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THE AR-PROPERTY FOR ROBERTS' EXAMPLE OF A COMPACT CONVEX SET WITH NO EXTREME POINTS PART 2: APPLICATION TO THE EXAMPLE

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ABSTRACT. In this second part of our paper, we apply the result of Part 1 to show that the compact convex set with no extreme points, constructed by Roberts (1977), is an AR.

This is the second part of our paper. In this part, we prove that the compact convex set with no extreme points, constructed by Roberts [R1], is an AR.

All notation, used in this part, comes from Part 1.

5. Roberts' example

We are going to describe the linear metric space which contains a compact convex set with no extreme points, constructed by Roberts [R1].

Let $\{d_n\}$ be a sequence of natural numbers. Put $m(1) = 1$ and inductively define $m(n+1) = d_n m(n)$. Let $\pi_1 = \{[0,1)\}$. Assume that π_n is a partition of $[0,1)$ into $m(n)$ equal length intervals of the form $[a,b)$. For each $S \in \pi_n$, let $\pi_{n+1}(S)$ denote the partition of S into d_n equal length subintervals. Then we define the partition $\pi_{n+1} = \cup\{\pi_{n+1}(S) : S \in \pi_n\}$ of $[0,1)$ into $m(n+1)$ equal length intervals. Consider the linear space consisting of all functions on $[0,1]$ which are finite linear combinations of characteristic functions of the form $\chi_{[a,b)}$. We denote

$$E_n = \operatorname{span}\{\chi_S : S \in \pi_n\};\ E = \bigcup_{n=1}^{\infty} E_n \tag{30}$$

(χ_S denotes the characteristic function of S);

$$E_{n+1}(S) = \operatorname{span}\{\chi_T : T \in \pi_{n+1}(S)\};$$

$$A_n = \{m(n)\chi_S : S \in \pi_n\} = \{a_i^n : i = 1, \dots, m(n)\} \text{ (compare with (10))}, \tag{31}$$

where $a_i^n = m(n)\chi_{S_i^n},\ i = 1, \dots, m(n); \pi_n = \{S_1^n, \dots, S_{m(n)}^n\}$. Observe that

$$E_1 \subset E_2 \subset \cdots \subset E, \quad \text{see (30)}.$$

Note that paranorms constructed in [R1] are monotone, norm bounded and total, hence they are monotone F-norms. Then "paranorms" can be replaced by

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"monotone F-norms". Observe that a paranorm is an F-norm if it is norm bounded and total. The following theorem was proved by Roberts [R1, §3].

Theorem 4. *For a suitable sequence of natural numbers $\{d_n\}$, there exist sequences of paranorms $\{N_n\}$ and $\{N_{n+1}(S) : S \in \pi_n\}$ on E_n and on $E_{n+1}(S)$ respectively, such that the following conditions hold:*

(i) $N_1(x) = \int_0^1 |x(t)|dt$ *for every* $x \in E_1$*;*
(ii) $N_{n+1}(x) = \inf\{N_n(y) + \sum\{N_{n+1}(S)(x(S)) : S \in \pi_n\}\}$ *for every* $x \in E_{n+1}$*, where the infimum is taken over all the expressions of x of the form:*

$$x = y + \sum\{x(S) : S \in \pi_n\},\ y \in E_n,\ x(S) \in E_{n+1}(S);$$

(iii) $\{N_n(x)\}$ *is a decreasing sequence for every* $x \in E$ *and the formula*

$$N(x) = \lim_{n\to\infty} N_n(x) \text{ for } x \in E$$

defines a monotone F-norm of E;
(iv) *If $x \in E_n$ and $N_n(x) < 4$, then $N_m(x) = N_n(x)$ for every $m > n$;*
(v) *The sequence $\{A_n\}$, defined by* (31)*, satisfies the conditions* (4) *and* (5) *of Theorem 1, Part 1, where $A_{n+1}(a) = \{m(n+1)\chi_{S'} : S' \in \pi_{n+1}(S)\}$ if $a = m(n)\chi_S \in A_n,\ S \in \pi_n$.*

Therefore $C = \overline{\cup_{n=o}^{\infty}\hat{A}_n} \subset X$, see (6), is a compact convex set with no extreme points, where X is the completion of (E, N).

