PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 2843–2846
S 0002-9939(97)04038-0

ON 'CLIFFORD'S THEOREM' FOR PRIMITIVE FINITARY GROUPS

B. A. F. WEHRFRITZ

(Communicated by Ronald M. Solomon)

Abstract. Let V be an infinite-dimensional vector space over any division ring D, and let G be an irreducible primitive subgroup of the finitary group $\mathrm{FGL}(V)$. We prove that every non-identity ascendant subgroup of G is also irreducible and primitive. For D a field, this was proved earlier by U. Meierfrankenfeld.

Let V be a left vector space over some division ring D, and let G be an irreducible subgroup of the full finitary general linear group $\mathrm{FGL}(V)$ on V. A subgroup H of a group G is ascendant if there is an ascending (possibly infinite) series

$$H = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq H_\gamma = G$$

of subgroups of G, each normal in its successor, joining H to G. In [1], see 7.6, U. Meierfrankenfeld proved that if D is a field, then every ascendant subgroup H of G is completely reducible, the classical Clifford's Theorem being the case where V is finite dimensional and H is normal in G. Independently, and using a substantially different approach, the author derived the same conclusion for any division ring D; see Prop. 9 of [5].

Meierfrankenfeld's approach studies primitive groups first and the major step in his proof ([1], 7.4) is to show that, if D is a field, if $\dim_D V$ is infinite and if G is also primitive with $H \neq \langle 1\rangle$, then H is irreducible and primitive. The author's approach did not consider primitive groups separately. Thus two obvious questions arise. Does this same conclusion hold for any division ring D, and if so, can it be obtained relatively quickly from the results of [5] without, for example, repeating and adapting as necessary the lengthy analysis in [1]? (Incidentally I assume that the latter is feasible, though I have not seriously tried to carry it out.) The object of this note is to show that the answer to both questions is yes.

Theorem. *Let V be an infinite-dimensional left vector space over the division ring D, and let H be a non-trivial ascendant subgroup of the irreducible primitive subgroup G of* $\mathrm{FGL}(V)$. *Then H too is irreducible and primitive.*

Proof of the Theorem. We prove first that H is irreducible. By 'Clifford's Theorem' (see [5], Prop. 9) the group H is completely reducible. Suppose H is reducible. Let $\{H_\alpha\colon\ 0 \leq \alpha \leq \gamma\}$ be an ascending series with $H = H_0$ and $H_\gamma = G$. By hypothesis $H \neq \langle 1\rangle$; choose $h \in H\backslash\langle 1\rangle$ and set $d = \dim_D[V, h]$. Then $0 < d < \infty$. For each

Received by the editors April 25, 1996.
1991 *Mathematics Subject Classification.* Primary 20H25.

$\alpha \leq \gamma$ set

$$K_\alpha = \langle g \in H_\alpha : \ \dim_D[V, g] \leq d \rangle.$$

Then K_α is normal in H_α, indeed if $\alpha < \gamma$ then K_α is normal in $H_{\alpha+1}$, and

$$K = K_0 \leq K_1 \cdots K_\alpha \leq K_{\alpha+1} \cdots K_\gamma \leq G$$

is an ascending series. Moreover $h \in K \leq H$, so K is non-trivial and reducible.

Let U be an irreducible D-K submodule of V and suppose $\dim_D U > d$. If $UK_\alpha = U$ and $g \in K_{\alpha+1}$ with $\dim_D[V, g] \leq d$, then g normalizes K_α, the modules U and Ug are both D-K_α irreducible, $U \cap Ug \neq \{0\}$ and $U = Ug$. Therefore $UK_{\alpha+1} = U$. A simple induction yields that $UK_\gamma = U$. But K_γ is normal in the primitive group G, so K_γ is irreducible ([4], 3.1, and [5], Prop. 8). Consequently $U = V$, a contradiction of the reducibility of K. Therefore $\dim_D U \leq d$. The same argument yields that V is a direct sum of irreducible D-K_α modules of dimension at most d for any $\alpha \leq \gamma$ with K_α reducible.

Choose α and an irreducible D-K_α submodule U of V with $[U, K_\alpha] \neq \{0\}$, with $\dim_D U \leq d$ and with $\dim_D U$ maximal. By the above such α and U exist. From all such choices pick α and U such that the D-K_α homogeneous component W of V containing U has $\dim_D W$ minimal; note that by finitariness $\infty > \dim_D W \geq \dim_D U > 0$.

