal of \( R \). Since \( q_i^{-1} u_i \in B_1 \setminus J(B_1) \), \( u_i B_1 = B_1 u_i \) is a two-sided ideal of \( B_1 \). By [D1, Theorem 4] the \( u_i B_1 \) are totally ordered by inclusion. Thus there exists some \( u_i \) in \( \{ u_1, \ldots, u_n \} \) such that \( I_1 = u_i B_1 \) and \( B_1 u_i = q_i B_1 q_i^{-1} \). Similarly, we have that for each \( B_k \in \{ B_1, \ldots, B_n \} \) there exists a \( u_k \in \text{st}(B_k) \) such that \( B_k = q_k B_k q_k^{-1} \) and \( u_k B_k \) is a two-sided ideal of \( R = B_1 \cap \cdots \cap B_n \) and \( q_k q_i^{-1} u_k \in B_k \setminus J(B_k) \).
\( \{ P_1, \ldots, P_n \} \subseteq \{ P_1, \ldots, P_n \} \), we conclude that each \( P_k \in \{ P_1, \ldots, P_n \} \) properly contains some one of \( \{ P_1, \ldots, P_n \} \). This is impossible. Indeed, since \( \{ P_1, \ldots, P_n \} \) is finite, we may choose \( P_k \) minimal in \( \{ P_1, \ldots, P_n \} \). But \( P_k \) strictly contains \( P_{k_1} \), a contradiction. Hence there is \( B_k \in \{ B_1, \ldots, B_n \} \) such that \( u_{i_k} \cap B_k \subseteq P_{i_k} \). Then \( q_{i_k}^{-1} u_{i_k} B_{q_{i_k}} \subseteq q_{i_k}^{-1} P_{i_k} B_{q_{i_k}} \subseteq q_{i_k}^{-1} \{ q_{i_k} = J(B_k) \}. \) Since \( q_{i_k}^{-1} u_{i_k} q_{i_k} \in B_k \setminus J(B_k) \), it follows that \( B_k = B_k q_{i_k}^{-1} u_{i_k} B_{k q_{i_k}} B_k \subseteq B_k J(B_k) B_k = J(B_k) \), a contradiction. Therefore, \( B_1 = \cdots = B_n \).

By [G1, Corollary 5.3], we know that each proper Dubrovin valuation ring is contained in a proper valuation ring which is integral over its center.

**Lemma 4.** Let \( B_1, \ldots, B_n \) be incomparable Dubrovin valuation rings of \( Q \) having property \( P \) such that \( D = F \cap B_1 \cap \cdots \cap B_n \) is a valuation ring of \( F \). Let \( R_i \) be the minimal Dubrovin valuation ring of \( Q \) containing \( B_i \) such that \( R_i \) is integral over its center \( D_i = R_i \cap F \). Then \( R_1 = \cdots = R_n \).

**Proof.** Since \( D \) is a valuation ring of \( F \), overrings of \( D \) in \( F \) are totally ordered by inclusion. Now \( D \subseteq D_i \) for \( i = 1, \ldots, n \), so \( D_1, \ldots, D_n \) are totally ordered, say, \( D_1 \subseteq \cdots \subseteq D_n \). Suppose that \( D_1 \neq D_n \). Write \( V_i = F \cap B_i \). Then either \( D_i \subseteq V_n \) or \( V_n \subseteq D_i \) contain \( D_i \). If \( V_n \subseteq D_i \), then \( Z(D_i B_n) = Z(R_i) = D_i \). So \( D_i B_n \) and \( R_i \) are conjugate and then \( D_i B_n \) is integral over \( D_i \). However, \( Z(D_i B_n) = D_i \subseteq D_n = Z(R_i) \). It follows that \( B_n \subseteq D_i B_n \subseteq D_n R_i = R_n \). This is contrary to the minimality of \( R_n \). Thus we have \( D_i \subseteq V_n \). Since \( R_i \) is integral over \( D_i \), \( R_i V_n \) is integral over \( V_n \). Since \( Z(R_i V_n) = Z(B_n) = V_n \), \( B_n \) and \( R_i V_n \) are conjugate by [W, Theorem A]. Hence \( B_n \) is integral over \( V_n \). It follows that \( B_n = R_n \). Let \( P = J(D_n) \); then \( D_n = D_p \) and \( (B_1)_{[F]} \cap F = \cdots = (B_n)_{[F]} \cap F = D_n \) as \( D_n \subseteq D_i \cap F \subseteq D_n \). Since \( B_n \) is integral over \( D_n \), each \( B_n \) is integral over \( D_n \) by [W, Theorem A]. Furthermore, by Lemma 1(i) \( B_{[F]} \) have property \( P \). Then by Lemma 3, \( B_1 \) is incomparable. Thus \( B_1 \) is integral over \( D_n \). Hence \( D_1 = \cdots = D_n \) and then \( R_i = B_i \) for all \( i \). By Lemma 1(i) \( R_1, \ldots, R_n \) have property \( P \) and by Lemma 3, \( R_1 = \cdots = R_n \).