Let $\|.\|$ denote the F-norm on X induced by N. Our aim is to show that the compact convex set C, defined by Theorem 4, satisfies the conditions of Theorem 2, Part 1. We need the following fact:

Lemma 6. *Using the notation* (31)*, for each $i = 1, \dots, m(n)$, if $x \in \operatorname{span} C_n(a_i^n) \cap E$, see* (8)*, then $\operatorname{supp} x \subset S_i^n$.*

Proof. Let $x \in \operatorname{span} C_n(a_i^n) \cap E$. Take $q > n$ such that $x \in E_q$. Then we have

$$x = \sum_{j=1}^{m(q)} \alpha_j a_j^q,\ \text{where } a_j^q = m(q)\chi_{S_j^q},\ j = 1, \dots, m(q). \tag{32}$$

Assume on the contrary that $\alpha_j \neq 0$ for some $S_j^q \subset [0,1]\backslash S_i^n$. We may assume that $j = 1$. Take a sequence $\{x_k\} \subset \operatorname{span} \cup_{j=1}^{\infty} \hat{A}_{n+j}(a_i^n) \cap E$ such that $x_k \to x$, see (8). Then, we have

$$\text{supp } x_k \subset S_i^n \text{ for every } k \in \mathbb{N}, \text{ see (30) and (31).} \tag{33}$$

For any function x on $[0,1]$ and $A \subset [0,1]$, denote $x|_A = \chi_A x$. Then from (33) we get

$$x - x_k|[0,1]\backslash S_i^n = x|[0,1]\backslash S_i^n \in E_q \text{ for every } k \in \mathbb{N}.$$

Since $\|x - x_k\| \to 0$, by Theorem 4(ii)–(iv), there exists an expression

$$x - x_k = z_q(k) + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k),\ k \in \mathbb{N}, \tag{34}$$

where $n(k) > q$ is chosen so that $x_k \in E_{n(k)}$, and $z_q(k) \in E_q$, $y_i^j(k) \in E_{j+1}(S_i^j)$ for $i = 1, \dots, m(j)$, $j = q, \dots, n(k)-1$, such that

$$N_q(z_q(k)) + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} N_{j+1}(S_i^j)(y_i^j(k)) \to 0 \text{ as } k \to \infty. \tag{35}$$

Since $z_q(k) \in E_q$ we have, see (30) and (31),

$$z_q(k) = \sum_{t=1}^{m(q)} y_t(k), \text{ where } y_t(k) = \lambda_t(k) m(q) \chi_{S_t^q}, \ t = 1, \dots, m(q). \tag{36}$$

From (35) we get $N_q(z_q(k)) \to 0$ as $k \to \infty$. Since $\{z_q(k)\}$ is a sequence in the finite dimensional space E_q, equipped with the Hamel base $\{\chi_{S_t^q} : t = 1, \dots, m(q)\}$ and the F-norm N_q, it follows from (36) that

$$N_q(y_t(k)) \to 0 \text{ as } k \to \infty, \text{ for every } t = 1, \dots, m(q). \tag{37}$$

Observe that for every $j \in \{q, \dots, n(k)-1\}$ and $i \in \{1, \dots, m(j)\}$ we have $S_i^j \subset S_t^q$ for some $t \in \{1, \dots, m(q)\}$. Denote

$$u_1(k) = \sum_{S_i^j \subset S_1^q} y_i^j(k). \tag{38}$$

Then we have

Claim 6. $y_1(k) + u_1(k) = \alpha_1 m(q) \chi_{S_1^q} \in E_q$ for every $k \in \mathbb{N}$.