Now suppose $K_{\alpha+1}$ is reducible. If U_1 is an irreducible D-$K_{\alpha+1}$ submodule of V containing a copy of U, then the choice of α and U ensures that $\dim_D U = \dim_D U_1$ and that U and U_1 are isomorphic as D-K_α modules. It follows that W is a direct sum of D-$K_{\alpha+1}$ homogeneous components of V and the minimal choice of W yields that W is a D-$K_{\alpha+1}$ homogeneous component of V. If $\lambda \leq \gamma$ is a limit ordinal with W a D-K_β homogeneous component of V for all β with $\alpha \leq \beta < \lambda$, then $WK_\lambda = \bigcup_{\beta<\lambda} WK_\beta = W$, so K_λ is reducible, the D-K_λ irreducible submodules of W are D-K_β irreducible and isomorphic for $\alpha \leq \beta < \lambda$ (by the choice of α and U), W is a sum of D-K_λ homogeneous components of V and, by the minimal choice of W, W is a D-K_λ homogeneous component of V. Since K_γ is irreducible ([4], 3.1 again), so $WK_\gamma \neq W$ and the above yields the existence of $\beta < \gamma$ with K_β reducible and $K_{\beta+1}$ irreducible.

To simplify notation assume $\beta = 0$; that is, assume K_1 is irreducible. Let $\alpha \geq 1$, let $L \neq \langle 1 \rangle$ be a reducible normal subgroup of K_α and let U be an irreducible D-L submodule of V with $[U, L] \neq \{0\}$. (Note that such α and L exist, for example 1 and K.) By an argument we have seen before $\dim_D U \leq d$. Let W be the D-L homogeneous component of V containing U. Choose α, L and U so firstly that $\dim_D U$ is maximal and secondly that $\dim_D W$ is minimal. For simplicity of notation assume $\alpha = 1$.

The D-L homogeneous components of V form a system $V = \bigoplus_{\omega\in\Omega} V_\omega$ of imprimitivity for K_1. Here Ω is infinite and permuted transitively and finitarily by K_1 and the $\dim_D V_\omega$ are finite. By definition K_1 is generated by elements with support of bounded dimension ($\leq d$). Consequently $K_1|_\Omega$ is generated by elements with support of bounded cardinality (at most $2d$). It follows, in the terminology of P. M. Neumann ([3], p. 563), that $K_1|_\Omega$ cannot be totally imprimitive and therefore must be almost primitive. Thus there is a K_1-invariant congruence $\mathfrak{q}$ on Ω such that $K_1|_{\Omega/\mathfrak{q}}$ is primitive and hence is either $\mathrm{Alt}(\Omega/\mathfrak{q})$ or $\mathrm{FSym}(\Omega/\mathfrak{q})$; see [3], 2.3. Moreover each $\omega\mathfrak{q} = \{\sigma \in \Omega : \ \omega\mathfrak{q}\sigma\}$ is finite. Let L_1 be the kernel of the action of K_1 on $\Omega/\mathfrak{q}$, so $L \leq L_1 \lhd K_1$. For any subset Σ of Ω, write V_Σ for $\bigoplus_{\sigma\in\Sigma} V_\sigma$.

Let $g \in K_2$ with $L_1^g \neq L_1$. Then $(K_1\colon\ L_1^g L_1) \leq 2$, since $\mathrm{Alt}(\Omega/\mathfrak{q})$ is simple of index 2 in $\mathrm{FSym}(\Omega/\mathfrak{q})$. Also for any ω in Ω there is a finite subset Σ of Ω with

$$V_{\omega\mathfrak{q}} g(L_1^g)L_1 = V_{\omega\mathfrak{q}} g L_1 \leq V_{\Sigma\mathfrak{q}} = V_\Sigma.$$

But $V_\omega g K_1 = V$, so

$$\dim_D V \leq 2 \cdot \dim_D(V_{\omega\mathfrak{q}} g(L_1^g)L_1) \leq 2 \cdot |\Sigma| \cdot \dim_D V_\omega < \infty.$$

This contradiction yields that L_1 is normal in K_2. Hence the D-L_1 homogeneous components of V form a system of imprimitivity for K_2 in V. But the choice of L, U and W above ensures that these are just the V_ω. Therefore $V = \bigoplus_\Omega V_\omega$ is also a system of imprimitivity for K_2. We may repeat the above arguments with K_2 in place of K_1. A simple transfinite induction yields that $V = \bigoplus_\Omega V_\omega$ is a system of imprimitivity for G. This contradiction of the primitivity of G completes the proof that H is irreducible. The primitivity of H follows at once from the following lemma.

