ON ‘CLIFFORD’S THEOREM’
FOR PRIMITIVE FINITARY GROUPS

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Abstract. Let \( V \) be an infinite-dimensional vector space over any division ring \( D \), and let \( G \) be an irreducible primitive subgroup of the finitary group \( \text{FGL}(V) \). We prove that every non-identity ascendant subgroup of \( G \) is also irreducible and primitive. For \( D \) a field, this was proved earlier by U. Meierfrankenfeld.

Let \( V \) be a left vector space over some division ring \( D \), and let \( G \) be an irreducible subgroup of the full finitary general linear group \( \text{FGL}(V) \) on \( V \). A subgroup \( H \) of a group \( G \) is ascendant if there is an ascending (possibly infinite) series

\[
H = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq H_\gamma = G
\]

of subgroups of \( G \), each normal in its successor, joining \( H \) to \( G \). In [1], see 7.6, U. Meierfrankenfeld proved that if \( D \) is a field, then every ascendant subgroup \( H \) of \( G \) is completely reducible, the classical Clifford’s Theorem being the case where \( V \) is finite dimensional and \( H \) is normal in \( G \). Independently, and using a substantially different approach, the author derived the same conclusion for any division ring \( D \); see Prop. 9 of [5].

Meierfrankenfeld’s approach studies primitive groups first and the major step in his proof ([1], 7.4) is to show that, if \( D \) is a field, if \( \dim_D V \) is infinite and if \( G \) is also primitive with \( H \not= \langle 1 \rangle \), then \( H \) is irreducible and primitive. The author’s approach did not consider primitive groups separately. Thus two obvious questions arise. Does this same conclusion hold for any division ring \( D \), and if so, can it be obtained relatively quickly from the results of [5] without, for example, repeating and adapting as necessary the lengthy analysis in [1]? (Incidentally I assume that the latter is feasible, though I have not seriously tried to carry it out.) The object of this note is to show that the answer to both questions is yes.

**Theorem.** Let \( V \) be an infinite-dimensional left vector space over the division ring \( D \), and let \( H \) be a non-trivial ascendant subgroup of the irreducible primitive subgroup \( G \) of \( \text{FGL}(V) \). Then \( H \) too is irreducible and primitive.

**Proof of the Theorem.** We prove first that \( H \) is irreducible. By ‘Clifford’s Theorem’ (see [5], Prop. 9) the group \( H \) is completely reducible. Suppose \( H \) is reducible. Let \( \{H_\alpha : 0 \leq \alpha \leq \gamma \} \) be an ascending series with \( H = H_0 \) and \( H_\gamma = G \). By hypothesis \( H \not= \langle 1 \rangle \); choose \( h \in H \setminus \langle 1 \rangle \) and set \( d = \dim_D [V, h] \). Then \( 0 < d < \infty \). For each
\[ \alpha \leq \gamma \text{ set} \]

\[ K_\alpha = \langle g \in H_\alpha : \dim_D[V, g] \leq d \rangle. \]

Then \( K_\alpha \) is normal in \( H_\alpha \), indeed if \( \alpha < \gamma \) then \( K_\alpha \) is normal in \( H_{\alpha+1} \), and

\[ K = K_0 \leq K_1 \cdots K_\alpha \leq K_{\alpha+1} \cdots K_\gamma \leq G \]

is an ascending series. Moreover \( \dim H_\alpha \), indeed if \( \alpha < \gamma \) then \( K_\alpha \) is normal in \( H_{\alpha+1} \), and

\[ K = K_0 \leq K_1 \cdots K_\alpha \leq K_{\alpha+1} \cdots K_\gamma \leq G \]

Let \( U \) be an irreducible \( D-K \) submodule of \( V \) and suppose \( \dim_D U > d \). If \( UK_\alpha = U \) and \( g \in K_{\alpha+1} \) with \( \dim_D[V, g] \leq d \), then \( g \) normalizes \( K_\alpha \), the modules \( U \) and \( Ug \) are both \( D-K_\alpha \) irreducible, \( U \cap Ug \neq \{0\} \) and \( U = Ug \). Therefore \( UK_{\alpha+1} = U \). A simple induction yields that \( UK_\gamma = U \). But \( K_\gamma \) is normal in the primitive group \( G \), so \( K_\gamma \) is irreducible ([4], 3.1, and [5], Prop. 8). Consequently \( U = V \), a contradiction of the reducibility of \( K \). Therefore \( \dim_D U \leq d \). The same argument yields that \( V \) is a direct sum of irreducible \( D-K_\alpha \) modules of dimension at most \( d \) for any \( \alpha \leq \gamma \) with \( K_\alpha \) reducible.