Proof. Recall that $x|_A = \chi_A x$. Since $S_1^q \subset [0,1] \setminus S_i^n$, from (33) and (34) we get

$$x|S_1^q = x - x_k|S_1^q = z_q(k)|S_1^q + \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k)|S_1^q. \tag{39}$$

From (32) and (36) we have

$$x|S_1^q = \alpha_1 m(q) \chi_{S_1^q} \text{ and } z_q(k)|S_1^q = y_1(k).$$

Since $y_i^j(k) \in E_{j+1}(S_i^j)$, we get

$$\sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} y_i^j(k)|S_1^q = \sum_{S_i^j \subset S_1^q} y_i^j(k) = u_1(k), \text{ see (38)}.$$

Consequently, the claim follows from (39).

Since $y_1(k) \in E_q$, from Claim 6 we get $u_1(k) \in E_q$ for every $k \in \mathbb{N}$. Therefore, from Theorem 4(ii)–(iv) and from (35) we get

$$\begin{aligned} N_q(u_1(k)) &\le \sum_{S_i^j \subset S_1^q} N_{j+1}(S_i^j)(y_i^j(k)) \le \sum_{t=1}^{m(q)} \sum_{S_i^j \subset S_t^q} N_{j+1}(S_i^j)(y_i^j(k)) \\ &= \sum_{j=q}^{n(k)-1} \sum_{i=1}^{m(j)} N_{j+1}(S_i^j)(y_i^j(k)) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence, from Claim 6 and from (37) we obtain

$$N_q(\alpha_1 m(q) \chi_{S_1^q}) \le N_q(y_1(k)) + N_q(u_1(k)) \to 0 \text{ as } k \to \infty.$$

Consequently, $N_q(\alpha_1 m(q)\chi_{S_1^q}) = 0$ and therefore $\alpha_1 = 0$. So, we get a contradiction. The lemma is proved.

6. The result

Now we come to our result in this paper:

Theorem 5. *The compact convex set C, defined by Theorem 4, is an AR.*

Proof. We are going to verify the conditions of Theorem 2. First, observe that Condition (ii) follows from the construction. Let us check Condition (i): if $x_i \in \operatorname{span} C_n(a_i^n)\backslash\{\theta\}$, $i = 1, \dots, m(n)$, then the set $\{x_1, \dots, x_{m(n)}\}$ is linearly independent in X.

Let $x_i \in \operatorname{span} C_n(a_i^n)\backslash\{\theta\}$ and $\lambda_i \in \mathbb{R},\ i = 1, \dots, m(n)$, such that

$$\lambda_1 x_1 + \cdots + \lambda_{m(n)} x_{m(n)} = \theta \text{ where } \lambda_i \in \mathbb{R},\ i = 1, \dots, m(n). \tag{40}$$

We shall prove that $\lambda_i = 0$ for every $i = 1, \dots, m(n)$. Assume on the contrary that $\lambda_i \neq 0$ for some $i \in \{1, \dots, m(n)\}$. We may assume that $i = 1$. Then we get

$$x_1 = \sum_{i=2}^{m(n)} \alpha_i x_i, \text{ where } \alpha_i = -\lambda_i/\lambda_1,\ i = 2, \dots, m(n).$$

Let $0 < \varepsilon < 4$. We take $y_i \in \operatorname{span} C_n(a_i^n) \cap E$ such that

$$\|y_i - x_i\| < (2m(n)\alpha)^{-1}\varepsilon < \varepsilon/2 \text{ for every } i = 1, \dots, m(n), \tag{41}$$

where $\alpha = \max\{[|\alpha_i|] + 1 : i = 2, \dots, m(n)\}$. (Recall that $[\alpha]$ denotes the greatest integer which is smaller than α.) From Lemma 6 we get

$$\operatorname{supp} y_i \subset S_i^n \text{ for every } i = 1, \dots, m(n). \tag{42}$$

Take $k \in \mathbb{N}$ such that $y_i \in E_{n+k}$ for every $i = 1, \dots, m(n)$. Let $z_i^1 = \alpha_i y_i$ for $i = 2, \dots, m(n)$. Then we can denote

$$z_1 = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i = y_1 - \sum_{i=2}^{m(n)} z_i^1 \in E_{n+k},$$

whence

$$y_1 = z_1 + \sum_{i=2}^{m(n)} z_i^1. \tag{43}$$

Note that

$$\operatorname{supp} y_1 \subset S_1^n \text{ and } \operatorname{supp} z_i^1 \subset S_i^n \text{ for } i = 2, \dots, m(n). \tag{44}$$