Lemma. *Let V be an infinite-dimensional left vector space over the division ring D and G a subgroup of* $\mathrm{FGL}(V)$*. The following are equivalent.*

a) *G is irreducible and primitive.*

b) *$G \neq \langle 1\rangle$ and every non-trivial normal subgroup of G is irreducible.*

Proof. As we have seen above a) implies b) by [4], 3.1, and [5], Prop. 8. Suppose b) holds. Clearly G is irreducible. Consider a non-trivial system $V = \bigoplus_{\omega\in\Omega} V_\omega$ of imprimitivity for G. Then G acts transitively and finitarily on Ω and each $\dim_D V_\omega$ is finite. Also $N = \bigcap_\Omega N_G(V_\omega)$ is a reducible normal subgroup of G and hence by b) is $\langle 1\rangle$. This holds for any such system of imprimitivity. If $G|_\Omega$ is totally imprimitive, then every element of G lies in some such N. Hence $G|_\Omega$ is almost primitive and hence for some such system of imprimitivity $G|_\Omega$ is $\mathrm{Alt}(\Omega)$ or $\mathrm{FSym}(\Omega)$.

Let $\omega \in \Omega$ and $g \in N_G(\omega)$. Then the support $\mathrm{supp}_\Omega(g)$ of g in Ω is finite and $C_G(g) \geq \mathrm{Alt}(\Omega\backslash \mathrm{supp}_\Omega(g))$. The latter is simple and does not lie in $N_G(\omega)$. Hence

$$|\omega C_G(g)| = (C_G(g)\colon\ C_G(g) \cap N_G(\omega)),$$

which is infinite. If $[V_\omega, g] \neq \{0\}$, then $[V_\omega x, g] \neq \{0\}$ for all x in $C_G(g)$ and $\dim_D[V, g] \geq |\omega C_G(g)|$ is infinite. Consequently g and $N_G(\omega)$ centralize V_ω. Let $v_\omega \in V_\omega\backslash\{0\}$ and let X be a right transversal of $N_G(\omega)$ to G. Then $U = \bigoplus_{x\in X} Dv_\omega x \leq V$ is a permutation module for G and $\sum_{x,y\in X} D(v_\omega x - v_\omega y)$ is a proper D-G submodule of U and hence of V. This contradicts the irreducibility of G and so G is primitive as claimed.

Now suppose $\dim_D V$ is finite but otherwise assume the notation of the theorem. Clearly H now need not be irreducible; just let H be the center of $G = GL(V)$. If H is normal, then H is homogeneous and hence has no non-zero fixed-points in V. However if H is only subnormal, then H can have non-zero fixed-points. For example let G be the group (E) of [2], p. 239. Then G is an irreducible primitive subgroup of $GL(3, \mathbb{C})$ of order 108 and is the split extension of a non-abelian normal subgroup N of order 27 and exponent 3 (it is a copy of $\mathrm{Tr}_1(3, 3)$) containing $a = \mathrm{diag}(1, \exp(2\pi/3), \exp(4\pi/3))$ and a cyclic group of order 4. Also N is nilpotent of class 2, the subgroup $\langle a\rangle$ is subnormal in G of subnormal depth 3 and $(1, 0, 0)$ is a non-zero fixed-point of a.

References

[1] U. Meierfrankenfeld, *Ascending subgroups of irreducible finitary linear group*, J. London Math. Soc. **51** (1995), 75–92. MR **96c:**20092
[2] G. A. Miller, H. F. Blichfeldt and L. E. Dickson, *Theory and applications of finite groups*, Dover Reprint, New York, 1961. MR **23:**A925
[3] P. M. Neumann, *The lawlessness of groups of finitary permutations*, Arch. Math. **26** (1975), 561–566. MR **54:**406
[4] B. A. F. Wehrfritz, *Primitive finitary skew linear groups*, Arch. Math. **62** (1994), 393–400. MR **95i:**20071
[5] ———, *Locally soluble primitive finitary skew linear groups*, Communications in Algebra **23** (1995), 803–817. MR **96a:**20070

School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, England
E-mail address: b.a.f.wehrfritz@qmw.ac.uk