

A COMMUTING PAIR IN HOPF ALGEBRAS

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ABSTRACT. We prove that if H is a semisimple Hopf algebra, then the action of the Drinfeld double $D(H)$ on H and the action of the character algebra on H form a commuting pair. This result and a result of G. I. Kats imply that the dimension of every simple $D(H)$ -submodule of H is a divisor of $\dim(H)$.

Let H be a finite dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic 0, $D(H)$ be the Drinfeld double of H , and $C(H)$ be the character algebra of H . $C(H)$ is spanned by the characters of H -modules and is an associative subalgebra of H^* . It is known that $D(H)$ acts on H and that $C(H)$ acts on H by the restriction of the action “ \dashv ” of H^* on H (these actions will be recalled below). The purpose of this note is to prove that these two actions form a commuting pair. Using this result, we prove that the dimension of every simple $D(H)$ -submodule of H is a divisor of $\dim(H)$. It would be interesting if there exists an analog of this commuting pair in the context of Poisson Lie groups.

We first recall the construction of the Drinfeld double (cf. [D], [M]) and fix necessary notations. Let H be a finite dimensional Hopf algebra over a field k (here we do not need any additional assumptions on H and k). The Drinfeld double of H , denoted by $D(H)$, as a vector space, is the tensor space $H^* \otimes H$. The comultiplication of $D(H)$ is given by

$$\Delta(f \otimes a) = \sum (f_{(2)} \otimes a_{(1)}) \otimes (f_{(1)} \otimes a_{(2)}) \in D(H) \otimes D(H),$$

where $\Delta f = f_{(1)} \otimes f_{(2)}$, $\Delta a = a_{(1)} \otimes a_{(2)}$ are comultiplications in H and H^* respectively. The multiplication in $D(H)$ is defined as follows: for $f \otimes a$ and $g \otimes b$ in $D(H)$,

$$(1) \quad (f \otimes a)(g \otimes b) = \sum f(a_{(1)} \triangleright g_{(2)}) \otimes (a_{(2)} \triangleleft g_{(1)})b,$$

where $a \triangleright g$ is the action of H on H^* given by

$$a \triangleright g = a_{(1)} \dashv g \leftarrow S^{-1}a_{(2)}$$

and $a \triangleleft g$ is the right action of H^* on H given by

$$a \triangleleft g = S^{-1}g_{(1)} \dashv a \leftarrow g_{(2)}.$$

The notations \dashv and \leftarrow mean the usual left and right actions of H on H^* , i.e., for $a \in H$ and $g \in H^*$,

$$a \dashv g = \sum g_{(1)} \langle g_{(2)}, a \rangle \in H^*, \quad g \leftarrow a = \sum g_{(2)} \langle g_{(1)}, a \rangle.$$

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We view H and H^* as subspaces of $D(H)$ by the embeddings $a \in H \mapsto 1 \otimes a$ and $f \in H^* \mapsto f \otimes 1$.

The antipode S and the counit of $D(H)$ are given by

$$S(f \otimes a) = S(a)S^{-1}(f), \quad \epsilon(f \otimes a) = \epsilon(f)\epsilon(a);$$

here $\epsilon(f)$ and $\epsilon(a)$ denote the counit maps for H^* and H .

The above operations give $D(H)$ a structure of Hopf algebra. Moreover $D(H)$ is quasitriangular with the R -matrix $R = \sum_i e_i^* \otimes e_i$, where $\{e_i\}$ is a basis for H , and $\{e_i^*\}$ is its dual basis for H^* . The quasitriangular structure will not play a role here. Notice that H and H^{*cop} (H^{*cop} is H^* with the opposite coproduct) are Hopf subalgebras of $D(H)$.

