MILNOR FIBRATIONS
OF LATTICE-ISOTOPIC ARRANGEMENTS

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ABSTRACT. We show that the associated Milnor fibrations are equivalent in a smooth family of arrangements with constant intersection lattice.

1. INTRODUCTION

We consider *arrangements* $\mathcal{A} = \{H_1, \ldots, H_n\}$ of linear codimension one subspaces of an ℓ-dimensional complex vector space V, with complement $M = V - \cup H_i$. In [4] it was shown that the lattice of intersections of subsets of the hyperplanes determines the cohomology ring of the complement. In [5] we showed that the diffeomorphism type of the complement is constant in a smooth one-parameter family of arrangements in which the lattice of intersections is unchanged (a "lattice isotopy," defined in Section 2.) Each subspace H_i is defined as the zero locus of a linear form α_i, defined up to a constant factor, and we may consider M as the complement of the divisor associated to the product $Q = \prod \alpha_i$.

There is a Milnor fibration [3]

$$Q^{-1}(1) = F \to M \to \mathbb{C} - \{0\} \ ,$$

where the right-hand map is simply Q. Note that Q is defined up to a constant, but that by the homogeneity of Q, the fiber bundle one gets is independent of the choice of constant. We call this the Milnor fibration associated to the arrangement.

In this paper we will apply Thom's second isotopy lemma to prove that the isomorphism type of the Milnor fibration is constant in a smooth one-parameter family of arrangements in which the intersection lattice is constant. This is a natural extension of the result of [5].

A question of considerable interest in the theory of arrangements is whether or not the intersection lattice determines the topology of the complement. The main result of this paper shows that fine invariants of the situation, such as the Milnor fiber, are determined by the intersection lattice, **within smooth families**.

In Section 2 we give the relevant definitions and make some basic observations. In particular we give a precise statement of Thom's second isotopy lemma. In Section 3 we give the proof of the main result. This involves setting up appropriate spaces and verifying that the hypotheses of Thom's lemma hold. As noted, these results focus interest on the Milnor fiber.

2. DEFINITIONS AND NECESSARY BACKGROUND

Our theorem for lattice-isotopic arrangements follows from the second isotopy lemma of R. Thom (as proved by J. Mather [2]). This result, which allows one to conclude that a family of maps is "locally trivial", is perhaps not as accessible as it would merit, so we give here the relevant definitions and a precise statement.

Suppose that $f : M \to S$ is a smooth map of smooth manifolds, and let X be a closed subset of M. A *stratification* of X is a partition of X into a finite collection of submanifolds (called strata X_α) of M so that the following frontier condition holds:

$$\text{if} \quad X_\beta \cap \bar{X}_\alpha \neq \emptyset , \quad \text{then} \quad X_\beta \subset \bar{X}_\alpha .$$

In [7] H. Whitney introduced two extra conditions on stratifications, which he called conditions a) and b):

a) Whenever $X_\beta \subset \bar{X}_\alpha$ and $x_i \in X_\alpha$ is a sequence converging to $y \in X_\beta$, with $TX_\alpha|_{x_i} \to T \subset TM_y$, then $TX_\beta|_y \subset T$.

b) For every $y \in X_\beta$, direction ℓ, and plane T of dimension $d_\alpha = \dim X_\alpha$, if there are sequences $x_i \in X_\alpha$, $y_i \in X_\beta$ so that $x_i \to y$, $y_i \to y$, $x_i - y_i \to \ell$ and $T_{x_i} X_\alpha \to T$, then $\ell \in T$.

A stratification is called a *Whitney stratification* if it satisfies condition b) for all $y \in X_\beta \subset \bar{X}_\alpha$. We note that condition b) actually implies condition a). See [1] or [7] for further discussion of Whitney stratifications.

The smooth map $f : M \to S$ is said to be *stratified* provided that one has Whitney stratifications $\Sigma = \{X_\alpha\}$ and $\sigma = \{S_\alpha\}$ of X and S respectively, that f maps stratum into stratum, and that $f|X_\alpha$ is a submersion for every stratum X_α.