Then from (1) and (41) we have

$$\begin{aligned}\|z_1\| &= \|y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i\| \le \|y_1 - x_1\| + \|x_1 - \sum_{i=2}^{m(n)} \alpha_i y_i\| \\ &< 2^{-1}\varepsilon + \|\sum_{i=2}^{m(n)} \alpha_i (x_i - y_i)\| \le 2^{-1}\varepsilon + \sum_{i=2}^{m(n)} \|\alpha_i (x_i - y_i)\| \\ &\le 2^{-1}\varepsilon + \sum_{i=2}^{m(n)} \max\{[\alpha_i] + 1 : i = 2, \dots, m(n)\}\|x_i - y_i\| \\ &< 2^{-1}\varepsilon + m(n)\alpha(2m(n)\alpha)^{-1}\varepsilon = \varepsilon.\end{aligned}$$

Now, recall that $y_1 \in E_{n+k}$. Our strategy is to "cut off" the "top" of y_1 step by step until we get an element in the space E_n.

The first step. In this step we cut a small element u_1 off from y_1 so that $y_1 - u_1 \in E_{n+k-1}$.

Since $\|z_1\| < \varepsilon < 4$, it follows from Theorem 4(iii) that there exists an $r \in \mathbb{N}$ such that $N_r(z_1) < 4$. Replacing r or $n + k$ by $\max\{r, n + k\}$ if necessary, we may assume that $r = n + k$. Hence from Theorem 4(iv) we get $N_{n+k}(z_1) = \|z_1\| < \varepsilon$. Therefore by Theorem 4(ii) there exist

$$y_i^{n+k-1} \in E_{n+k}(S_i^{n+k-1}),\ i = 1, \dots, m(n + k - 1),\ \text{and } z_2 \in E_{n+k-1} \text{ such that}$$

$$z_1 = z_2 + \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1};$$

and

$$N_{n+k-1}(z_2) + \sum_{i=1}^{m(n+k-1)} N_{n+k}(S_i^{n+k-1})(y_i^{n+k-1}) < \varepsilon. \tag{45}$$

Denote

$$\varepsilon_1 = \sum_{i=1}^{m(n+k-1)} N_{n+k}(S_i^{n+k-1})(y_i^{n+k-1}). \tag{46}$$

Then from (45) we have

$$N_{n+k-1}(z_2) < \varepsilon - \varepsilon_1. \tag{47}$$

Let

$$v_i = \sum_{j \in T(i)} y_j^{n+k-1},\ \text{where } T(i) = \{j : S_j^{n+k-1} \subset S_i^n\} \text{ for } i = 1, \dots, m(n). \tag{48}$$

Let $u_1 = v_1$. Then we have $\mathrm{supp}(y_1 - u_1) \subset S_1^n$. We prove:

Claim 7. $y_1 - u_1 \in E_{n+k-1}$.

Proof. Observe that

$$\begin{aligned} z_2 &= z_1 - \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1} = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n+k-1)} y_i^{n+k-1} \\ &= y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n)} \sum_{j \in T(i)} y_j^{n+k-1} = y_1 - \sum_{i=2}^{m(n)} \alpha_i y_i - \sum_{i=1}^{m(n)} v_i \\ &= y_1 - v_1 - \sum_{i=2}^{m(n)} (\alpha_i y_i + v_i) = - \sum_{i=1}^{m(n)} (\alpha_i y_i + v_i), \text{ where } \alpha_1 = -1. \end{aligned}$$

Since $z_2 \in E_{n+k-1}$, we can write $-z_2 = \sum_{j=1}^{m(n+k-1)} w_j$, where

$$w_j = \lambda_j m(n+k-1) \chi_{S_j^{n+k-1}}, \ \lambda_j \in \mathbb{R}, \ j = 1, \dots, m(n+k-1).$$