We also recall some basic notions about modules and module algebras of a Hopf algebra. A (left) module of H means a left module of H as an associative algebra. An associative algebra A is called a (left) module algebra of H if A is a H -module such that the algebra structure and H -module structure for A are compatible in the following sense: for $h \in H$, $u, v \in A$ and the unit 1_A in A ,

$$(2) \quad h \cdot (uv) = \sum (h_{(1)} \cdot u)(h_{(2)} \cdot v), \quad h \cdot 1_A = \epsilon(h)1_A.$$

Similarly a right module algebra of H is an associative algebra A together with a right H -module structure satisfying the conditions

$$(uv) \cdot h = (u \cdot h_{(1)})(v \cdot h_{(2)}), \quad 1_A \cdot h = \epsilon(h)1_A.$$

For a finite dimensional H -module V , the character χ_V of V is an element of H^* defined by $\langle \chi_V, a \rangle = \text{Tr}|_V(a)$ for every $a \in H$. Because $\chi_{W \otimes V} = \chi_W \chi_V$ for H -modules W, V , the characters of H span an associative subalgebra of H^* . This algebra is called the character algebra of H and denoted by $C(H)$. If H is semisimple and the ground field is algebraically closed and of characteristic 0, then $C(H)$ consists of the elements $v \in H^*$ that are cocommutative, i.e., $\sum v_{(1)} \otimes v_{(2)} = \sum v_{(2)} \otimes v_{(1)}$.

H is an H^* -module algebra under the action \rightarrow given by $g \rightarrow a = \langle g, a_{(2)} \rangle a_{(1)}$. We will be concerned with the restriction of \rightarrow on the character algebra $C(H)$.

There is $D(H)$ -action on H defined by

$$(3) \quad (f \otimes a) \cdot b = (a_{(1)} b S(a_{(2)})) \leftarrow S^{-1}(f).$$

Lemma 1. H is a module algebra of $D(H)$ under the action (3).

Proof. To prove H is a $D(H)$ -module under (3), We need to prove that

$$(4) \quad (xy) \cdot v = x \cdot (y \cdot v)$$

for every $x, y \in D(H)$ and $v \in H$. This is true for the cases $x, y \in H \subset D(H)$, $x, y \in H^* \subset D(H)$ and $x \in H^*, y \in H$. It is known that the definition of the multiplication of $D(H)$ is equivalent to the following (cf. [M])

$$(f \otimes a)(g \otimes b) = f(a_{(1)} \rightarrow g \leftarrow S^{-1}a_{(3)}) \otimes a_{(2)}b.$$

To prove (3), we only need to prove for $a \in H \subset D(H)$, $g \in H^* \subset D(H)$ and $v \in H$,

$$(5) \quad a \cdot (g \cdot v) = (ag) \cdot v = (a_{(1)} \rightarrow g \leftarrow S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v).$$

This is proved in the following computation:

$$\begin{aligned}
 &(a_{(1)} \rightharpoonup g \leftarrow S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v) \\
 &= \langle g_{(3)}, a_{(1)} \rangle \langle g_{(1)}, S^{-1}a_{(3)} \rangle g_{(2)} \cdot (a_{(2)} \cdot v) \\
 &= \langle g_{(3)}, a_{(1)} \rangle \langle g_{(1)}, S^{-1}a_{(4)} \rangle g_{(2)} \cdot (a_{(2)}vSa_{(3)}) \\
 &= \langle g_{(3)}, a_{(1)} \rangle \langle g_{(1)}, S^{-1}a_{(4)} \rangle (a_{(2)}vSa_{(3)}) \leftarrow S^{-1}g_{(2)} \\
 &= \langle g_{(3)}, a_{(1)} \rangle \langle g_{(1)}, S^{-1}a_{(6)} \rangle \langle S^{-1}g_{(2)}, a_{(2)}v_{(1)}Sa_{(5)} \rangle a_{(3)}v_{(2)}Sa_{(4)} \\
 &= \langle g_{(3)}, a_{(1)} \rangle \langle g_{(1)}, S^{-1}a_{(6)} \rangle \langle g_{(2)}, a_{(5)}S^{-1}v_{(1)}S^{-1}a_{(2)} \rangle a_{(3)}v_{(2)}Sa_{(4)} \\
 &= \langle g, S^{-1}a_{(6)}a_{(5)}S^{-1}v_{(1)}S^{-1}a_{(2)}a_{(1)} \rangle a_{(3)}v_{(2)}Sa_{(4)} \\
 &= \langle S^{-1}g, v_{(1)} \rangle a_{(1)}v_{(2)}Sa_{(2)} \\
 &= a \cdot (g \cdot v).
 \end{aligned}$$

