A COMMUTING PAIR IN HOPF ALGEBRAS

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Abstract. We prove that if $H$ is a semisimple Hopf algebra, then the action of the Drinfeld double $D(H)$ on $H$ and the action of the character algebra on $H$ form a commuting pair. This result and a result of G. I. Kats imply that the dimension of every simple $D(H)$-submodule of $H$ is a divisor of $\dim(H)$.

Let $H$ be a finite dimensional semisimple Hopf algebra over an algebraically closed field $k$ of characteristic 0, $D(H)$ be the Drinfeld double of $H$, and $C(H)$ be the character algebra of $H$. $C(H)$ is spanned by the characters of $H$-modules and is an associative subalgebra of $H^*$. It is known that $D(H)$ acts on $H$ and that $C(H)$ acts on $H$ by the restriction of the action “$\rightarrow$” of $H^*$ on $H$ (these actions will be recalled below). The purpose of this note is to prove that these two actions form a commuting pair. Using this result, we prove that the dimension of every simple $D(H)$-submodule of $H$ is a divisor of $\dim(H)$. It would be interesting if there exists an analog of this commuting pair in the context of Poisson Lie groups.

We first recall the construction of the Drinfeld double (cf. [D], [M]) and fix necessary notations. Let $H$ be a finite dimensional Hopf algebra over a field $k$ (here we do not need any additional assumptions on $H$ and $k$). The Drinfeld double of $H$, denoted by $D(H)$, as a vector space, is the tensor space $H^* \otimes H$. The comultiplication of $D(A)$ is given by

$$\Delta(f \otimes a) = \sum (f_{(2)} \otimes a_{(1)}) \otimes (f_{(1)} \otimes a_{(2)}) \in D(H) \otimes D(H),$$

where $\Delta f = f_{(1)} \otimes f_{(2)}$, $\Delta a = a_{(1)} \otimes a_{(2)}$ are comultiplications in $H$ and $H^*$ respectively. The multiplication in $D(H)$ is defined as follows: for $f \otimes a$ and $g \otimes b$ in $D(H)$,

$$(f \otimes a)(g \otimes b) = \sum f(a_{(1)} \triangleright g_{(2)}) \otimes (a_{(2)} \triangleleft g_{(1)})b,$$

where $a \triangleright g$ is the action of $H$ on $H^*$ given by

$$a \triangleright g = a_{(1)} \longrightarrow g \leftarrow S^{-1}a_{(2)}$$

and $a \triangleleft g$ is the right action of $H^*$ on $H$ given by

$$a \triangleleft g = S^{-1}g_{(1)} \longrightarrow a \leftarrow g_{(2)}.$$
We view $H$ and $H^*$ as subspaces of $D(H)$ by the embeddings $a \in H \mapsto 1 \otimes a$ and $f \in H^* \mapsto f \otimes 1$.

The antipode $S$ and the counit of $D(H)$ are given by

$$S(f \otimes a) = S(a)S^{-1}(f), \quad \epsilon(f \otimes a) = \epsilon(f)\epsilon(a);$$

here $\epsilon(f)$ and $\epsilon(a)$ denote the counit maps for $H^*$ and $H$.

The above operations give $D(H)$ a structure of Hopf algebra. Moreover $D(H)$ is quasitriangular with the $R$-matrix $R = \sum_i e_i^* \otimes e_i$, where $\{e_i\}$ is a basis for $H$, and $\{e_i^*\}$ is its dual basis for $H^*$. The quasitriangular structure will not play a role here. Notice that $H$ and $H^{*\text{cop}}$ ($H^{*\text{cop}}$ is $H^*$ with the opposite coproduct) are Hopf subalgebras of $D(H)$.

