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A COMMUTING PAIR IN HOPF ALGEBRAS

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Abstract. We prove that if H is a semisimple Hopf algebra, then the action of the Drinfeld double $D(H)$ on H and the action of the character algebra on H form a commuting pair. This result and a result of G. I. Kats imply that the dimension of every simple $D(H)$-submodule of H is a divisor of dim (H).

Let H be a finite dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic 0, $D(H)$ be the Drinfeld double of H, and $C(H)$ be the character algebra of H. $C(H)$ is spanned by the characters of H-modules and is an associative subalgebra of H^*. It is known that $D(H)$ acts on H and that $C(H)$ acts on H by the restriction of the action "$\rightharpoonup$" of H^* on H (these actions will be recalled below). The purpose of this note is to prove that these two actions form a commuting pair. Using this result, we prove that the dimension of every simple $D(H)$-submodule of H is a divisor of $\dim(H)$. It would be interesting if there exists an analog of this commuting pair in the context of Poisson Lie groups.

We first recall the construction of the Drinfeld double (cf. [D], [M]) and fix necessary notations. Let H be a finite dimensional Hopf algebra over a field k (here we do not need any additional assumptions on H and k). The Drinfeld double of H, denoted by $D(H)$, as a vector space, is the tensor space $H^* \otimes H$. The comultiplication of $D(A)$ is given by

$$\Delta(f \otimes a) = \sum \left(f_{(2)} \otimes a_{(1)}\right) \otimes \left(f_{(1)} \otimes a_{(2)}\right) \in D(H) \otimes D(H),$$

where $\Delta f = f_{(1)} \otimes f_{(2)}$, $\Delta a = a_{(1)} \otimes a_{(2)}$ are comultiplictions in H and H^* respectively. The multiplication in $D(H)$ is defined as follows: for $f \otimes a$ and $g \otimes b$ in $D(H)$,

$$(f \otimes a)(g \otimes b) = \sum f(a_{(1)} \triangleright g_{(2)}) \otimes (a_{(2)} \triangleleft g_{(1)})b, \tag{1}$$

where $a \triangleright g$ is the action of H on H^* given by

$$a \triangleright g = a_{(1)} \rightharpoonup g \leftharpoonup S^{-1}a_{(2)}$$

and $a \triangleleft g$ is the right action of H^* on H given by

$$a \triangleleft g = S^{-1}g_{(1)} \rightharpoonup a \leftharpoonup g_{(2)}.$$

The notations $\rightharpoonup$ and $\leftharpoonup$ mean the usual left and right actions of H on H^*, i.e., for $a \in H$ and $g \in H^*$,

$$a \rightharpoonup g = \sum g_{(1)} \langle g_{(2)}, a\rangle \in H^*, \qquad g \leftharpoonup a = \sum g_{(2)} \langle g_{(1)}, a\rangle.$$

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We view H and H^* as subspaces of $D(H)$ by the embeddings $a \in H \mapsto 1 \otimes a$ and $f \in H^* \mapsto f \otimes 1$.

The antipode S and the counit of $D(H)$ are given by

$$S(f \otimes a) = S(a)S^{-1}(f), \quad \epsilon(f \otimes a) = \epsilon(f)\epsilon(a);$$

here $\epsilon(f)$ and $\epsilon(a)$ denote the counit maps for H^* and H.

The above operations give $D(H)$ a structure of Hopf algebra. Moreover $D(H)$ is quasitriangular with the R-matrix $R = \sum_i e_i^* \otimes e_i$, where $\{e_i\}$ is a basis for H, and $\{e_i^*\}$ is its dual basis for H^*. The quasitriangular structure will not play a role here. Notice that H and H^{*cop} (H^{*cop} is H^* with the opposite coproduct) are Hopf subalgebras of $D(H)$.