Thom's first isotopy lemma [2, p.62] says that a proper smooth stratified map is the projection of a smooth fiber bundle. This result was used in [5, Theorem 1] to show that lattice-isotopic arrangements are topologically isomorphic.

We next recall some definitions from [5]. For any arrangement, one has the associated *lattice*, which consists of the various intersections of collections of hyperplanes H_i, ordered by reverse inclusion. Suppose that I is an open interval of real numbers, containing 0 and 1.

Definition 1. A smooth 1-parameter family of arrangements $\mathcal{A} = \{A_t\}$ is a finite collection $\{H_{it}\}, i = 1, 2, \ldots, n$, of subspaces for each $t \in I$ so that H_{it} is the locus in $V \times \{t\} \subset V \times I$ of an equation linear in the coordinates of V with coefficients smooth functions of t. Thus, the defining equations have the form

$$\alpha_i(z, t) = \alpha_{i1}(t)z_1 + \alpha_{i2}(t)z_2 + \ldots + \alpha_{i\ell}(t)z_\ell ,$$

where the $\alpha_{ij} : I \to \mathbb{C}$ are smooth functions.

A one-parameter family gives arrangements $\mathcal{A}_t$ for any $t \in I$ in the obvious way.

Definition 2. Arrangements $\mathcal{A}_0 = \{H_1, H_2, \ldots, H_n\}$ and $\mathcal{A}_1 = \{G_1, G_2, \ldots, G_n\}$ have the same lattice if for all $J \subset \{1, 2, \ldots, n\}$,

$$\dim \cap_{i \in J} H_i = \dim \cap_{i \in J} G_i .$$

Definition 3. A one-parameter family $\{\mathcal{A}_t\}$ is a lattice isotopy provided that for any t_1, t_2 the arrangements $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ have the same lattice.

Let Q_{it} be the products of the linear forms defining the hyperplanes H_{it} and set

$$Q_t(z) = \prod_{i=1}^{n} \alpha_i(z, t)$$
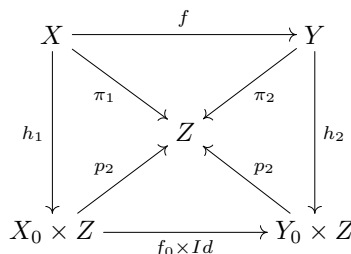
Then there is the map $Q : \mathbb{C}^\ell \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ defined by $Q(z, t) = (Q_t(z), t)$. For a fixed $t \in I$, one has the Milnor fibration Q_t from the complement of the hyperplanes of the arrangement $\mathcal{A}_t$ into $\mathbb{C}^* = \mathbb{C} - \mathbf{0}$, with Milnor fiber $F_t = Q_t^{-1}(1)$. The main result of this note is that this fibration is independent of t.

To make this notion of independence of t precise we adopt the following definitions of [2, p.63].

Suppose we are given the following diagram of spaces and mappings:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \pi_1 \searrow \quad \swarrow \pi_2 & \\ & Z & \end{array}$$
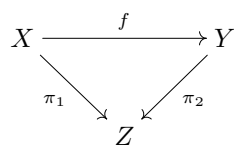
We say that f is *trivial over Z* provided that there are spaces X_0 and Y_0 and a mapping $f_0 : X_0 \to Y_0$ with homeomorphisms $h_1 : X \to X_0 \times Z$, $h_2 : Y \to Y_0 \times Z$ so that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{\quad\quad f \quad\quad} & & & Y \\ \downarrow h_1 & \pi_1 \searrow \quad \swarrow \pi_2 & & & \downarrow h_2 \\ & Z & & & \\ & p_2 \nearrow \quad \nwarrow p_2 & & & \\ X_0 \times Z & \xrightarrow{\quad f_0 \times Id \quad} & & & Y_0 \times Z \end{array}$$

f is *locally trivial over Z* if for all $z \in Z$ there is a neighborhood U of $z \in Z$ so that in the diagram

$$\begin{array}{ccc} \pi_1^{-1}(U) & \xrightarrow{\quad f \quad} & \pi_2^{-1}(U) \\ & \pi_1 \searrow \quad \swarrow \pi_2 & \\ & U & \end{array}$$
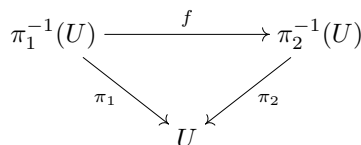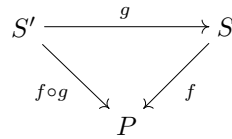
we have f trivial over U.