We also recall some basic notions about modules and module algebras of a Hopf algebra. A (left) module of $H$ means a left module of $H$ as an associative algebra. An associative algebra $A$ is called a (left) module algebra of $H$ if $A$ is a $H$-module such that the algebra structure and $H$-module structure for $A$ are compatible in the following sense: for $h \in H$, $u, v \in A$ and the unit $1_A$ in $A$,

$$h \cdot (uv) = \sum (h_{(1)} \cdot u)(h_{(2)} \cdot v), \quad h \cdot 1_A = \epsilon(h)1_A.$$

Similarly a right module algebra of $H$ is an associative algebra $A$ together with a right $H$-module structure satisfying the conditions

$$(uv) \cdot h = (u \cdot h_{(1)})(v \cdot h_{(2)}), \quad 1_A \cdot h = \epsilon(h)1_A.$$

For a finite dimensional $H$-module $V$, the character $\chi_V$ of $V$ is an element of $H^*$ defined by $(\chi_V, a) = Tr|_V(a)$ for every $a \in H$. Because $\chi_{W \otimes V} = \chi_W \chi_V$ for $H$-modules $W, V$, the characters of $H$ span an associative subalgebra of $H^*$. This algebra is called the character algebra of $H$ and denoted by $C(H)$. If $H$ is semisimple and the ground field is algebraically closed and of characteristic 0, then $C(H)$ consists of the elements $v \in H^*$ that are cocommutative, i.e., $\sum v_{(1)} \otimes v_{(2)} = \sum v_{(2)} \otimes v_{(1)}$.

$H$ is an $H^*$-module algebra under the action $\rightarrow$ given by $g \rightarrow a = (g, a_{(2)})a_{(1)}$. We will be concerned with the restriction of $\rightarrow$ on the character algebra $C(H)$.

There is $D(H)$-action on $H$ defined by

$$(f \otimes a) \cdot b = (a_{(1)}bS(a_{(2)})) \rightarrow S^{-1}(f).$$

**Lemma 1.** $H$ is a module algebra of $D(H)$ under the action (3).

**Proof.** To prove $H$ is a $D(H)$-module under (3), We need to prove that

$$(xy) \cdot v = x \cdot (y \cdot v)$$

for every $x, y \in D(H)$ and $v \in H$. This is true for the cases $x, y \in H \subset D(H)$, $x, y \in H^* \subset D(H)$ and $x \in H^*, y \in H$. It is known that the definition of the multiplication of $D(H)$ is equivalent to the following (cf. [M])

$$(f \otimes a)(g \otimes b) = f(a_{(1)} \rightarrow g \leftarrow S^{-1}a_{(3)}) \otimes a_{(2)}b.$$

To prove (3), we only need to prove for $a \in H \subset D(H)$, $g \in H^* \subset D(H)$ and $v \in H$,

$$a \cdot (g \cdot v) = (ag) \cdot v = (a_{(1)} \rightarrow g \leftarrow S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v).$$
This is proved in the following computation:
\[
(a_{(1)} - g - S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v) \\
= (g(3), a_{(1)}) (g(1), S^{-1}a_{(3)}g(2)) (a_{(2)} \cdot v) \\
= (g(3), a_{(1)}) (g(1), S^{-1}a_{(4)}g(2)) (a_{(2)}vSa(3)) \\
= (g(3), a_{(1)}) (g(1), S^{-1}a_{(4)})(a_{(2)}vSa(3)) - S^{-1}g(2) \\
= (g(3), a_{(1)}) (g(1), S^{-1}a_{(4)})(S^{-1}g(2), a_{(2)}v(1)Sa(5)a(3)v(2)Sa(4)) \\
= (g(3), a_{(1)}) (g(1), S^{-1}a_{(4)})(g(2), a_{(5)}S^{-1}v(1)a(3)v(2)a_{(2)}a(3)v(2)Sa(4)) \\
= (g, S^{-1}a_{(5)}S^{-1}v(1)S^{-1}a_{(2)}a_{(1)}a(3)v(2)Sa(4)) \\
= (S^{-1}g, v(1))a_{(1)}v(2)Sa(2) \\
= a \cdot (g \cdot v).
\]
Thus \( H \) is a \( D(H) \)-module under (3). It is clear that (2) is true for \( h \) in \( H \) or \( H^* \); since \( H \) and \( H^* \) generate \( D(H) \), (2) is also true for \( h \in D(H) \). This proves that \( H \) is a module algebra of \( D(H) \) under the action (3).

