

EXACT TOPOLOGICAL ANALOGS TO ORTHOPOSETS

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ABSTRACT. An arbitrary orthoposet E is shown to be isomorphic to $(\mathcal{E}, \mathcal{C}, \mathcal{C}^c)$, \mathcal{E} being a subbasis of a Hausdorff topological space \mathcal{S} satisfying 1) $S \in \mathcal{E}$, 2) $\alpha \in \mathcal{E} \Rightarrow \alpha^c \in \mathcal{E}$, and 3) every covering of \mathcal{S} by elements of \mathcal{E} possesses an at most 2-element subcovering. The couple $(\mathcal{S}, \mathcal{E})$ turns out to be unique.

1. INTRODUCTION

A well-known theorem by M. Stone (cf. [2, 8]) establishes a bijective correspondence between the Boolean algebras and the zero-dimensional compact topological spaces. Thus these spaces serve to be exact topological analogs to the Boolean algebras. Let us widen the class of the Boolean algebras to the class of all orthoposets (cf. [2, 4, 5]). Which will be an exact topological “double” to the latter class?

An important class of orthoposets wider than the Boolean algebras is the class of orthomodular lattices (cf. [4]). It was proved in [1] that for every orthomodular lattice L there exists a compact Hausdorff closure (not necessarily topological) space \mathcal{L} such that the orthomodular lattice $CO(\mathcal{L})$ of all clopen subsets of \mathcal{L} is isomorphic to L .

There are many orthoposets which are not Boolean algebras or even orthomodular lattices (cf. [3, 4, 5, 7]). As is well-known, they are of importance within the foundations of quantum mechanics (cf. [3, 7]).

In Section 2 of this note, we introduce a purely topological concept of an orthopair which enables one to consider a bijective correspondence between the orthoposets and the orthopairs.

2. ORTHOPAIRS

Recall that an *orthoposet* (cf. [2, 4, 5]) is a poset E with the greatest element 1 and an involutive antiautomorphism $\prime: E \rightarrow E$ satisfying $x \vee x' = 1$ for every $x \in E$.

Also recall that two orthoposets, E_1 and E_2 , are said to be *isomorphic* if there exists an *isomorphism* between them, i.e., a bijection $\varphi: E_1 \rightarrow E_2$ satisfying $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$ ($x, y \in E_1$) and $\varphi(x') = \varphi(x)'$ ($x \in E_1$).

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Definition 2.1. An *orthopair* is a couple $(\mathcal{S}, \mathcal{E})$, wherein \mathcal{S} is a Hausdorff topological space and \mathcal{E} is a subbasis of \mathcal{S} satisfying

- 1) $\mathcal{S} \in \mathcal{E}$,
- 2) $\alpha \in \mathcal{E} \Rightarrow \alpha^c \in \mathcal{E}$ (here, $\alpha^c = \mathcal{S} \setminus \alpha$), and
- 3) every covering of \mathcal{S} by elements of \mathcal{E} possesses an at most 2-element subcovering.

Remark 2.2. Suppose that $(\mathcal{S}, \mathcal{E})$ is an orthopair. Then $(\mathcal{E}, \subset, ^c)$ is obviously an orthoposet and \mathcal{S} is zero-dimensional (i.e., has a basis consisting of clopen sets). By the Alexander Lemma (cf. [2, 6]), \mathcal{S} is compact.

Theorem 2.3. For an arbitrary orthoposet E there exists an orthopair $(\mathcal{S}, \mathcal{E})$ such that E and \mathcal{E} , as orthoposets, are isomorphic. Moreover, if $(\mathcal{S}_1, \mathcal{E}_1)$ and $(\mathcal{S}_2, \mathcal{E}_2)$ are orthopairs such that \mathcal{E}_1 and \mathcal{E}_2 are both isomorphic to E , then there exists a homeomorphism $u: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ satisfying $\alpha \in \mathcal{E}_1 \Leftrightarrow u(\alpha) \in \mathcal{E}_2$ for every $\alpha \subset \mathcal{S}_1$.