Choose \( \alpha \) and an irreducible \( D-K_\alpha \) submodule \( U \) of \( V \) with \( \{U, K_\alpha\} \neq \{0\} \), with \( \dim_D U \leq d \) and with \( \dim_D U \) maximal. By the above such \( \alpha \) and \( U \) exist. From all such choices pick \( \alpha \) and \( U \) such that the \( D-K_\alpha \) homogeneous component \( W \) of \( V \) containing \( U \) has \( \dim_D W \) minimal; note that by finitariness \( \infty > \dim_D W \geq \dim_D U > 0 \).

Now suppose \( K_{\alpha+1} \) is reducible. If \( U_1 \) is an irreducible \( D-K_{\alpha+1} \) submodule of \( V \) containing a copy of \( U \), then the choice of \( \alpha \) and \( U \) ensures that \( \dim_D U = \dim_D U_1 \) and that \( U \) and \( U_1 \) are isomorphic as \( D-K_\alpha \) modules. It follows that \( W \) is a direct sum of \( D-K_{\alpha+1} \) homogeneous components of \( V \) and the minimal choice of \( W \) yields that \( W \) is a \( D-K_{\alpha+1} \) homogeneous component of \( V \). If \( \lambda \leq \gamma \) is a limit ordinal with \( W \) a \( D-K_\beta \) homogeneous component of \( V \) for all \( \beta \) with \( \alpha \leq \beta < \lambda \), then \( WK_\lambda = \bigcup_{\beta<\lambda} WK_\beta = W \), so \( K_\lambda \) is reducible, the \( D-K_\lambda \) irreducible submodules of \( W \) are \( D-K_\beta \) irreducible and isomorphic for \( \alpha \leq \beta < \lambda \) (by the choice of \( \alpha \) and \( U \)), \( W \) is a sum of \( D-K_\lambda \) homogeneous components of \( V \) and, by the minimal choice of \( W \), \( W \) is a \( D-K_\lambda \) homogeneous component of \( V \). Since \( K_\gamma \) is irreducible ([4], 3.1 again), so \( WK_\gamma \neq W \) and the above yields the existence of \( \beta < \gamma \) with \( K_\beta \) reducible and \( K_{\beta+1} \) irreducible.

To simplify notation assume \( \beta = 0 \); that is, assume \( K_1 \) is irreducible. Let \( \alpha \geq 1 \), let \( L \neq \{1\} \) be a reducible normal subgroup of \( K_\alpha \) and let \( U \) be an irreducible \( D-L \) submodule of \( V \) with \( \{U, L\} \neq \{0\} \). (Note that such \( \alpha \) and \( L \) exist, for example \( 1 \) and \( K_1 \)) By an argument we have seen before \( \dim_D U \leq d \). Let \( W \) be the \( D-L \) homogeneous component of \( V \) containing \( U \). Choose \( \alpha \), \( L \) and \( U \) so firstly that \( \dim_D U \) is maximal and secondly that \( \dim_D W \) is minimal. For simplicity of notation assume \( \alpha = 1 \).

The \( D-L \) homogeneous components of \( V \) form a system \( V = \bigoplus_{\sigma \in \Gamma} V_\sigma \) of imprimitivity for \( K_1 \). Here \( \Omega \) is infinite and permuted transitively and finitarily by \( K_1 \) and the \( \dim_D V_\sigma \) are finite. By definition \( K_1 \) is generated by elements with support of bounded dimension \( \leq d \). Consequently \( K_1|\Omega \) is generated by elements with support of bounded cardinality (at most \( 2d \)). It follows, in the terminology of P. M. Neumann ([3], p. 563), that \( K_1|\Omega \) cannot be totally imprimitive and therefore must be almost primitive. Thus there is a \( K_1 \)-invariant congruence \( q \) on \( \Omega \) such that \( K_1|\Omega/q \) is primitive and hence is either \( \text{Alt}(\Omega/q) \) or \( \text{FSym}(\Omega/q) \); see [3], 2.3. Moreover each \( \omega q = \{\sigma \in \Omega : \omega q \sigma \} \) is finite. Let \( L_1 \) be the kernel of the action of \( K_1 \) on \( \Omega/q \), so \( L_1 \leq L_1 < K_1 \). For any subset \( \Sigma \) of \( \Omega \), write \( V_\Sigma \) for \( \bigoplus_{\sigma \in \Sigma} V_\sigma \).
Let $g \in K_2$ with $L_1^g \neq L_1$. Then $(K_1: L_1^g L_1) \leq 2$, since $\text{Alt}(\Omega/\omega)$ is simple of index 2 in $\text{FSym}(\Omega/\omega)$. Also for any $\omega$ in $\Omega$ there is a finite subset $\Sigma$ of $\Omega$ with 

$$V_\omega \cdot g(L_1^\omega L_1) = V_\omega \cdot g L_1 \leq V_\Sigma = V_\Sigma.$$

