

## CALIBRATED THIN $\Pi_1^1$ $\sigma$ -IDEALS ARE $G_\delta$

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**ABSTRACT.** Let  $E$  be a compact metric space, and let  $I \subset \mathcal{K}(E)$  be a calibrated thin  $\Pi_1^1$   $\sigma$ -ideal. Then  $I$  is  $G_\delta$ . This solves an open problem, which was posed by Kechris, Louveau and Woodin. Using our result we obtain a new proof of Kaufman's theorem concerning  $U$ -sets and  $U_0$ -sets.

The study of  $\sigma$ -ideals of compact sets has been motivated by problems in harmonic analysis, namely by problems concerning  $U$ -sets and  $U_0$ -sets. The theory of  $\sigma$ -ideals of compact sets was developed by Kechris, Louveau and Woodin in [3]. Their results on the structure of  $\sigma$ -ideals of compact sets were used by Debs and Saint-Raymond ([1]) to give a positive answer to the old question of whether every Borel  $U$ -set is meager. The main aim of this paper is to give an answer to an open question which was posed in [3], namely to prove the assertion from the title. Using our result we obtain a new proof of Kaufman's theorem concerning  $U$ -sets and  $U_0$ -sets.

Let  $E$  be a compact metric space, and  $\mathcal{K}(E)$  be the space of all closed subsets of  $E$  with the Hausdorff metric

$$\begin{aligned}\delta(K, L) &= \sup\{\max(\text{dist}(x, K), \text{dist}(y, L)); x \in L, y \in K\}, \text{ if } K, L \neq \emptyset, \\ &= \text{diam}(E) + 1, \text{ if } K = \emptyset, L \neq \emptyset, \text{ or } K \neq \emptyset, L = \emptyset, \\ &= 0, \text{ if } K = L = \emptyset.\end{aligned}$$

If  $A \subset E$ , then  $\mathcal{K}(A)$  stands for the set of all compact subsets of  $A$ . A set  $I \subset \mathcal{K}(E)$  is called  $\sigma$ -ideal if

- (i)  $K, L \in \mathcal{K}(E)$ ,  $K \in I$ ,  $L \subset K$ , then  $L \in I$ ,
- (ii)  $K, K_1, K_2, \dots \in \mathcal{K}(E)$ ,  $K_n \in I$  for all  $n \in \mathbb{N}$  and  $K = \bigcup_{n=1}^{+\infty} K_n$ , then  $K \in I$ .

We say that a  $\sigma$ -ideal  $I$  is *calibrated* if, whenever  $F \in \mathcal{K}(E)$ ,  $F_n \in I$  for every  $n \in \mathbb{N}$ ,  $\mathcal{K}(F \setminus \bigcup_{n=1}^{+\infty} F_n) \subset I$ , then  $F \in I$ . A  $\sigma$ -ideal  $I \subset \mathcal{K}(E)$  is said to be *thin* if  $E$  contains no uncountable family of pairwise disjoint closed sets which are not in  $I$ . A  $\sigma$ -ideal  $I$  is called *locally non-Borel* if for every closed set  $F \notin I$ ,  $I \cap \mathcal{K}(F)$  is not Borel.

Let  $X$  be a Polish space and  $P \subset X$  be  $\Pi_1^1$ . A mapping  $\varphi : X \rightarrow [0, \omega_1]$  is said to be  $\Pi_1^1$ -rank on  $P$  if

- (i)  $\forall x \in X \setminus P : \varphi(x) = \omega_1$ ,
- (ii)  $\forall x \in P : \varphi(x) < \omega_1$ ,

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- (iii)  $\{(x, y) \in X \times X; \varphi(x) < \varphi(y)\}$  is  $\Pi_1^1$  in  $X \times X$ ,
- (iv)  $\{(x, y) \in X \times X; x \in P, \varphi(x) \leq \varphi(y)\}$  is  $\Pi_1^1$  in  $X \times X$ .

*Remark.* In the following, only properties described below in Theorems C, D and E of  $\Pi_1^1$ -rank will be important for us.

