A COMMUTING PAIR IN HOPF ALGEBRAS

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ABSTRACT. We prove that if \( H \) is a semisimple Hopf algebra, then the action of the Drinfeld double \( \mathcal{D}(H) \) on \( H \) and the action of the character algebra on \( H \) form a commuting pair. This result and a result of G. I. Kats imply that the dimension of every simple \( \mathcal{D}(H) \)-submodule of \( H \) is a divisor of \( \dim(H) \).

Let \( H \) be a finite dimensional semisimple Hopf algebra over an algebraically closed field \( k \) of characteristic 0, \( \mathcal{D}(H) \) be the Drinfeld double of \( H \), and \( C(H) \) be the character algebra of \( H \). \( C(H) \) is spanned by the characters of \( H \)-modules and is an associative subalgebra of \( H^* \). It is known that \( \mathcal{D}(H) \) acts on \( H \) and that \( C(H) \) acts on \( H \) by the restriction of the action \( \leftarrow \) of \( H^* \) on \( H \) (these actions will be recalled below). The purpose of this note is to prove that these two actions form a commuting pair. Using this result, we prove that the dimension of every simple \( \mathcal{D}(H) \)-submodule of \( H \) is a divisor of \( \dim(H) \). It would be interesting if there exists an analog of this commuting pair in the context of Poisson Lie groups.

We first recall the construction of the Drinfeld double (cf. [D], [M]) and fix necessary notations. Let \( H \) be a finite dimensional Hopf algebra over a field \( k \) (here we do not need any additional assumptions on \( H \) and \( k \)). The Drinfeld double of \( H \), denoted by \( \mathcal{D}(H) \), as a vector space, is the tensor space \( H^* \otimes H \). The comultiplication of \( \mathcal{D}(A) \) is given by

\[
\Delta(f \otimes a) = \sum (f(2) \otimes a(1)) \otimes (f(1) \otimes a(2)) \in D(H) \otimes D(H),
\]

where \( \Delta f = f(1) \otimes f(2) \), \( \Delta a = a(1) \otimes a(2) \) are comultiplications in \( H \) and \( H^* \) respectively. The multiplication in \( D(H) \) is defined as follows: for \( f \otimes a \) and \( g \otimes b \) in \( D(H) \),

\[
(f \otimes a)(g \otimes b) = \sum f(a_1 \triangleright g(2)) \otimes (a_2 \triangleleft g(1)) b,
\]

where \( a \triangleright g \) is the action of \( H \) on \( H^* \) given by

\[
a \triangleright g = a(1) \rightarrow g \leftarrow S^{-1}a(2)
\]

and \( a \triangleleft g \) is the right action of \( H^* \) on \( H \) given by

\[
a \triangleleft g = S^{-1}g(1) \rightarrow a \leftarrow g(2).
\]

The notations \( \rightarrow \) and \( \leftarrow \) mean the usual left and right actions of \( H \) on \( H^* \), i.e., for \( a \in H \) and \( g \in H^* \),

\[
a \rightarrow g = \sum g(1) \langle g(2), a \rangle \in H^*, \quad g \leftarrow a = \sum g(2) \langle g(1), a \rangle.
\]
We view $H$ and $H^*$ as subspaces of $D(H)$ by the embeddings $a \in H \mapsto 1 \otimes a$ and $f \in H^* \mapsto f \otimes 1$.

The antipode $S$ and the counit of $D(H)$ are given by

$$S(f \otimes a) = S(a)S^{-1}(f), \quad \epsilon(f \otimes a) = \epsilon(f)\epsilon(a);$$

here $\epsilon(f)$ and $\epsilon(a)$ denote the counit maps for $H^*$ and $H$.

The above operations give $D(H)$ a structure of Hopf algebra. Moreover $D(H)$ is quasitriangular with the $R$-matrix $R = \sum_i e_i^* \otimes e_i$, where $\{e_i\}$ is a basis for $H$, and $\{e_i^*\}$ is its dual basis for $H^*$. The quasitriangular structure will not play a role here. Notice that $H$ and $H^{cop}$ ($H^{cop}$ is $H^*$ with the opposite coproduct) are Hopf subalgebras of $D(H)$.