We also recall some basic notions about modules and module algebras of a Hopf algebra. A (left) module of H means a left module of H as an associative algebra. An associative algebra A is called a (left) module algebra of H if A is a H-module such that the algebra structure and H-module structure for A are compatible in the following sense: for $h \in H$, $u, v \in A$ and the unit 1_A in A,

$$h \cdot (uv) = \sum (h_{(1)} \cdot u)(h_{(2)} \cdot v), \quad h \cdot 1_A = \epsilon(h)1_A. \tag{2}$$

Similarly a right module algebra of H is an associative algebra A together with a right H-module structure satisfying the conditions

$$(uv) \cdot h = (u \cdot h_{(1)})(v \cdot h_{(2)}), \quad 1_A \cdot h = \epsilon(h)1_A.$$

For a finite dimensional H-module V, the character χ_V of V is an element of H^* defined by $\langle \chi_V, a\rangle = Tr|_V(a)$ for every $a \in H$. Because $\chi_{W\otimes V} = \chi_W\chi_V$ for H-modules W, V, the characters of H span an associative subalgebra of H^*. This algebra is called the character algebra of H and denoted by $C(H)$. If H is semisimple and the ground field is algebraically closed and of characteristic 0, then $C(H)$ consists of the elements $v \in H^*$ that are cocommutative, i.e., $\sum v_{(1)} \otimes v_{(2)} = \sum v_{(2)} \otimes v_{(1)}$.

H is an H^*-module algebra under the action $\rightharpoonup$ given by $g \rightharpoonup a = \langle g, a_{(2)}\rangle a_{(1)}$. We will be concerned with the restriction of $\rightharpoonup$ on the character algebra $C(H)$.

There is $D(H)$-action on H defined by

$$(f \otimes a) \cdot b = (a_{(1)} b S(a_{(2)})) \leftharpoonup S^{-1}(f). \tag{3}$$

Lemma 1. *H is a module algebra of $D(H)$ under the action* (3)*.*

Proof. To prove H is a $D(H)$-module under (3), We need to prove that

$$(xy) \cdot v = x \cdot (y \cdot v) \tag{4}$$

for every $x, y \in D(H)$ and $v \in H$. This is true for the cases $x, y \in H \subset D(H)$, $x, y \in H^* \subset D(H)$ and $x \in H^*, y \in H$. It is known that the definition of the multiplication of $D(H)$ is equivalent to the following (cf. [M])

$$(f \otimes a)(g \otimes b) = f(a_{(1)} \rightharpoonup g \leftharpoonup S^{-1}a_{(3)}) \otimes a_{(2)}b.$$

To prove (3), we only need to prove for $a \in H \subset D(H)$, $g \in H^* \subset D(H)$ and $v \in H$,

$$a \cdot (g \cdot v) = (ag) \cdot v = (a_{(1)} \rightharpoonup g \leftharpoonup S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v). \tag{5}$$

This is proved in the following computation:

$$
\begin{aligned}
&(a_{(1)} \rightharpoonup g \leftharpoonup S^{-1}a_{(3)}) \cdot (a_{(2)} \cdot v) \\
&= \langle g_{(3)}, a_{(1)}\rangle\langle g_{(1)}, S^{-1}a_{(3)}\rangle g_{(2)} \cdot (a_{(2)} \cdot v) \\
&= \langle g_{(3)}, a_{(1)}\rangle\langle g_{(1)}, S^{-1}a_{(4)}\rangle g_{(2)} \cdot (a_{(2)} v S a_{(3)}) \\
&= \langle g_{(3)}, a_{(1)}\rangle\langle g_{(1)}, S^{-1}a_{(4)}\rangle (a_{(2)} v S a_{(3)}) \leftharpoonup S^{-1}g_{(2)} \\
&= \langle g_{(3)}, a_{(1)}\rangle\langle g_{(1)}, S^{-1}a_{(6)}\rangle\langle S^{-1}g_{(2)}, a_{(2)} v_{(1)} S a_{(5)}\rangle a_{(3)} v_{(2)} S a_{(4)} \\
&= \langle g_{(3)}, a_{(1)}\rangle\langle g_{(1)}, S^{-1}a_{(6)}\rangle\langle g_{(2)}, a_{(5)} S^{-1}v_{(1)} S^{-1}a_{(2)}\rangle a_{(3)} v_{(2)} S a_{(4)} \\
&= \langle g, S^{-1}a_{(6)} a_{(5)} S^{-1}v_{(1)} S^{-1}a_{(2)} a_{(1)}\rangle a_{(3)} v_{(2)} S a_{(4)} \\
&= \langle S^{-1}g, v_{(1)}\rangle a_{(1)} v_{(2)} S a_{(2)} \\
&= a \cdot (g \cdot v).
\end{aligned}
$$