Proof. Let E be an orthoposet. Put $\mathcal{S} = \{f: E \rightarrow \{0, 1\} \mid f \text{ is monotonic and } f(x) + f(x') = 1 \text{ for all } x \in E\}$. Note that $f \in \mathcal{S} \Rightarrow f(1) = 1$. We consider \mathcal{S} as a subspace of the product topological space $\{0, 1\}^E$. Obviously \mathcal{S} is Hausdorff. For every $x \in E$, we define a continuous mapping $\pi_x: \mathcal{S} \rightarrow \{0, 1\}$ as $\pi_x(f) = f(x)$ ($f \in \mathcal{S}$). Define a mapping $\varphi: E \rightarrow \mathcal{P}(\mathcal{S})$, $\mathcal{P}(\mathcal{S})$ being the set of all subsets of \mathcal{S} by $\varphi(x) = \pi_x^{-1}(1)$ ($x \in E$). Put $\mathcal{E} = \{\varphi(x) \mid x \in E\}$. Then \mathcal{E} is obviously a subbasis of \mathcal{S} . Next, $\mathcal{S} = \varphi(1) \in \mathcal{E}$. Since $\varphi(x)^c = \varphi(x')$ ($x \in E$), 2) follows. Obviously $x, y \in E$, $x \leq y \Rightarrow \varphi(x) \subset \varphi(y)$.

We aim to show that $x, y \in E$, $\varphi(x) \subset \varphi(y) \Rightarrow x \leq y$ and to establish 3). Prior to this, we need a lemma.

For $x, y \in E$, we write $x \perp y$ provided that $x \leq y'$ and $x \not\leq y$ otherwise.

Lemma 2.4. Let $A \subset E$ satisfy $x, y \in A \Rightarrow x \not\leq y$. Then there exists $f \in \mathcal{S}$ with $A \subset f^{-1}(1)$.

Proof. By the Zorn Lemma, there is a maximal, with respect to inclusion, $B \subset E$ such that $A \subset B$ and $x, y \in B \Rightarrow x \not\leq y$. Put

$$f(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in E \setminus B. \end{cases}$$

It suffices to show that

- (i) $x \in B$, $y \in E$, $x \leq y \Rightarrow y \in B$ and
- (ii) if $x \in E$, then either $x \in B$ or $x' \in B$.

Suppose that (ii) fails to be true. Then there obviously exists $x \in E$ with $x \notin B$ and $x' \notin B$. Since B is maximal, there exist $z, a \in B$ with $z \perp x$ and $x' \perp a$. Then $z \perp a$. This is a contradiction. Thus (ii) follows. Let us prove (i). Suppose that $x \in B$, $y \in E$, and $x \leq y$. Since $x \perp y'$, we get $y' \notin B$. By (ii), $y \in B$. \square

Let us continue the proof of the theorem. Let $x, y \in E$ satisfy $\varphi(x) \subset \varphi(y)$. Suppose $x \not\leq y$. Since $x \not\leq y'$, by Lemma 2.4, there exists $f \in \mathcal{S}$ satisfying $f(x) = f(y') = 1$. Obviously $f \in \varphi(x) \setminus \varphi(y)$. This is a contradiction. Since we have verified 1) and 2) and have shown φ to satisfy $x \leq y \Leftrightarrow \varphi(x) \subset \varphi(y)$ ($x, y \in E$) and $\varphi(x') = \varphi(x)^c$ ($x \in E$), it follows that $(\mathcal{E}, \subset, ^c)$ is an orthoposet and φ is an isomorphism between E and \mathcal{E} . Let us show 3). Suppose that $\mathcal{A} \subset \mathcal{E}$ is a covering of \mathcal{S} which contains no at most 2-element subcovering. Put $\mathcal{B} = \{\alpha^c \mid \alpha \in \mathcal{A}\}$. Since $\beta \cap \gamma \neq \emptyset$ ($\beta, \gamma \in \mathcal{B}$), it follows that $\varphi^{-1}(\beta) \not\leq \varphi^{-1}(\gamma)$ ($\beta, \gamma \in \mathcal{B}$). By