We also recall some basic notions about modules and module algebras of a Hopf algebra. A (left) module of $H$ means a left module of $H$ as an associative algebra. An associative algebra $A$ is called a (left) module algebra of $H$ if $A$ is a $H$-module such that the algebra structure and $H$-module structure for $A$ are compatible in the following sense: for $h \in H$, $u, v \in A$ and the unit $1_A$ in $A$,

$$(2) \quad h \cdot (uv) = \sum (h_{(1)} \cdot u)(h_{(2)} \cdot v), \quad h \cdot 1_A = \epsilon(h)1_A.$$  

Similarly a right module algebra of $H$ is an associative algebra $A$ together with a right $H$-module structure satisfying the conditions

$$(uv)\cdot h = (u \cdot h_{(1)})(v \cdot h_{(2)}), \quad 1_A \cdot h = \epsilon(h)1_A.$$  

For a finite dimensional $H$-module $V$, the character $\chi_V$ of $V$ is an element of $H^*$ defined by $\langle \chi_V, a \rangle = \text{Tr}[V(a)]$ for every $a \in H$. Because $\chi_W \otimes V = \chi_W \chi_V$ for $H$-modules $W, V$, the characters of $H$ span an associative subalgebra of $H^*$.

This algebra is called the character algebra of $H$ and denoted by $C(H)$. If $H$ is semisimple and the ground field is algebraically closed and of characteristic 0, then $C(H)$ consists of the elements $v \in H^*$ that are cocommutative, i.e., $\sum v_{(1)} \otimes v_{(2)} = \sum v_{(2)} \otimes v_{(1)}$.

$H$ is an $H^*$-module algebra under the action $\rightarrow$ given by $g \rightarrow a = (g, a_{(2)})a_{(1)}$. We will be concerned with the restriction of $\rightarrow$ on the character algebra $C(H)$.

There is $D(H)$-action on $H$ defined by

$$\langle f \otimes a , b \rangle = (a_{(1)}bS(a_{(2)})) \rightarrow S^{-1}(f).$$  

**Lemma 1.** $H$ is a module algebra of $D(H)$ under the action (3).

**Proof.** To prove $H$ is a $D(H)$-module under (3), We need to prove that

$$\langle xy \cdot v, x \cdot (y \cdot v) \rangle$$

for every $x, y \in D(H)$ and $v \in H$. This is true for the cases $x, y \in H \subset D(H)$, $x, y \in H^* \subset D(H)$ and $x \in H^*, y \in H$. It is known that the definition of the multiplication of $D(H)$ is equivalent to the following (cf. [M])

$$(f \otimes a)(g \otimes b) = f(a_{(1)} \rightarrow g \rightarrow S^{-1}a_{(3)}) \otimes a_{(2)}b.$$  

To prove (3), we only need to prove for $a \in H \subset D(H)$, $g \in H^* \subset D(H)$ and $v \in H$,

$$(5) \quad a \cdot (g \cdot v) = (ag) \cdot v = (a_{(1)} \rightarrow g \rightarrow S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v).$$
This is proved in the following computation:
\[
(a_{(1)} - g - S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v) = (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(3)}g_{(2)} \cdot (a_{(2)} \cdot v).
\]
\[
= (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(4)}g_{(2)} \cdot (a_{(2)}vSa_{(3)})
\]
\[
= (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(4)}(a_{(2)}vSa_{(3)}) - S^{-1}g_{(2)}
\]
\[
= (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(4)}(S^{-1}g_{(2)}a_{(2)}v_{(1)}S_{a(5)}a_{(3)}v_{(2)}Sa_{(4)}
\]
\[
= (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(4)}(g_{(2)}a_{(5)}S^{-1}v_{(1)}S^{-1}a_{(2)}a_{(3)}v_{(2)}Sa_{(4)}
\]
\[
= (g_{(3)}a_{(1)}) \langle g_{(1)}S^{-1}a_{(4)}(b_{(2)}a_{(1)}a_{(3)}v_{(2)}Sa_{(4)}
\]
\[
= (S^{-1}g_{(1)}a_{(1)}v_{(2)}Sa_{(2)}
\]
\[
= a \cdot (g \cdot v).
\]
Thus \(H\) is a \(D(H)\)-module under (3). It is clear that (2) is true for \(h \in H\) or \(H^{*}\); since \(H\) and \(H^{*}\) generate \(D(H)\), (2) is also true for \(h \in D(H)\). This proves that \(H\) is a module algebra of \(D(H)\) under the action (3).