Thus H is a $D(H)$ -module under (3). It is clear that (2) is true for h in H or H^* ; since H and H^* generate $D(H)$, (2) is also true for $h \in D(H)$. This proves that H is a module algebra of $D(H)$ under the action (3). \square

We outline a more conceptual proof of Lemma 1 that explains formula (3). For this, we need the following formula for the coproduct of the dual Hopf algebra of $D(H)$, $D(H)^*$ (cf. [M]), identifying $D(H)^*$ with $H \otimes H^*$:

$$(6) \quad \Delta(a \otimes g) = \sum (a_{(1)} \otimes e_i^*g_{(1)}e_j^*) \otimes (S^{-1}(e_j)a_{(2)}e_i \otimes g_{(2)}),$$

where $\{e_i\}$ is a basis of H and $\{e_i^*\}$ is its dual basis of H^* . Now $D(H)^*$ is a right module algebra of $D(H)$ under the action \leftarrow . It is clear from (6) that $H \subset D(H)^*$ is stable under \leftarrow , so H^{op} is a right module algebra of $D(H)$ (here H^{op} is H with the opposite multiplication). Using (6), it is easy to prove that this right action of $D(H)$ on H^{op} is given by the formula

$$b \cdot (g \otimes a) = S^{-1}(a_{(2)})(b \leftarrow g)a_{(1)}.$$

Now we use the following general fact: if A^{op} is a right H -module algebra with the action $b \cdot h$ for $b \in A$, $h \in H$, then A is an H -module algebra with action $h \cdot b = b \cdot S(h)$. So H is a module algebra of $D(H)$ with the action

$$(a \otimes f) \cdot b = b \cdot (S(f \otimes a)) = (a_{(1)}bS(a_{(2)})) \leftarrow (S^{-1}f).$$

We see that this action is precisely the one defined in (3).

Now we are in the position to state our main theorem.

Theorem 1. *If H is a semisimple Hopf algebra over an algebraically closed field k of characteristic 0, then the action of $D(H)$ given by (3) and the action \rightharpoonup of $C(H)$ form a commuting pair, i.e., an operator $T \in \text{End}_k(H)$ commutes with the action of $D(H)$ if and only if T is in the image of $C(H)$ in $\text{End}_k(H)$; and $T \in \text{End}_k(H)$ commutes with the action of $C(H)$ if and only if T is the image of $D(H)$ in $\text{End}_k(H)$.*

Proof. We note that the semisimplicity of H implies that $S^2 = 1$ and $D(H)$ is semisimple ([LR], [R]). The semisimplicity of H also implies that $C(H)$ is a semisimple algebra (cf. [Z]). In particular the images of $D(H)$ and $C(H)$ in $\text{End}_k(H)$ are semisimple algebras. Therefore it suffices to prove that $T \in \text{End}_k(H)$ commutes with the action of $D(H)$ if and only if T is in the image of $C(H)$.

Assume T commutes with the action of $D(H)$; we need to prove T is in the image of $C(H)$. We note that $H^* \subset D(H)$ acts on H by restriction: this action is just the action “ \leftarrow ” of H^* on H twisted by S^{-1} . By Lemma 2 below, there exists a unique $v \in H^*$ such that

$$(7) \quad T(b) = v \rightarrow b = \langle v, b_{(2)} \rangle b_{(1)}$$

for every $b \in H$.

For T as in (7), T commutes with the action of $H \subset D(H)$ implies that

$$(8) \quad \langle v, a_{(2)}b_{(2)}Sa_{(3)} \rangle a_{(1)}b_{(1)}Sa_{(4)} = \langle v, b_{(2)} \rangle a_{(1)}b_{(1)}Sa_{(2)}$$

for every $a, b \in H$. Apply the counit map to both sides of (8), we obtain

$$\langle v, a_{(1)}bS(a_{(2)}) \rangle = \langle v, b \rangle \epsilon(a);$$

this further implies that

$$(9) \quad \langle v, ab \rangle = \langle v, a_{(1)}ba_{(3)}Sa_{(2)} \rangle = \langle v, ba_{(2)} \rangle \epsilon(a_{(1)}) = \langle v, ba \rangle.$$

This proves that v is cocommutative or $v \in C(H)$. Note that in (9), we use the fact that $a_{(2)}Sa_{(1)} = \epsilon(a)$ which is true for the Hopf algebras with the property $S^2 = 1$.