Our next goal is to define a class of mappings which are locally trivial. Suppose $g : M' \to M$ is a smooth mapping, and suppose that S' and S are closed, Whitney-stratified subsets of M' and M respectively, with $g(S') \subseteq S$.

Suppose that X_α and X_β are strata of the Whitney stratification of S', with $y \in X_\beta$. Suppose that $g|X_\alpha$ and $g|X_\beta$ are of constant rank. We then say that (X_α, X_β) *satisfies condition a_g at y* provided that the following holds:

Let x_i be any sequence of points in X_α converging to y. Suppose that the sequence of planes $\ker(d(g|X_\alpha)_{x_i}) \subset TM'_{x_i}$ converges to a plane $\tau \subseteq TM'_y$ in the appropriate Grassmannian bundle. Then $\ker(d(g|X_\beta)_y) \subset \tau$.

We say that (X_α, X_β) satisfies condition a_g if it satisfies condition a_g at every point y of X_β.

Suppose we then have a diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\ g\ } & S \\
& {\scriptstyle f \circ g} \searrow \quad \swarrow {\scriptstyle f} & \\
& P &
\end{array}
$$

Definition 4. g is a Thom mapping over P if and only if the following four conditions are satisfied:
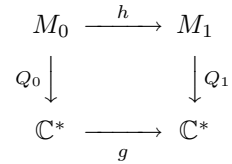
(a) $g|S'$ and $f|S$ are proper.

(b) For each stratum Y of the stratification of S, $f|Y$ is a submersion.

(c) For each stratum X_α of S', $g(X_\alpha)$ lies in a stratum Y of S, and $g : X_\alpha \to Y$ is a submersion.

(d) Any pair (X_α, X_β) of strata of S' satisfies condition a_g.

Theorem (Thom's second isotopy lemma, [2, p. 66]). *If g is a Thom mapping over P, then g is locally trivial over P.*

3. MAIN RESULT

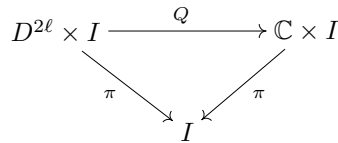**Theorem.** *Lattice-isotopic arrangements have smoothly equivalent Milnor fibrations.*

In other words, we have a commuting diagram

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ h\ } & M_1 \\
{\scriptstyle Q_0} \downarrow & & \downarrow {\scriptstyle Q_1} \\
\mathbb{C}^* & \xrightarrow[\ g\ ]{} & \mathbb{C}^*
\end{array}
$$

where h and g are diffeomorphisms.

Proof. We wish to obtain a set-up so that we can apply the second isotopy lemma. One difficulty is that the map $Q_t : \mathbb{C}^\ell \to \mathbb{C}$ is not proper. Thus we consider the restriction of Q_t to the unit ball of $\mathbb{C}^\ell$, obtaining proper maps $Q_t : D_t^{2\ell} \to \mathbb{C}$ and $Q : D^{2\ell} \times I \to \mathbb{C} \times I$. Consider the diagram

$$
\begin{array}{ccc}
D^{2\ell} \times I & \xrightarrow{\ Q\ } & \mathbb{C} \times I \\
& {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\
& I &
\end{array}
$$

with π denoting the projections. Recall I is an open interval containing 0 and 1.

We wish to show that Q is trivial over I. We stratify $\mathbb{C} \times I$ by $\{(\mathbb{C} - \{0\}) \times I, \{0\} \times I\}$, clearly a Whitney stratification.