Denote $\varphi_i = \sum_{S_j^{n+k-1} \subset S_i^n} w_j$. Since

$$\varphi_i \in E_{n+k-1} \text{ and } \operatorname{supp} \varphi_i \subset S_i^n \text{ for every } i = 1, \dots, m(n),$$

we get

$$-z_2 = \sum_{i=1}^{m(n)} \varphi_i = \sum_{i=1}^{m(n)} (\alpha_i y_i + v_i).$$

Therefore

$$\sum_{i=1}^{m(n)} (\alpha_i y_i + v_i - \varphi_i) = \theta. \tag{49}$$

From (42) and (48) we have

$$\operatorname{supp}(\alpha_i y_i + v_i) \subset S_i^n \text{ for every } i = 1, \dots, m(n).$$

Hence

$$\operatorname{supp}(\alpha_i y_i + v_i - \varphi_i) \subset S_i^n \text{ for every } i = 1, \dots, m(n).$$

Therefore from (49) we obtain

$$\alpha_i y_i + v_i - \varphi_i = \theta \text{ for } i = 1, \dots, m(n).$$

It follows that

$$\alpha_i y_i + v_i = \varphi_i \in E_{n+k-1} \text{ for } i = 1, \dots, m(n).$$

Since $\alpha_1 = 1$ we get

$$\alpha_1 y_1 + v_1 = -y_1 + u_1 = -(y_1 - u_1) \in E_{n+k-1}.$$

Consequently $y_1 - u_1 \in E_{n+k-1}$, and the claim is proved.

Remark 7. Denote $u_i^1 = v_i$ and $z_i^2 = z_i^1 + v_i = z_i^1 + u_i^1$ for $i = 1, \dots, m(n)$. Then from (44) and (48) we get

$$y_1 - u_1 = z_2 + \sum_{i=2}^{m(n)} z_i^2 \in E_{n+k-1};$$

moreover

$$\operatorname{supp}(y_1 - u_1) \subset S_1^n \text{ and } \operatorname{supp} z_i^2 \subset S_i^n \text{ for } i = 2, \dots, m(n). \tag{50}$$

Therefore (43) and (44) hold true if y_1, z_i^1 are replaced by $y_1 - u_1$ and z_i^2, respectively.

The induction step. We continue to cut a small element u_2 off from $y_1 - u_1$ so that $y_1 - u_1 - u_2 \in E_{n+k-2}$ and so on. In the k-th step we get

$$y_1 - u_1 - u_2 - \cdots - u_k \in E_n.$$

This process is done by induction as follows: Since

$$z_2 \in E_{n+k-1} \text{ and } N_{n+k-1}(z_2) < \varepsilon - \varepsilon_1 < 4,$$

we can obtain $z_3 \in E_{n+k-2}$ and $y_i^{n+k-2} \in E_{n+k-1}(S_i^{n+k-2}),\ i = 1, \dots, m(n+k-1)$, such that

$$z_2 = z_3 + \sum_{i=1}^{m(n+k-2)} y_i^{n+k-2},$$

and

$$N_{n+k-2}(z_3) + \sum_{i=1}^{m(n+k-2)} N_{n+k-2}(S_i^{n+k-2})(y_i^{n+k-2}) < \varepsilon - \varepsilon_1. \tag{51}$$

Denote

$$T(2,i) = \{t : S_t^{n+k-2} \subset S_i^n\},\ i = 1, \dots, m(n);$$
$$u_i^2 = \sum_{t \in T(2,i)} y_t^{n+k-2};\ i = 1, \dots, m(n).$$

Notice $u_2 = u_1^2$. Let $z_i^3 = z_i^2 + u_i^2$ for $i = 2, \dots, m(n)$. Then it is easy to see that $y_1 - u_1, z_2, z_i^2$ and E_{n+k-1} in Remark 7 can be replaced by $y_1 - u_1 - u_2, z_3, z_i^3$ and E_{n+k-2} respectively.