Conversely, if $v \in C(H)$, we need to prove that the action “ $v \rightarrow$ ” commutes with the action of $D(H)$. It is clear that “ $v \rightarrow$ ” commutes with the restriction action of $H^* \subset D(H)$. Because v is cocommutative,

$$(10) \quad \begin{aligned} v \rightarrow (a \cdot b) &= \langle v, a_{(2)}b_{(2)}Sa_{(3)} \rangle a_{(1)}b_{(1)}Sa_{(4)} \\ &= \langle v, Sa_{(3)}a_{(2)}b_{(2)} \rangle a_{(1)}b_{(1)}Sa_{(4)} \\ &= \langle v, b_{(2)} \rangle a_{(1)}b_{(1)}S(a_{(2)}) = a \cdot (v \rightarrow b). \end{aligned}$$

This proves that “ $v \rightarrow$ ” commutes with the restriction action of $H \subset D(H)$. Because $D(H)$ is generated by $H^* \subset D(H)$ and $H \subset D(H)$, so “ $v \rightarrow$ ” commutes with the action of $D(H)$. □

Lemma 2. *If $T \in \text{End}_k(H)$ commutes with the action \leftarrow of H^* on H , then there exists $v \in H^*$ such that $T(a) = v \rightarrow a$ for all $a \in H$.*

Proof. This is a version of the following well-known fact: if A is an associative algebra, $T \in \text{End}_k(A)$ commutes with the left multiplication r_a for all $a \in A$, then T is a right multiplication for some $b \in A$. To apply this fact, we notice that the transpose action of \rightarrow is the left multiplication of H^* on H^* , while the transpose action of \leftarrow is the right multiplication of H^* on H^* . T commutes with the action \leftarrow of H^* on H , implies that $T^* \in \text{End}_k(H^*)$ commutes with the left multiplications on H^* . Therefore T^* is given by a right multiplication, and therefore there exists $v \in H^*$ such that $T(a) = (T^*)^*(a) = v \rightarrow a$ for all $a \in H$. □

Before giving a corollary of Theorem 1 concerning the dimension of the simple $D(H)$ -submodules in H , we recall a theorem in [K] (cf. [Z] for an exposition suitable for the discussion here). We assume the conditions in Theorem 1. Since $C(H)$ is semisimple, it is a sum of full matrix algebras M_1, \dots, M_s . We choose a minimal idempotent e_i in M_i . Then $\text{tr}(e_i)$, the trace of the operator on H^* given by $g \mapsto ge_i$, is a divisor of $\dim(H)$.

Corollary. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0, and let H be the $D(H)$ -module defined above. Then the dimension of every simple $D(H)$ -submodule in H is a divisor of $\dim(H)$.*

Proof. Let V_1, \dots, V_s be the simple $C(H)$ -modules correspondent to M_1, \dots, M_s respectively. Note that the $C(H)$ -action on H is faithful, since this action is the restriction of the H^* -action “ \dashv ”. All V_i 's appear as submodules in H . Because $D(H)$ -action and $C(H)$ -action form a commuting pair, and both $D(H)$ and $C(H)$ are semisimple, simple $D(H)$ -submodules in H and simple $C(H)$ -submodules in H are bijectively correspondent. Let W_i ($i = 1, \dots, s$) be the simple $D(H)$ -module correspondent to V_i . As a $D(H) \otimes C(H)$ -module, H is isomorphic to $H = \bigoplus_{i=1}^s (W_i \otimes V_i)$. Because e_i is a minimal idempotent of $M_i \subset C(H)$, its trace on V_i is 1, and its trace on H is $\dim(W_i)$ by the above decomposition of H . On the other hand, since the $C(H)$ -action “ \dashv ” on H is the transpose action of the action of left multiplication on H^* , the trace of e_i on H is $\text{tr}(e_i)$ above. This proves $\dim(W_i) = \text{tr}(e_i)$. It follows that $\dim(W_i)$ is a divisor of $\dim(H)$. \square

In the case that H is the group algebra of a finite group G over \mathbb{C} , each simple $D(H)$ -submodule of $\mathbb{C}G$ is spanned by the elements in a conjugacy class of G .

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