We use the arrangement to stratify $D^{2\ell} \times I$. For any fixed arrangement $\mathcal{A}$, there is a natural stratification obtained by taking as strata the elements of the intersection lattice with all other lattice elements removed. That is, any element X of the lattice determines an arrangement $\mathcal{A}^X$ in X, whose hyperplanes are all $H_i \cap X$ for hyperplanes H_i of the original arrangement which do not contain X. Then one obtains a stratification of $\mathbb{C}^\ell$ by taking as strata all sets $M^X = X - \cup(H_i \cap X)$ together with the top-dimensional stratum M. By taking intersections of these strata with $\mathrm{Int}(D^{2\ell})$ and $\partial D^{2\ell}$ one obtains a stratification of $D^{2\ell}$.

Finally, notice that since in a lattice isotopy of arrangements we require unchanging intersection lattice, we may take $V_X = \bigcup\limits_t M_t^X$ intersected with $\mathrm{int}(D^{2\ell}) \times I$ and with $\partial(D^{2\ell}) \times I$ to form a stratification of $D^{2\ell} \times I$.

As in [5, p.557], this forms a Whitney stratification of $D^{2\ell} \times I$. (The proof uses the additional and obvious fact that for any lattice element X, the manifold $\bigcup\limits_t X_t$ intersects $\partial D^{2\ell} \times I$ transversely.)

We now check the conditions of Thom's second isotopy lemma for the diagram

$$
\begin{array}{ccc}
D^{2\ell} \times I & \xrightarrow{\ \ Q\ \ } & B \times I \subset \mathbb{C} \times I \\
& \searrow{\scriptstyle \pi'} \quad \swarrow{\scriptstyle \pi} & \\
& I &
\end{array}
$$

where B is a sufficiently large closed disk in $\mathbb{C}$, chosen so that $Q(D^{2\ell} \times I) \subset (\mathrm{Int}B) \times I$

(a) $Q|_{D^{2\ell} \times I}$ and $\pi|_{B \times I}$ are proper, because Q and B are compact.

(b) We stratify $B \times I$ with three strata; $\partial B \times I$, $(\mathrm{Int}B - \{0\}) \times I$, and $\{0\} \times I$. This is clearly a Whitney stratification. Clearly π, as projection to the I factor, is a submersion on these strata.

(c) We need to check that Q maps strata to strata submersively. The closed disk B was chosen so that Q maps into $\mathrm{Int}(B) \times I$. Thus each stratum of $D^{2\ell} \times I$ maps either into $(\mathrm{Int}\ B - \{0\}) \times I$, or into $\{0\} \times I$. Strata containing points lying on hyperplanes map to $\{0\} \times I$, and since the lattice is constant throughout the isotropy, the restriction of Q to such strata is a submersion.

Consider a point (v, t_0) on the top-dimensional stratum X. That is, v lies on no hyperplane of the arrangement $\mathcal{A}_{t_0}$. Clearly the same holds for (v, t) for all t in some small open interval $I' \subset I$, centered at t_0. Thus we may find an open set $V \times I'$, containing (v, t), with $V \times I' \subset X$. Consider $Q|_{V \times I'} : V \times I' \to (\mathrm{Int}B - \{0\}) \times I$. The Jacobian of this map has the form

$$
\begin{bmatrix}
\dfrac{\partial Q_t}{\partial z_1} & \cdots & \dfrac{\partial Q_t}{\partial z_\ell} & \dfrac{\partial Q_t}{\partial t} \\[2ex]
0 \ \ 0 & \cdots & 0 & 1
\end{bmatrix}
$$

and hence is a submersion on $V \times I'$, since $\frac{\partial Q_t}{\partial z_1} = \frac{\partial Q_t}{\partial z_2} = \cdots = \frac{\partial Q_t}{\partial z_\ell} = 0$ at some (v, t) implies $Q_t(v) = 0$. (That is, 0 is the *only* critical value of Q_t, for all $t \in I$.)