Now, for every $j = 1, \dots, k$ and $i = 1, \dots, m(n+k-j)$ we choose $y_i^{n+k-j} \in E_{n+k-j+1}(S_i^{n+k-j})$ and $z_{j+1} \in E_{n+k-j}$ such that

$$z_j = z_{j+1} + \sum_{i=1}^{m(n+k-j)} y_i^{n+k-j};$$

and

$$\begin{aligned} &N_{n+k-j}(z_{j+1}) + \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) \\ &\quad < \varepsilon - \sum_{t=1}^{j-1} \varepsilon_t, \end{aligned} \tag{52}$$

where

$$\varepsilon_t = \sum_{i=1}^{m(n+k-t)} N_{n+k-t+1}(S_i^{n+k-t})(y_i^{n+k-t}).$$

(Observe that $\varepsilon - \varepsilon_1 > 0$, by (47), and $\varepsilon - \sum_{t=1}^{j-1} \varepsilon_t > 0$ by induction.) Then from (52) we have $N_{n+k-j}(z_{j+1}) < \varepsilon - \sum_{t=1}^{j} \varepsilon_t$. Observe that

$$\sum_{j=1}^{k} \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) = \sum_{j=1}^{k} \varepsilon_j < \varepsilon. \tag{53}$$

Denote

$$\begin{aligned} T(j,i) &= \{t : S_t^{n+k-j} \subset S_i^n\},\ i = 1, \ldots, m(n);\\ u_i^j &= \sum_{t \in T(j,i)} y_t^{n+k-j}. \end{aligned}$$

Notice $u_j = u_1^j$. Let $z_i^j = z_i^{j-1} + u_i^{j-1}$ for $i = 2, \ldots, m(n)$. Then from Theorem 4(ii)–(iv) we get

$$\begin{aligned} \|u_j\| &\leq \sum_{i \in T(j,1)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j})\\ &\leq \sum_{i=1}^{m(n+k-j)} N_{n+k-j+1}(S_i^{n+k-j})(y_i^{n+k-j}) = \varepsilon_j. \end{aligned} \tag{54}$$

Let $z = \sum_{j=1}^{k} u_j$. Since $\{N_n(x)\}$ is a decreasing sequence for every $x \in E$, from (53) and (54) we have

$$\|z\| \leq \sum_{j=1}^{k} \|u_j\| \leq \sum_{j=1}^{k} \varepsilon_j < \varepsilon.$$

Then it is easy to see that $y_1 - u_1,\ z_2, z_i^2$ and E_{n+k-1} in Remark 7 can be replaced by $y_1 - \sum_{j=1}^{k} u_j = y_1 - z,\ z_{k+1},\quad z_i^{k+1}$ and $E_{n+k-k} = E_n$ respectively. Therefore we get $y_1 - z = y_1 - \sum_{j=1}^{k} u_j \in E_n$. Observe that

$$\|x_1 - (y_1 - z)\| \leq \|x_1 - y_1\| + \|z\| < 2\varepsilon.$$

Therefore $\|x_1 - E_n\| < 2\varepsilon$. Since ε is arbitrarily small we infer that $x_1 \in E_n$. So, $x_1 \in E_n \cap \operatorname{span} C_n(a_1^n)$. Hence from Lemma 6 we get

$$x_1 = t_1 a_1^n = t_1 m(n) \chi_{S_1^n} \text{ for some } t_1 \in \mathbb{R}.$$

Thus we have shown that if $\lambda_i \neq 0$, then $x_i = t_i m(n) \chi_{S_i^n}$ for some $t_i \in \mathbb{R}$. Since $\{m(n)\chi_{S_i^n} : i = 1, \ldots, m(n)\}$ is linearly independent, it follows from (40) that $t_i = 0$, if $\lambda_i \neq 0$. This contradicts the fact that $x_i \in \operatorname{span} C_n(a_i^n) \backslash \{\theta\}$ for every $i = 1, \ldots, m(n)$. Theorem 5 is proved.

From Theorem 5 and from [CDM] we obtain the following result which answers affirmatively a problem of Dobrowolski and Mogilski; see [DM], Question 5-5 (Question 575).

Corollary. *For any dense σ-fd-compact convex subset W of C, we have $(C, W) \simeq (Q, Q^f)$, where Q denotes the Hilbert cube, and*

$$Q^f = \{x = (x_n) \in Q :\ x_n = 0 \text{ for almost all } n \in \mathbb{N}\}$$

and "σ-fd-compact" means a countable union of finite dimensional compact sets.