Lemma 2.4, there exists $f \in \mathcal{S}$ with $f(\varphi^{-1}(\beta)) = 1$ for all $\beta \in \mathcal{B}$. Thus we obtain $f \in \bigcap_{\beta \in \mathcal{B}} \beta = (\bigcup_{\alpha \in \mathcal{A}} \alpha)^c$. This is a contradiction.

Now, take an arbitrary orthopair (T, \mathcal{F}) . Put $\mathcal{S}_1 = \{f: \mathcal{F} \rightarrow \{0, 1\} \mid \alpha, \beta \in \mathcal{F}, \alpha \subset \beta \Rightarrow f(\alpha) \leq f(\beta) \text{ and } \alpha \in \mathcal{F} \Rightarrow f(\alpha) + f(\alpha^c) = 1\}$. We endow \mathcal{S}_1 with the topology inherited from $\{0, 1\}^{\mathcal{F}}$. By the Tikhonoff Theorem, \mathcal{S}_1 is compact. For each $t \in T$ define $f_t \in \mathcal{S}_1$ by

$$f_t(\alpha) = \begin{cases} 1 & \text{if } t \in \alpha, \\ 0 & \text{if } t \notin \alpha \end{cases} \quad (\alpha \in \mathcal{F}).$$

Since the sets $\alpha \in \mathcal{F}$ are clopen and T is Hausdorff, it follows that the mapping $\phi: T \rightarrow \mathcal{S}_1$ defined as $\phi(t) = f_t$ ($t \in T$) is a continuous injection. Suppose $f \in \mathcal{S}_1$. Put $\mathcal{A}_f = \{\alpha \in \mathcal{F} \mid f(\alpha) = 0\}$. If $\alpha, \beta \in \mathcal{A}_f$, then $f(\alpha^c) = 1 > f(\beta)$ and thus $\alpha^c \not\subset \beta$, i.e., $\alpha \cup \beta \neq T$. Hence \mathcal{A}_f is not a covering of T . If we take $t \in T \setminus (\bigcup_{\alpha \in \mathcal{A}_f} \alpha)$, then $f_t = f$. Thus ϕ is a bijection. As T and \mathcal{S}_1 are compact, ϕ is a homeomorphism.

To conclude the proof of the theorem, suppose that $\psi: \mathcal{F} \rightarrow E$ is an isomorphism. Put $u(f) = \phi^{-1}(f \circ \psi)$ ($f \in \mathcal{S}$). Obviously $u: \mathcal{S} \rightarrow T$ is a homeomorphism, and if $\alpha \subset \mathcal{S}$, then $\alpha \in \mathcal{E} \Leftrightarrow u(\alpha) \in \mathcal{F}$. \square

Remark 2.5. By Lemma 2.4, we may identify the functions $f \in \mathcal{S}$ with the maximal, with respect to inclusion, subsets $A \subset E$ with $x, y \in A \Rightarrow x \not\leq y$.

Example 2.6. Let E be the Boolean algebra of all subsets of $\{1, 2, 3\}$. Observe that $(\{1, 2, 3\}, E)$, $\{1, 2, 3\}$ being endowed with the discrete topology is not an orthopair, though $\{1, 2, 3\}$ is a Stone space for E . By making use of Remark 2.5, let us construct an orthopair, $(\mathcal{S}, \mathcal{E})$, for E . Put $A = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, $B = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, $C = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, and $D = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. Then $\mathcal{S} = \{A, B, C, D\}$, the topology in \mathcal{S} is discrete, and $\mathcal{E} = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B, D\}, \{B, C, D\}, \{A, C, D\}, \mathcal{S}\}$.

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