(d) Finally, and most crucially, we need to verify that any pair (X_α, X_β) of strata of $D^{2\ell} \times I$ satisfy the condition A_Q. This condition only has content in the case that there is a sequence $\{x_i\} \subset X_\alpha$ converging to $y \in X_\beta$. Our strata for $D^{2\ell} \times I$ are

$$
M_i = (\bigcup\limits_t M_t) \cap (\mathrm{int}(D^{2\ell}) \times I),
$$

$$
M_b = (\bigcup\limits_t M_t) \cap ((\partial D^{2\ell}) \times I),
$$

and for any X in the (common) lattice (X not minimal)

$$
M_i^X = (\bigcup\limits_t M_t^X \cap (\mathrm{int}(D^{2\ell}) \times I),
$$

$$
M_b^X = (\bigcup\limits_t M_t) \cap ((\partial D^{2\ell}) \times I) .
$$

Here of course the subscripts "i" and "b" refer to interior and boundary respectively. Actually, M_i and M_b correspond to the lowest lattice element, the intersection of the empty set of hyperplanes.

Thus we must verify non-vacuous cases (i) $(M_i^X, M_i^{X'})$, (ii) $(M_i^X, M_b^{X'})$ and (iii) $(M_b^X, M_b^{X'})$ where $X \leq X'$ in the lattice partial order (i.e., $X' \subseteq X$). Each of the three cases splits further into two subcases, depending on whether or not the first stratum in the pair is M_i or M_b, or is something else.

Case (i)a. $(M_i, M_i^{X'})$. We suppose $x_j \to y$, where $\{x_j\}$ is a sequence in M_i, $y \in M_i^{X'}$. Further, we suppose that the sequence of planes

$$\tau_j = \ker(d(Q|M_i)_{x_j}) \subset T(D^{2\ell} \times I)_{x_j}$$

converges to a plane $\tau \subseteq T(D^{2\ell} \times I)_y$. We consider $\kappa = \ker(d(Q|M_i^{X'})_y)$. Since $Q(x, t) = (0, t)$ for all $x \in M_i^{X'}$, $\kappa = X' \times \{0\} \subset \mathbb{C}^\ell \times \mathbb{R} = T(D^{2\ell} \times I)_y$. We wish to show that $\kappa \subset \tau$.

Denote by $(grad\, Q)_{x_j}$ the vector $\left(\frac{\partial Q}{\partial z_i}\right)_{x_j}$. Now $\tau_j = (grad\, Q)_{x_j}^\perp \times \{0\} \subset \mathbb{C}^\ell \times \mathbb{R}$. Clearly $\tau = (grad\, Q)_y^\perp \times \{0\} \subset \mathbb{C}^\ell \times \mathbb{R}$. A point $v \in \kappa = X' \times \{0\}$ has the form $(v_1, \dots, v_\ell, 0)$, with $(v_1, v_2, \dots, v_\ell, \pi(y)) \in M_i^{X'}$ and $Q(v_1, \dots, v_\ell, \pi(y)) = 0$. For t fixed, Q is a homogeneous function of the variables $z_1, \dots, z_\ell$, of degree n. Thus one has the functional ("Euler") identity

$$Q_t(z) = \sum_i \frac{z_i}{n} \frac{\partial Q_t}{\partial z_i}.$$

Thus,

$$0 = Q_{\pi(y)}(v) = \frac{1}{n} v \cdot (grad\, Q)_y$$

and so $v \in (grad\, Q)_y^\perp \times \{0\}$. Thus $\kappa \subset \tau$ as desired.

The cases (ii)a and (iii)a are similar.

Case (i)b. $(M_i^X, M_i^{X'})$. Then

$$\ker(d(Q|M_i^X)_{x_i}) = X \times \{0\} \subset (\mathbb{C}^\ell \times \mathbb{R})_{x_i} = T(D^{2\ell} \times I)_{x_i}$$

certainly converges to $\tau = X \times \{0\} \subset T(D^{2\ell} \times I)_y$. But $\ker(d(Q|M_i^{X'})_y) = X' \times \{0\} \subset \tau = X \times \{0\} \subset T(D^{2\ell} \times I)_y$, so that a_Q holds in this case.

The cases (ii)b and (iii)b are similar.

Thus Q is a Thom mapping and the theorem follows by Thom's second isotopy lemma. □

Corollary. *Lattice-isotopic arrangements have diffeomorphic Milnor fibers.*

Finally, we point out that Rybnikov [6] has exhibited arrangements with the same lattice and non-isomorphic fundamental group. Thus the requirement of a lattice *isotopy* in the above results is necessary.

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