A subset  $P$  of  $X$  is called  $\Pi_1^1$ -complete if it is  $\Pi_1^1$  and, for any Polish space  $Y$  and any  $\Pi_1^1$  subset  $Q$  of  $Y$ , there is a Borel mapping  $f : Y \rightarrow X$  such that  $Q = f^{-1}(P)$ . It is easy to see that no  $\Pi_1^1$ -complete set is analytic.

The following problem was posed in [3]: Is every calibrated thin  $\Pi_1^1$   $\sigma$ -ideal  $I \subset \mathcal{K}(E)$ , where  $E$  is a compact metric space, necessarily  $\mathbf{G}_\delta$ ?

A partial answer was given in [5]. We will prove:

**Theorem 1.** *Let  $E$  be a compact metric space and  $I \subset \mathcal{K}(E)$  be a calibrated thin  $\Pi_1^1$   $\sigma$ -ideal. Then  $I$  is a  $\mathbf{G}_\delta$  subset of  $\mathcal{K}(E)$ .*

We will use the following theorems.

**Theorem A** ([2], p. 133). *Let  $E$  be a compact metric space. Let  $P, B$  be two disjoint subsets of  $E$  with  $P$  in  $\Sigma_1^1$ . If there is no  $\mathbf{F}_\sigma$  set  $C$  separating  $P$  from  $B$  (i.e.  $P \subset C, B \cap C = \emptyset$ ), then there is a homeomorphic copy  $F$  of  $2^{\mathbb{N}}$  with  $F \subset P \cup B$ , and  $F \cap B$  is countable dense in  $F$ .*

**Theorem B** ([2], p. 132). *Let  $E$  be a compact metric space. Then every  $\Pi_1^1$   $\sigma$ -ideal  $I \subset \mathcal{K}(E)$  is either  $\mathbf{G}_\delta$  or else  $\Pi_1^1$ -complete.*

**Theorem C** ([2], p. 144). *Any  $\Pi_1^1$  subset of a Polish space admits a  $\Pi_1^1$ -rank.*

**Theorem D** ([2], p. 144). *Let  $X$  be a Polish space,  $P \subset X$  be a  $\Pi_1^1$  set and  $\varphi$  be a  $\Pi_1^1$ -rank on  $P$ . Then for every ordinal  $\alpha < \omega_1$  the set  $\{x \in X; \varphi(x) < \alpha\}$  is Borel.*

**Theorem E** ([2], p. 148). *Let  $X$  be a Polish space,  $P \subset X$  be a  $\Pi_1^1$  set and  $\varphi$  be a  $\Pi_1^1$ -rank on  $P$ . If  $Q \subset P$  is  $\Sigma_1^1$ , then  $\varphi$  is bounded on  $Q$ , i.e.  $\sup\{\varphi(x); x \in Q\} < \omega_1$ .*

Let  $E$  be a compact metric space,  $K \in \mathcal{K}(E)$ ,  $r > 0$ . Then a ball in the space  $\mathcal{K}(E)$  with the center  $K$  and with the radius  $r > 0$  is denoted by  $B(K, r)$ .

We start with an easy observation.

**Lemma 2.** *Let  $E$  be a compact metric space,  $K, L \in \mathcal{K}(E)$ ,  $K \neq L, K \cap L = \emptyset$ . Then there exists  $\varepsilon > 0$  such that  $\overline{B(K, \varepsilon)} \cap \overline{B(L, \varepsilon)} = \emptyset$  and if  $K' \in \overline{B(K, \varepsilon)}, L' \in \overline{B(L, \varepsilon)}$ , then  $K' \cap L' = \emptyset$ .*

*Proof.* If  $K = \emptyset$  or  $L = \emptyset$ , then we put  $\varepsilon = \frac{1}{4}(\text{diam}(E) + 1)$  and we are done. Otherwise  $\varepsilon = \frac{1}{4}\text{dist}(K, L)$ . If  $K' \in \overline{B(K, \varepsilon)}, L' \in \overline{B(L, \varepsilon)}$ , then  $\text{dist}(K', L') \geq \frac{1}{2}\varepsilon > 0$  and the proof is complete.  $\square$

**Theorem 3.** *Let  $E$  be a compact metric space,  $I \subset \mathcal{K}(E)$ ,  $I \neq \mathcal{K}(E)$  be a calibrated locally non-Borel  $\Pi_1^1$   $\sigma$ -ideal. Then there exists a family of pairwise disjoint elements from  $\mathcal{K}(E) \setminus I$  with the cardinality of the continuum.*