Thus H is a $D(H)$-module under (3). It is clear that (2) is true for h in H or H^*; since H and H^* generate $D(H)$, (2) is also true for $h \in D(H)$. This proves that H is a module algebra of $D(H)$ under the action (3). □

We outline a more conceptual proof of Lemma 1 that explains formula (3). For this, we need the following formula for the coproduct of the dual Hopf algebra of $D(H)$, $D(H)^*$ (cf. [M]), identifying $D(H)^*$ with $H \otimes H^*$:

$$
\Delta(a \otimes g) = \sum \left(a_{(1)} \otimes e_i^* g_{(1)} e_j^*\right) \otimes \left(S^{-1}(e_j) a_{(2)} e_i \otimes g_{(2)}\right), \tag{6}
$$

where $\{e_i\}$ is a basis of H and $\{e_i^*\}$ is its dual basis of H^*. Now $D(H)^*$ is a right module algebra of $D(H)$ under the action $\leftharpoonup$. It is clear from (6) that $H \subset D(H)^*$ is stable under $\leftharpoonup$, so H^{op} is a right module algebra of $D(H)$ (here H^{op} is H with the opposite multiplication). Using (6), it is easy to prove that this right action of $D(H)$ on H^{op} is given by the formula

$$
b \cdot (g \otimes a) = S^{-1}(a_{(2)})(b \leftharpoonup g) a_{(1)}.
$$

Now we use the following general fact: if A^{op} is a right H-module algebra with the action $b \cdot h$ for $b \in A$, $h \in H$, then A is an H-module algebra with action $h \cdot b = b \cdot S(h)$. So H is a module algebra of $D(H)$ with the action

$$
(a \otimes f) \cdot b = b \cdot (S(f \otimes a)) = (a_{(1)} b S(a_{(2)})) \leftharpoonup (S^{-1} f).
$$

We see that this action is precisely the one defined in (3).

Now we are in the position to state our main theorem.

Theorem 1. *If H is a semisimple Hopf algebra over an algebraically closed field k of characteristic 0, then the action of $D(H)$ given by (3) and the action $\rightharpoonup$ of $C(H)$ form a commutating pair, i.e., an operator $T \in End_k(H)$ commutes with the action of $D(H)$ if and only if T is in the image of $C(H)$ in $End_k(H)$; and $T \in End_k(H)$ commutes with the action of $C(H)$ if and only if T is the image of $D(H)$ in $End_k(H)$.*

Proof. We note that the semisimplicity of H implies that $S^2 = 1$ and $D(H)$ is semisimple ([LR], [R]). The semisimplicity of H also implies that $C(H)$ is a semisimple algebra (cf. [Z]). In particular the images of $D(H)$ and $C(H)$ in $\mathrm{End}_k(H)$ are semisimple algebras. Therefore it suffices to prove that $T \in \mathrm{End}_k(H)$ commutes with the action of $D(H)$ if and only if T is in the image of $C(H)$.

Assume T commutes with the action of $D(H)$; we need to prove T is in the image of $C(H)$. We note that $H^* \subset D(H)$ acts on H by restriction: this action is just the action "$\leftharpoonup$" of H^* on H twisted by S^{-1}. By Lemma 2 below, there exists a unique $v \in H^*$ such that

$$T(b) = v \rightharpoonup b = \langle v, b_{(2)} \rangle b_{(1)} \tag{7}$$

for every $b \in H$.

For T as in (7), T commutes with the action of $H \subset D(H)$ implies that

$$\langle v, a_{(2)} b_{(2)} S a_{(3)} \rangle a_{(1)} b_{(1)} S a_{(4)} = \langle v, b_{(2)} \rangle a_{(1)} b_{(1)} S a_{(2)} \tag{8}$$

for every $a, b \in H$. Apply the counit map to both sides of (8), we obtain

$$\langle v, a_{(1)} b S(a_{(2)}) \rangle = \langle v, b \rangle \epsilon(a);$$

this further implies that

$$\langle v, ab \rangle = \langle v, a_{(1)} b a_{(3)} S a_{(2)} \rangle = \langle v, b a_{(2)} \rangle \epsilon(a_{(1)}) = \langle v, ba \rangle. \tag{9}$$

This proves that v is cocommutative or $v \in C(H)$. Note that in (9), we use the fact that $a_{(2)} S a_{(1)} = \epsilon(a)$ which is true for the Hopf algebras with the property $S^2 = 1$.