We outline a more conceptual proof of Lemma 1 that explains formula (3). For this, we need the following formula for the coproduct of the dual Hopf algebra of \(D(H)\), \((D(H))^{*}\) (cf. [M]), identifying \((D(H))^{*}\) with \(H \otimes H^{*}\):

\[
\Delta(a \otimes g) = \sum (a_{(1)} \otimes e_{i}^{*}g_{(1)}e_{j}^{*}) \otimes (S^{-1}(e_{j})a_{(2)}e_{i} \otimes g_{(2)}),
\]

where \(\{e_{i}\}\) is a basis of \(H\) and \(\{e_{i}^{*}\}\) is its dual basis of \(H^{*}\). Now \((D(H))^{*}\) is a right module algebra of \(D(H)\) under the action \(\Delta\). It is clear from (6) that \(H \subset (D(H))^{*}\) is stable under \(\Delta\), so \(H^{op}\) is a right module algebra of \(D(H)\) (here \(H^{op}\) is \(H\) with the opposite multiplication). Using (6), it is easy to prove that this right action of \(D(H)\) on \(H^{op}\) is given by the formula

\[
b \cdot (g \otimes a) = S^{-1}(a_{(2)})(b - g)a_{(1)}.
\]

Now we use the following general fact: if \(A^{op}\) is a right \(H\)-module algebra with the action \(b \cdot h\) for \(b \in A, h \in H\), then \(A\) is an \(H\)-module algebra with action \(h \cdot b = b \cdot S(h)\). So \(H\) is a module algebra of \(D(H)\) with the action

\[
(a \otimes f) \cdot b = b \cdot (S(f \otimes a)) = (a_{(1)}bS(a_{(2)})) \rightarrow (S^{-1}f).
\]
We see that this action is precisely the one defined in (3).

Now we are in the position to state our main theorem.

**Theorem 1.** If \(H\) is a semisimple Hopf algebra over an algebraically closed field \(k\) of characteristic 0, then the action of \(D(H)\) given by (3) and the action \(\Delta\) of \(C(H)\) form a commutating pair, i.e., an operator \(T \in \text{End}_{k}(H)\) commutes with the action of \(D(H)\) if and only if \(T\) is in the image of \(C(H)\) in \(\text{End}_{k}(H)\); and \(T \in \text{End}_{k}(H)\) commutes with the action of \(C(H)\) if and only if \(T\) is the image of \(D(H)\) in \(\text{End}_{k}(H)\).

**Proof.** We note that the semisimplicity of \(H\) implies that \(S^{2} = 1\) and \(D(H)\) is semisimple ([LR], [R]). The semisimplicity of \(H\) also implies that \(C(H)\) is a semisimple algebra (cf. [Z]). In particular the images of \(D(H)\) and \(C(H)\) in \(\text{End}_{k}(H)\) are semisimple algebras. Therefore it suffices to prove that \(T \in \text{End}_{k}(H)\) commutes with the action of \(D(H)\) if and only if \(T\) is in the image of \(C(H)\).
Assume $T$ commutes with the action of $D(H)$; we need to prove $T$ is in the image of $C(H)$. We note that $H^* \subset D(H)$ acts on $H$ by restriction: this action is just the action $\sim$ of $H^*$ on $H$ twisted by $S^{-1}$. By Lemma 2 below, there exists a unique $v \in H^*$ such that

$$T(b) = v \to b = \langle v, b(2) \rangle b(1)$$

(7) for every $b \in H$.

For $T$ as in (7), $T$ commutes with the action of $H \subset D(H)$ implies that

$$\langle v, a(2) b(2) S a(3) a(1) b(1) S a(4) \rangle = \langle v, b(2) a(1) b(1) S a(2) \rangle$$

(8) for every $a, b \in H$. Apply the counit map to both sides of (8), we obtain

$$\langle v, a(1) b S(a(2)) \rangle = \langle v, b \epsilon(a) \rangle$$

this further implies that

$$\langle v, a b \rangle = \langle v, a(1) b a(3) S a(2) \rangle = \langle v, b a(2) \rangle \epsilon(a(1)) = \langle v, b a \rangle.$$ 

(9) This proves that $v$ is cocommutative or $v \in C(H)$. Note that in (9), we use the fact that $a(2) S a(1) = \epsilon(a)$ which is true for the Hopf algebras with the property $S^2 = 1$.