We outline a more conceptual proof of Lemma 1 that explains formula (3). For this, we need the following formula for the coproduct of the dual Hopf algebra of \( D(H) \), \( D(H)^* \) (cf. [M]), identifying \( D(H)^* \) with \( H \otimes H^* \):
\[
\Delta(a \otimes g) = \sum (a_{(1)} \otimes e_i^*g(1)e_{j}) \otimes (S^{-1}(e_j)a_{(2)}e_i \otimes g(2)),
\]
where \( \{e_i\} \) is a basis of \( H \) and \( \{e_i^*\} \) is its dual basis of \( H^* \). Now \( D(H)^* \) is a right module algebra of \( D(H) \) under the action \( \cdot \). It is clear from (6) that \( H \subset D(H)^* \) is stable under \( \cdot \), so \( H^{op} \) is a right module algebra of \( D(H) \) (here \( H^{op} \) is \( H \) with the opposite multiplication). Using (6), it is easy to prove that this right action of \( D(H) \) on \( H^{op} \) is given by the formula
\[
b \cdot (g \otimes a) = S^{-1}(a_{(2)})(b - g)a_{(1)}.
\]
Now we use the following general fact: if \( A^{op} \) is a right \( H \)-module algebra with the action \( b \cdot h \) for \( b \in A, h \in H \), then \( A \) is an \( H \)-module algebra with action \( h \cdot b = b \cdot S(h) \). So \( H \) is a module algebra of \( D(H) \) with the action
\[
(a \otimes f) \cdot b = b \cdot (S(f \otimes a)) = (a_{(1)}bS(a_{(2)})) \cdot (S^{-1}f).
\]
We see that this action is precisely the one defined in (3).

Now we are in the position to state our main theorem.

**Theorem 1.** If \( H \) is a semisimple Hopf algebra over an algebraically closed field \( k \) of characteristic 0, then the action of \( D(H) \) given by (3) and the action \( \cdot \) of \( C(H) \) form a commutating pair, i.e., an operator \( T \in \text{End}_k(H) \) commutes with the action of \( D(H) \) if and only if \( T \) is in the image of \( C(H) \) in \( \text{End}_k(H) \); and \( T \in \text{End}_k(H) \) commutes with the action of \( C(H) \) if and only if \( T \) is the image of \( D(H) \) in \( \text{End}_k(H) \).

**Proof.** We note that the semisimplicity of \( H \) implies that \( S^2 = 1 \) and \( D(H) \) is semisimple ([LR], [R]). The semisimplicity of \( H \) also implies that \( C(H) \) is a semisimple algebra (cf. [Z]). In particular the images of \( D(H) \) and \( C(H) \) in \( \text{End}_k(H) \) are semisimple algebras. Therefore it suffices to prove that \( T \in \text{End}_k(H) \) commutes with the action of \( D(H) \) if and only if \( T \) is in the image of \( C(H) \).
Assume $T$ commutes with the action of $D(H)$; we need to prove $T$ is in the
image of $C(H)$. We note that $H^* \subset D(H)$ acts on $H$ by restriction: this action is
just the action “$\rightarrow$” of $H^*$ on $H$ twisted by $S^{-1}$. By Lemma 2 below, there exists
a unique $v \in H^*$ such that

(7) \[ T(b) = v \rightarrow b = \langle v, b_{(2)} \rangle b_{(1)} \]

for every $b \in H$. 

For $T$ as in (7), $T$ commutes with the action of $H \subset D(H)$ implies that

(8) \[ \langle v, a_{(2)} b_{(2)} S(a_{(3)}) a_{(1)} b_{(1)} S(a_{(4)}) \rangle = \langle v, b_{(2)} a_{(1)} b_{(1)} S(a_{(2)}) \rangle \]

for every $a, b \in H$. Apply the counit map to both sides of (8), we obtain

\[ \langle v, a_{(1)} b S(a_{(2)}) \rangle = \langle v, b \cdot (a) \rangle; \]

this further implies that

(9) \[ \langle v, a \cdot b \rangle = \langle v, a_{(2)} b_{(2)} S(a_{(3)}) a_{(1)} b_{(1)} S(a_{(4)}) \rangle = \langle v, b_{(2)} a_{(1)} b_{(1)} S(a_{(2)}) \rangle = \langle v, b \cdot a \rangle. \]

This proves that $v$ is cocommutative or $v \in C(H)$. Note that in (9), we use the fact that $a_{(2)} S a_{(1)} = \epsilon(a)$ which is true for the Hopf algebras with the property $S^2 = 1$.