*Proof.* There exists an unbounded  $\Pi_1^1$ -rank  $\varphi$  on  $I$  (see Theorems B, C and D). We will construct a transfinite sequence  $(K_\alpha)_{\alpha < \omega_1}$  such that

- (i)  $K_\alpha \in I$ ,
- (ii)  $K_\alpha \cap K_\beta = \emptyset$  for every  $\alpha, \beta < \omega_1, \alpha \neq \beta$ ,

(iii)  $\varphi(K_\alpha) > \alpha$ .

Choose  $K_0 \in I$  such that  $\varphi(K_0) > 0$ . Suppose that for  $\lambda < \omega_1$  we have constructed  $K_\alpha$ ,  $\alpha < \lambda$  fulfilling (i), (ii) and (iii). There exists  $T \in \mathcal{K}(E) \setminus I$  such that  $T \cap \bigcup_{\alpha < \lambda} K_\alpha = \emptyset$ ; otherwise the calibration of  $I$  gives  $E \in I$  and it implies  $I = \mathcal{K}(E)$ , a contradiction. If

$$\sup\{\varphi(K); K \in I, K \subset T\} < \xi < \omega_1,$$

then

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(E); \varphi(K) < \xi\} \subset I.$$

We obtain

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(T); \varphi(K) < \xi\} \subset I \cap \mathcal{K}(T).$$

This and Theorem D imply that  $I \cap \mathcal{K}(T)$  is a Borel set. It is a contradiction with locally non-Borelness of  $I$ . Thus we have

$$\sup\{\varphi(K); K \in I, K \subset T\} = \omega_1.$$

We choose  $K_\lambda \in I \cap \mathcal{K}(T)$  such that  $\varphi(K_\lambda) > \lambda$ . It finishes the construction of  $(K_\alpha)_{\alpha < \omega_1}$ . Denote  $B = \{K_\alpha; \alpha < \omega_1\}$ . Let  $C$  be a Borel set such that  $B \subset C \subset I$ . Then we have

$$\sup\{\varphi(K); K \in C\} \geq \sup\{\varphi(K); K \in B\} = \omega_1.$$

It is a contradiction, since  $\varphi$  must be bounded on each Borel subset of  $I$  (Theorem E). Now Theorem A gives that there exists a homeomorphic copy  $F$  of  $2^{\mathbb{N}}$  such that  $F \subset B \cup \mathcal{K}(E) \setminus I$  and  $Q = F \cap B$  is countable and dense in  $F$ .

Following [2], p. 111, we denote the set of all finite sequences from  $\{0, 1\}$  by  $\text{Seq}\{0, 1\}$  and if  $\iota \in \text{Seq}\{0, 1\}$ , then  $\iota \hat{\ } 0$  ( $\iota \hat{\ } 1$ , respectively) stands for the concatenation of the sequences  $\iota$  and  $(0)$  ( $\iota$  and  $(1)$ , respectively).

Now for every  $\iota \in \text{Seq}\{0, 1\}$  we construct  $P_\iota \in \mathcal{K}(E)$ ,  $\varepsilon_\iota > 0$  such that

- (i)  $P_\iota \in Q$ ,
- (ii) for every  $K_0 \in \overline{B(P_{\iota \hat{\ } 0}, \varepsilon_{\iota \hat{\ } 0})}$ ,  $K_1 \in \overline{B(P_{\iota \hat{\ } 1}, \varepsilon_{\iota \hat{\ } 1})}$  we have  $K_0 \cap K_1 = \emptyset$ ,
- (iii)  $\varepsilon_\iota < 2^{-|\iota|}$  ( $|\iota|$  stands for the length of  $\iota$ ),
- (iv)  $\overline{B(P_{\iota \hat{\ } 0}, \varepsilon_{\iota \hat{\ } 0})} \cap \overline{B(P_{\iota \hat{\ } 1}, \varepsilon_{\iota \hat{\ } 1})} = \emptyset$ ,
- (v)  $\overline{B(P_{\iota \hat{\ } 0}, \varepsilon_{\iota \hat{\ } 0})} \cup \overline{B(P_{\iota \hat{\ } 1}, \varepsilon_{\iota \hat{\ } 1})} \subset B(P_\iota, \varepsilon_\iota)$ .