But $V_\omega \cdot g K_1 = V$, so 

$$\dim_D V \leq 2 \cdot \dim_D (V_\omega \cdot g(L_1^\omega L_1)) \leq 2 \cdot |\Sigma| \cdot \dim_D V_\omega < \infty.$$ 

This contradiction yields that $L_1$ is normal in $K_2$. Hence the $D$-$L_1$ homogeneous components of $V$ form a system of imprimitivity for $K_2$ in $V$. But the choice of $L, U$ and $W$ above ensures that these are just the $V_\omega$. Therefore $V = \bigoplus_{\omega} V_\omega$ is also a system of imprimitivity for $K_2$. We may repeat the above arguments with $K_2$ in place of $K_1$. A simple transfinite induction yields that $V = \bigoplus_{\omega} V_\omega$ is a system of imprimitivity for $G$. This contradiction of the primitivity of $G$ completes the proof that $H$ is irreducible. The primitivity of $H$ follows at once from the following lemma.

**Lemma.** Let $V$ be an infinite-dimensional left vector space over the division ring $D$ and $G$ a subgroup of $\text{FGL}(V)$. The following are equivalent.

a) $G$ is irreducible and primitive.

b) $G \neq (1)$ and every non-trivial normal subgroup of $G$ is irreducible.

**Proof.** As we have seen above a) implies b) by [4], 3.1, and [5], Prop. 8. Suppose b) holds. Clearly $G$ is irreducible. Consider a non-trivial system $V = \bigoplus_{\omega} V_\omega$ of imprimitivity for $G$. Then $G$ acts transitively and finitarily on $\Omega$ and each $\dim_D V_\omega$ is finite. Also $N = \bigcap_{\omega} N_G(V_\omega)$ is a reducible normal subgroup of $G$ and hence by b) is $(1)$. This holds for any such system of imprimitivity. If $G|_\Omega$ is totally imprimitive, then every element of $G$ lies in some such $N$. Hence $G|_\Omega$ is almost primitive and hence for some such system of imprimitivity $G|_\Omega$ is $\text{Alt}(\Omega)$ or $\text{FSym}(\Omega)$.

Let $\omega \in \Omega$ and $g \in N_G(\omega)$. Then the support $\text{supp}_G(g)$ of $g$ in $\Omega$ is finite and $C_G(g) \geq \text{Alt}(\Omega \setminus \text{supp}_G(g))$. The latter is simple and does not lie in $N_G(\omega)$. Hence 

$$|\omega C_G(g)| = (C_G(g) : C_G(g) \cap N_G(\omega)),$$

which is infinite. If $[V_\omega, g] \neq \{0\}$, then $[V_\omega x, g] \neq \{0\}$ for all $x$ in $C_G(g)$ and $\dim_D [V, g] \geq |\omega C_G(g)|$ is infinite. Consequently $g$ and $N_G(\omega)$ centralize $V_\omega$. Let $v_\omega \in V_\omega \setminus \{0\}$ and let $X$ be a right transversal of $N_G(\omega)$ to $G$. Then $U = \bigoplus_{x \in X} Dv_\omega x \leq V$ is a permutation module for $G$ and $\sum_{x,y \in X} D(v_\omega x - v_\omega y)$ is a proper $D$-$G$ submodule of $U$ and hence of $V$. This contradicts the irreducibility of $G$ and so $G$ is primitive as claimed.

Now suppose $\dim_D V$ is finite but otherwise assume the notation of the theorem. Clearly $H$ now need not be irreducible; just let $H$ be the center of $G = \text{GL}(V)$. If $H$ is normal, then $H$ is homogeneous and hence has no non-zero fixed-points in $V$. However if $H$ is only subnormal, then $H$ can have non-zero fixed-points. For example let $G$ be the group $\langle E \rangle$ of [2], p. 239. Then $G$ is an irreducible primitive subgroup of $\text{GL}(3, \mathbb{C})$ of order 108 and is the split extension of a non-abelian normal subgroup $N$ of order 27 and exponent 3 (it is a copy of $\text{TR}_1(3, 3)$) containing $a = \text{diag}(1, \exp(2\pi/3), \exp(4\pi/3))$ and a cyclic group of order 4. Also $N$ is nilpotent of class 2, the subgroup $\langle a \rangle$ is subnormal in $G$ of subnormal depth 3 and $(1, 0, 0)$ is a non-zero fixed-point of $a$. 

References


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