Conversely, if $v \in C(H)$, we need to prove that the action “$\rightarrow$” commutes with the action of $D(H)$. It is clear that “$\rightarrow$” commutes with the restriction action of $H^* \subset D(H)$. Because $v$ is cocommutative,

(10) \[ v \rightarrow (a \cdot b) = \langle v, a_{(2)} b_{(2)} S(a_{(3)}) a_{(1)} b_{(1)} S(a_{(4)}) \rangle = \langle v, S(a_{(3)}) a_{(2)} b_{(2)} a_{(1)} b_{(1)} S(a_{(4)}) \rangle = \langle v, b_{(2)} a_{(1)} b_{(1)} S(a_{(2)}) \rangle = a \cdot (v \rightarrow b). \]

This proves that “$\rightarrow$” commutes with the restriction action of $H \subset D(H)$. Because $D(H)$ is generated by $H^* \subset D(H)$ and $H \subset D(H)$, so “$\rightarrow$” commutes with the action of $D(H)$.

\[ \square \]

Lemma 2. If $T \in \text{End}_k(H)$ commutes with the action $\rightarrow$ of $H^*$ on $H$, then there exists $v \in H^*$ such that $T(a) = v \rightarrow a$ for all $a \in H$.

Proof. This is a version of the following well-known fact: if $A$ is an associative
algebra, $T \in \text{End}_k(A)$ commutes with the left multiplication $r_a$ for all $a \in A$, then
$T$ is a right multiplication for some $b \in A$. To apply this fact, we notice that the transpose action of $\rightarrow$ is the left multiplication of $H^*$ on $H^*$, while the transpose action of $\rightarrow$ is the right multiplication of $H^*$ on $H^*$. $T$ commutes with the action $\rightarrow$ of $H^*$ on $H$, implies that $T^* \in \text{End}_k(H^*)$ commutes with the left multiplications on $H^*$. Therefore $T^*$ is given by a right multiplication, and therefore there exists $v \in H^*$ such that $T(a) = (T^*)^*(a) = v \rightarrow a$ for all $a \in H$.

\[ \square \]

Before giving a corollary of Theorem 1 concerning the dimension of the simple
$D(H)$-submodules in $H$, we recall a theorem in [K] (cf. [Z] for an exposition suitable
for the discussion here). We assume the conditions in Theorem 1. Since $C(H)$ is
semisimple, it is a sum of full matrix algebras $M_1, \ldots, M_s$. We choose a minimal
idempotent $e_i$ in $M_i$. Then $tr(e_i)$, the trace of the operator on $H^*$ given by $g \mapsto ge_i$, is a divisor of $\text{dim}(H)$.

Corollary. Let $H$ be a semisimple Hopf algebra over an algebraically closed field
$k$ of characteristic 0, and let $H$ be the $D(H)$-module defined above. Then the
dimension of every simple $D(H)$-submodule in $H$ is a divisor of $\text{dim}(H)$.
Proof. Let $V_1, \ldots, V_s$ be the simple $C(H)$-modules correspondent to $M_1, \ldots, M_s$ respectively. Note that the $C(H)$-action on $H$ is faithful, since this action is the restriction of the $H^*$-action “$ightharpoonup$”. All $V_i$’s appear as submodules in $H$. Because $D(H)$-action and $C(H)$-action form a commuting pair, and both $D(H)$ and $C(H)$ are semisimple, simple $D(H)$-submodules in $H$ and simple $C(H)$-submodules in $H$ are bijectively correspondent. Let $W_i$ ($i = 1, \ldots, s$) be the simple $D(H)$-module correspondent to $V_i$. As a $D(H) \otimes C(H)$-module, $H$ is isomorphic to $H = \bigoplus_{i=1}^s (W_i \otimes V_i)$. Because $e_i$ is a minimal idempotent of $M_i \subset C(H)$, its trace on $V_i$ is 1, and its trace on $H$ is $\dim(W_i)$ by the above decomposition of $H$. On the other hand, since the $C(H)$-action “$ightharpoonup$” on $H$ is the transpose action of the action of left multiplication on $H^*$, the trace of $e_i$ on $H$ is $\text{tr}(e_i)$ above. This proves $\dim(W_i) = \text{tr}(e_i)$. It follows that $\dim(W_i)$ is a divisor of $\dim(H)$.

In the case that $H$ is the group algebra of a finite group $G$ over $\mathbb{C}$, each simple $D(H)$-submodule of $CG$ is spanned by the elements in a conjugacy class of $G$.

REFERENCES


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