**Lemma 5.** Let \( B_1, \ldots, B_n, B \) be Dubrovin valuation rings of \( Q \) such that \( B_i \subseteq B \) for some \( i \). Then \( B_1, \ldots, B_n, B \) have property \( P \) if and only if \( B_1, \ldots, B_n \) have property \( P \).

**Proof.** It is trivial as \( B_1 \cap \cdots \cap B_n \cap B = B_1 \cap \cdots \cap B_n \) and \( B(B) \subseteq B(B_i) \).

**Lemma 6.** Let \( B_1, \ldots, B_n \) be Dubrovin valuation rings of \( Q \) having property \( P \). If \( B, B' \) are Dubrovin valuation rings of \( Q \) such that \( B_i \subseteq B, B_j \subseteq B' \) for some \( i, j = 1, \ldots, n \), then \( B, B' \) have property \( P \).

**Proof.** Set \( D = F \cap B_1 \cap \cdots \cap B_n \) and \( W = F \cap B \cap B' \). For any maximal ideal \( m \) of \( W \), by Lemma 1(ii), \( B_{[F]} \) have property \( P \), where \( P = m \cap D \). By [E, 11.4, 11.12], \( D_p = F \cap B_{[F]} \cap \cdots \cap B_{[F]} \) is a valuation ring of \( F \) and \( B_p \subseteq B_m \), \( B_{[F]} \subseteq B_{[F]} \). By Lemma 1(ii) again, we may assume that \( D = F \cap B_1 \cap \cdots \cap B_n \) is a valuation ring of \( F \), to prove that \( B, B' \) have property \( P \). By Lemma 5, it is enough to consider the situation that \( n \neq 1 \) and \( B_1, \ldots, B_n \) are incomparable and \( B, B' \) are incomparable. We prove the lemma by induction on \( [Q : F] \). Let \( [Q : F] > 1 \). Let \( C \) be the minimal Dubrovin valuation ring of \( Q \) containing \( B_j \) such that \( C \) is integral over its center \( V = F \cap C \). Then by Lemma 4, \( B_1, \ldots, B_n \subseteq C \) and by [G1, Corollary 5.6] \( Z(C/J(C)) \neq V/J(V) \). Thus \( [C/J(C) : Z(C/J(D))] < [Q : F] \). By
Lemma 2, $B_1/J(C), \ldots, B_n/J(C)$ are Dubrovin valuation rings of $C/J(C)$ having property P. If $B$ or $B' \supseteq C$, then $B \subseteq B'$ or $B' \subset B$ as $B(B_i)$ and $B(B_j)$ are totally ordered by inclusion. So we may assume that $B, B' \subset C$. Now, by induction, $B/J(C), B'/J(C)$ have property P. Hence $B, B'$ have property P by Lemma 2

**Lemma 7.** Let $B_1, B_2$ be Dubrovin valuation rings of $Q$ having property P. Then $B_1$ and $B_2$ are comaximal in $Q$ if and only if $Z(B_1)$ and $Z(B_2)$ are comaximal in $F$.

**Proof.** Assume that $B_1$ and $B_2$ are comaximal in $Q$. If $V \neq F$ is an overring of $Z(B_1)$ and $Z(B_2)$ in $F$, then

$$D = Z(B_1) \cap Z(B_2) = Z(B_1) \cap Z(B_2) \cap V.$$  

By [K, Theorem 107] $V = D_p$ for some prime ideal $P$ of $D$. By Lemma 1(i) $B_1, B_2$ have property P and $Z(B_1) = Z(B_2) = V$. By [G1, Corollary 5.6] and Lemma 4, there is a Dubrovin valuation ring $B \neq Q$ containing $B_1$ and $B_2$, a contradiction. Hence $Z(B_1)$ and $Z(B_2)$ are comaximal in $F$. The converse is trivial.

Now, we can give the main theorem of this paper.

**Theorem.** Let $B_1, \ldots, B_n$ be Dubrovin valuation rings of $Q$. If $B_1, \ldots, B_n$ have property P, then $B_1, \ldots, B_n$ have the intersection property.

**Proof.** Assume that $B_1, \ldots, B_n$ have property P. Then by Lemma 6 any $B_i, B_j$ have property P for $i, j = 1, \ldots, n$, and by Lemma 2, $B_i/J(B_{ij}), B_j/J(B_{ij})$ have property P, where $B_{ij}$ is the least overring of $B_i$ and $B_j$ in $Q$. By Lemma 7, $Z(B_i/J(B_{ij}))$ and $Z(B_j/J(B_{ij}))$ are comaximal and then by [G1, Corollary 6.2] $B_i/J(B_{ij}), B_j/J(B_{ij})$ have the intersection property. Hence any $B_i, B_j$ have the intersection property by [G1, Prop. 6.3] and then by [G1, Theorem 6.8], $B_1, \ldots, B_n$ have the intersection property.

**References.**


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