Related to Theorem 5 we ask:

Question 1 [NT1]. Let F denote the linear metric space constructed by Roberts [R1]:

(i) Is every convex subset of F an AR?
(ii) Does every compact convex subset of F have the fixed point property?

We do not know even if the *whole space* F is an AR. It is of interest to know whether Theorem 2 still holds in the general case:

Question 2. Is the compact convex set C defined by Theorem 1 an AR?

It should be noted that Condition (i) is essential in our proof of Theorem 2. However, as we observed in Remark 6, Condition (ii) can be dropped. By [NT2] C has the fixed point property. So we may ask Question 2 in the following more general situation:

Question 3 [NT1]. Assume that X is a compact convex set with the fixed point property. Is X an AR?

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References

[BD] C. Bessaga and T. Dobrowolski, Some open problems on the border of infinite dimensional topology and functional analysis, *Proceedings of the international conference on geometric topology*, PWN, Warszawa 1980.

[CDM] D. Curtis, T. Dobrowolski and J. Mogilski, Some applications of the topological characterizations of the sigma-compact spaces ℓ^2_f nad Σ, *Trans. Amer. Math. Soc.* **284**(1984), 837–847. MR **86i:**54035

[DM] T. Dobrowolski and J. Mogilski, Problems on topological classification of incomplete metric spaces, Open problems in topology, J. van Mill and G. M. Reed (Editors) Elsevier Science Publishers B. V. North-Holand 1990. MR **92c:**54001

[G] R. Geoghegan, Open problems in infinite dimensional topology, *Topology Proceedings*, **4**(1979), 287–330. MR **82a:**57015

[K1] V. Klee, Shrinkable neighbourhoods in Hausdorff linear spaces, *Math. Ann.* **141**(1960), 281–285. MR **24:**A1003

[K2] V. Klee, Leray-Schauder theory without local convexity, *Math. Ann.* **141**(1960), 286–296. MR **24:**A1004

[KM] M. G. Krein and D. P. Milman, On extreme points of regular convex sets, *Studia Math.* **9**(1940), 133–138. MR **3:**90a

[KP] N. J. Kalton and N. T. Peck, A re-examination of Roberts' example of a compact convex set with no extreme points, *Math. Ann.* **253**(1980), 89–101. MR **82h:**46055

[KPR] N. J. Kalton, N. T. Peck and J. W. Roberts, An F-space sampler, *London Math. Soc. Lecture Note Series* **89**(1984). MR **87c:**46002

[N1] Nguyen To Nhu, Investigating the ANR-property of metric spaces, *Fund. Math.* **124**(1984), 243–254; Correction, *Fund. Math.* **141**(1992), 297. MR **86d:**54018, MR **93k:**54042

[N2] Nguyen To Nhu, The finite dimensional approximation property and the AR-property in needle point spaces, *J. London Math. Soc.* (to appear).

[NS] Nguyen To Nhu and Katsuro Sakai, The compact neighborhood extension property and the local equi-connectedness, *Proc. Amer. Math. Soc.* **121**(1994), 259–265. MR **94g:**54009

[NT1] Nguyen To Nhu and Le Hoang Tri, Every needle point space contains a compact convex AR-set with no extreme points, *Proc. Amer. Math. Soc.* **120**(1994), 1261–1265. MR **94f:**54038

[NT2] Nguyen To Nhu and Le Hoang Tri, No Roberts space is a counter-example to Schauder's conjecture, *Topology*, **33**(1994), 371–378. MR **95h:**46014

[R1] J. W. Roberts, A compact convex set with no extreme points, *Studia Math.* **60**(1977), 255–266. MR **57:**10595

[R2] J. W. Roberts, Pathological compact convex sets in the spaces L_p, $0 \leq p < 1$, The Altgeld Book, University of Illinois, 1976.

[Re] S. Rolewicz, Metric linear spaces, PWN, Warszawa 1972; Second publication, PWN, Warszawa 1982. MR **55:**10993; MR **88i:**46004a; MR **88i:**46004b

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