Conversely, if $v \in C(H)$, we need to prove that the action "$v \rightharpoonup$" commutes with the action of $D(H)$. It is clear that "$v \rightharpoonup$" commutes with the restriction action of $H^* \subset D(H)$. Because v is cocommutative,

$$\begin{aligned} v \rightharpoonup (a \cdot b) &= \langle v, a_{(2)} b_{(2)} S a_{(3)} \rangle a_{(1)} b_{(1)} S a_{(4)} \\ &= \langle v, S a_{(3)} a_{(2)} b_{(2)} \rangle a_{(1)} b_{(1)} S a_{(4)} \\ &= \langle v, b_{(2)} \rangle a_{(1)} b_{(1)} S(a_{(2)}) = a \cdot (v \rightharpoonup b). \end{aligned} \tag{10}$$

This proves that "$v \rightharpoonup$" commutes with the restriction action of $H \subset D(H)$. Because $D(H)$ is generated by $H^* \subset D(H)$ and $H \subset D(H)$, so "$v \rightharpoonup$" commutes with the action of $D(H)$. □

Lemma 2. *If $T \in End_k(H)$ commutes with the action $\leftharpoonup$ of H^* on H, then there exists $v \in H^*$ such that $T(a) = v \rightharpoonup a$ for all $a \in H$.*

Proof. This is a version of the following well-known fact: if A is an associative algebra, $T \in \mathrm{End}_k(A)$ commutes with the left multiplication r_a for all $a \in A$, then T is a right multiplication for some $b \in A$. To apply this fact, we notice that the transpose action of $\rightharpoonup$ is the left multiplication of H^* on H^*, while the transpose action of $\leftharpoonup$ is the right multiplication of H^* on H^*. T commutes with the action $\leftharpoonup$ of H^* on H, implies that $T^* \in \mathrm{End}_k(H^*)$ commutes with the left multiplications on H^*. Therefore T^* is given by a right multiplication, and therefore there exists $v \in H^*$ such that $T(a) = (T^*)^*(a) = v \rightharpoonup a$ for all $a \in H$. □

Before giving a corollary of Theorem 1 concerning the dimension of the simple $D(H)$-submodules in H, we recall a theorem in [K] (cf. [Z] for an exposition suitable for the discussion here). We assume the conditions in Theorem 1. Since $C(H)$ is semisimple, it is a sum of full matrix algebras $M_1, \ldots, M_s$. We choose a minimal idempotent e_i in M_i. Then $tr(e_i)$, the trace of the operator on H^* given by $g \mapsto ge_i$, is a divisor of $dim(H)$.

Corollary. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0, and let H be the $D(H)$-module defined above. Then the dimension of every simple $D(H)$-submodule in H is a divisor of $dim(H)$.*

Proof. Let $V_1, \ldots, V_s$ be the simple $C(H)$-modules correspondent to $M_1, \ldots, M_s$ respectively. Note that the $C(H)$-action on H is faithful, since this action is the restriction of the H^*-action "$\rightharpoonup$". All V_i's appear as submodules in H. Because $D(H)$-action and $C(H)$-action form a commuting pair, and both $D(H)$ and $C(H)$ are semisimple, simple $D(H)$-submodules in H and simple $C(H)$-submodules in H are bijectively correspondent. Let W_i $(i = 1, \ldots, s)$ be the simple $D(H)$-module correspondent to V_i. As a $D(H) \otimes C(H)$-module, H is isomorphic to $H = \oplus_{i=1}^{s} (W_i \otimes V_i)$. Because e_i is a minimal idempotent of $M_i \subset C(H)$, its trace on V_i is 1, and its trace on H is $dim(W_i)$ by the above decomposition of H. On the other hand, since the $C(H)$-action "$\rightharpoonup$" on H is the transpose action of the action of left multiplication on H^*, the trace of e_i on H is $tr(e_i)$ above. This proves $dim(W_i) = tr(e_i)$. It follows that $dim(W_i)$ is a divisor of $dim(H)$. □

In the case that H is the group algebra of a finite group G over $\mathbb{C}$, each simple $D(H)$-submodule of $\mathbb{C}G$ is spanned by the elements in a conjugacy class of G.

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