Conversely, if $v \in C(H)$, we need to prove that the action $\sim$ commutes with the action of $D(H)$. It is clear that $\sim$ commutes with the restriction action of $H^* \subset D(H)$. Because $v$ is cocommutative,

$$v \to (a \cdot b) = \langle v, a(2) b(2) S a(3) a(1) b(1) S a(4) \rangle$$

$$= \langle v, S a(3) a(2) b(2) a(1) b(1) S a(4) \rangle$$

$$= \langle v, b(2) a(1) b(1) S a(2) \rangle = a \cdot (v \to b).$$

This proves that $\sim$ commutes with the restriction action of $H \subset D(H)$. Because $D(H)$ is generated by $H^* \subset D(H)$ and $H \subset D(H)$, so $\sim$ commutes with the action of $D(H)$.

\textbf{Lemma 2.} If $T \in \text{End}_k(H)$ commutes with the action $\sim$ of $H^*$ on $H$, then there exists $v \in H^*$ such that $T(a) = v \to a$ for all $a \in H$.

\textbf{Proof.} This is a version of the following well-known fact: if $A$ is an associative algebra, $T \in \text{End}_k(A)$ commutes with the left multiplication $r_a$ for all $a \in A$, then $T$ is a right multiplication for some $b \in A$. To apply this fact, we notice that the transpose action of $\sim$ is the left multiplication of $H^*$ on $H^*$, while the transpose action of $\to$ is the right multiplication of $H^*$ on $H^*$. $T$ commutes with the action $\sim$ of $H^*$ on $H$, implies that $T^* \in \text{End}_k(H^*)$ commutes with the left multiplications on $H^*$. Therefore $T^*$ is given by a right multiplication, and therefore there exists $v \in H^*$ such that $T(a) = (T^*)(a) = v \to a$ for all $a \in H$.

Before giving a corollary of Theorem 1 concerning the dimension of the simple $D(H)$-submodules in $H$, we recall a theorem in [K] (cf. [Z] for an exposition suitable for the discussion here). We assume the conditions in Theorem 1. Since $C(H)$ is semisimple, it is a sum of full matrix algebras $M_1, \ldots, M_s$. We choose a minimal idempotent $e_i$ in $M_i$. Then $tr(e_i)$, the trace of the operator on $H^*$ given by $g \mapsto ge_i$, is a divisor of $\text{dim}(H)$.

\textbf{Corollary.} Let $H$ be a semisimple Hopf algebra over an algebraically closed field $k$ of characteristic 0, and let $H$ be the $D(H)$-module defined above. Then the dimension of every simple $D(H)$-submodule in $H$ is a divisor of $\text{dim}(H)$.
Proof. Let $V_1, \ldots, V_s$ be the simple $C(H)$-modules correspondent to $M_1, \ldots, M_s$ respectively. Note that the $C(H)$-action on $H$ is faithful, since this action is the restriction of the $H^*$-action “—”. All $V_i$’s appear as submodules in $H$. Because $D(H)$-action and $C(H)$-action form a commuting pair, and both $D(H)$ and $C(H)$ are semisimple, simple $D(H)$-submodules in $H$ and simple $C(H)$-submodules in $H$ are bijectively correspondent. Let $W_i$ ($i = 1, \ldots, s$) be the simple $D(H)$-module correspondent to $V_i$. As a $D(H) \otimes C(H)$-module, $H$ is isomorphic to $H = \bigoplus_{i=1}^s (W_i \otimes V_i)$. Because $e_i$ is a minimal idempotent of $M_i \subset C(H)$, its trace on $V_i$ is 1, and its trace on $H$ is $\dim(W_i)$ by the above decomposition of $H$. On the other hand, since the $C(H)$-action “—” on $H$ is the transpose action of the action of left multiplication on $H^*$, the trace of $e_i$ on $H$ is $tr(e_i)$ above. This proves $\dim(W_i) = tr(e_i)$. It follows that $\dim(W_i)$ is a divisor of $\dim(H)$. \hfill \Box

In the case that $H$ is the group algebra of a finite group $G$ over $\mathbb{C}$, each simple $D(H)$-submodule of $CG$ is spanned by the elements in a conjugacy class of $G$.

REFERENCES


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