Choose  $P_\emptyset \in Q$ ,  $\varepsilon_\emptyset \in (0, 1)$ . Now suppose that  $P_\iota$ ,  $\varepsilon_\iota$  were defined. Since  $F$  has no isolated point and  $Q$  is dense in  $F$ , we can choose  $P_{\iota \hat{\ } 0}$ ,  $P_{\iota \hat{\ } 1} \in Q \cap B(P_\iota, \varepsilon_\iota)$  such that  $P_{\iota \hat{\ } 0} \neq P_{\iota \hat{\ } 1}$ . As  $Q \subset B$  we have  $P_{\iota \hat{\ } 0} \cap P_{\iota \hat{\ } 1} = \emptyset$ . According to Lemma 2 there exists  $\varepsilon \in (0, 2^{-|\iota|-1})$  such that the conditions (ii) and (iv) are fulfilled for  $P_{\iota \hat{\ } 0}$ ,  $P_{\iota \hat{\ } 1}$  and  $\varepsilon_{\iota \hat{\ } 0} = \varepsilon_{\iota \hat{\ } 1} = \varepsilon$ . If  $\varepsilon$  is sufficiently small, then the condition (v) is also fulfilled. Put

$$W = \bigcap_{n=1}^{+\infty} \bigcup_{|\iota|=n} B(P_\iota, \varepsilon_\iota).$$

Clearly  $W \subset F$  and  $W$  is a homeomorphic copy of  $2^{\mathbb{N}}$ . If  $L_0, L_1 \in W$ ,  $L_0 \neq L_1$ , then there exist  $\iota, \tau \in \text{Seq}\{0, 1\}$  such that  $|\iota| = |\tau|$ ,  $\iota \neq \tau$ ,  $L_0 \in B(P_\iota, \varepsilon_\iota)$ ,  $L_1 \in B(P_\tau, \varepsilon_\tau)$ . The property (ii) implies that  $L_0 \cap L_1 = \emptyset$ . Thus the elements of  $W$  form a family of pairwise disjoint sets. This implies that  $W \setminus Q$  is a family of pairwise disjoint sets from  $\mathcal{K}(E) \setminus I$  with the cardinality of the continuum and we are done.  $\square$

We introduce the notion of a  $I$ -thin set to exhibit an interesting corollary of Theorem 3.

Let  $E$  be a compact metric space, and let  $I \subset \mathcal{K}(E)$  be a  $\sigma$ -ideal. A set  $A \subset E$  is  $I$ -thin if there is no uncountable family  $\Phi \subset \mathcal{K}(A)$  of pairwise disjoint sets which are not in  $I$ .

**Corollary 4.** *Let  $E$  be a compact metric space, and let  $I \subset \mathcal{K}(E)$  be a calibrated locally non-Borel  $\Pi_1^1$   $\sigma$ -ideal. Then  $K \in \mathcal{K}(E)$  is  $I$ -thin if and only if  $K \in I$ .*

Since  $\sigma$ -ideals  $U$  (of closed sets of uniqueness in the unit circle  $\mathbb{T}$ ) and  $U_0$  (of closed sets of extended uniqueness in the unit circle  $\mathbb{T}$ ) (cf. [2]) fulfill the conditions in Theorem 3 we have the following corollary.

**Corollary 5** (Kaufman, see [2], p. 235). *Let  $K \in \mathcal{K}(\mathbb{T}) \setminus U$ . Then we can find a family  $\{K_x\}_{x \in \mathbb{Z}^{\mathbb{N}}}$  of pairwise disjoint sets from  $\mathcal{K}(\mathbb{T}) \setminus U$  contained in  $K$ . Similarly replace  $U$  by  $U_0$ .*

Corollary 4 and Corollary 5 follow from Theorem 3 and the following easy observation.

**Observation 6.** *Let  $E$  be a metric compact space,  $I \subset \mathcal{K}(E)$  be a calibrated locally non-Borel  $\Pi_1^1$   $\sigma$ -ideal and  $F \in \mathcal{K}(E) \setminus I$ . Then the  $\sigma$ -ideal  $I \cap \mathcal{K}(F)$  is a calibrated locally non-Borel  $\Pi_1^1$   $\sigma$ -ideal in the space  $\mathcal{K}(F)$ .*

**Lemma 7.** *Let  $E$  be a compact metric space and  $K \in \mathcal{K}(E)$ . Then*

$$f_K : L \mapsto L \cap K$$

*is a Borel mapping from  $\mathcal{K}(E)$  to  $\mathcal{K}(E)$ .*

*Proof.* It is well-known that  $\mathcal{K}(E)$  is a compact metric space and that the topology induced by the Hausdorff metric and the topology generated by the sets of the form  $\{L \in \mathcal{K}(E); L \subset V\}$ ,  $\{L \in \mathcal{K}(E); L \cap V \neq \emptyset\}$  for  $V$  open in  $E$  coincide on  $\mathcal{K}(E)$  (cf. [2], p. 117, [4]). Therefore it is sufficient to prove that the sets

$$X = \{L \in \mathcal{K}(E); L \cap K \cap G \neq \emptyset\}, \quad Y = \{L \in \mathcal{K}(E); L \cap K \subset G\}$$

are Borel for every  $G$  open in  $E$ . There exist closed sets  $F_n$ ,  $n \in \mathbb{N}$ , such that  $K \cap G = \bigcup_{n=1}^{+\infty} F_n$ . We have

$$\{L \in \mathcal{K}(E); L \cap K \cap G \neq \emptyset\} = \bigcup_{n=1}^{+\infty} \{L \in \mathcal{K}(E); L \cap F_n \neq \emptyset\}$$

and therefore  $X$  is Borel. The set  $Y$  is also Borel, since

$$\{L \in \mathcal{K}(E); L \cap K \subset G\} = \mathcal{K}(E) \setminus \{L \in \mathcal{K}(E); L \cap K \cap (E \setminus G) \neq \emptyset\}. \quad \square$$

*Proof of Theorem 1.* If  $E \in I$ , then we are done. Suppose that  $E \notin I$ . Let  $\mathcal{F} \subset \mathcal{K}(E)$  be a maximal system (with respect to the inclusion) of pairwise disjoint elements from  $\mathcal{K}(E) \setminus I$  such that  $I \cap \mathcal{K}(F)$  is  $\mathbf{G}_\delta$  in  $\mathcal{K}(E)$ , whenever  $F \in \mathcal{F}$ . The system  $\mathcal{F}$  is countable since  $I$  is thin. Put

$$I^* = \{K \in \mathcal{K}(E); \text{ for every } F \in \mathcal{F} \text{ we have } K \cap F \in I\}.$$

We claim  $I = I^*$ . Clearly  $I \subset I^*$ . Suppose that  $K \in I^*$ . Let  $L \in \mathcal{K}(E)$ ,  $L \cap \bigcup\{F; F \in \mathcal{F}\} = \emptyset$ . If  $L$  is an element from  $\mathcal{K}(E) \setminus I$ , then the  $\sigma$ -ideal  $\mathcal{K}(L) \cap I$  must be locally non-Borel in the space  $\mathcal{K}(L)$ ; otherwise we obtain a contradiction with the maximality of  $\mathcal{F}$ . Using Observation 6 and Theorem 3 we obtain that the

$\sigma$ -ideal  $\mathcal{K}(L) \cap I$  contains an uncountable family of pairwise disjoint closed sets which are not in  $I$ . This contradicts the thinness of  $I$ . Thus we can conclude

$$\mathcal{K}(K \setminus \bigcup\{F; F \in \mathcal{F}\}) \subset I.$$

Since  $K \setminus \bigcup\{F; F \in \mathcal{F}\} = K \setminus \bigcup\{F \cap K; F \in \mathcal{F}\}$ ,  $\mathcal{F}$  is countable and  $I$  is calibrated, we obtain  $K \in I$ . Using Lemma 7 and the following fact

$$I = I^* = \bigcap\{f_F^{-1}(I); F \in \mathcal{F}\} = \bigcap\{f_F^{-1}(I \cap \mathcal{K}(F)); F \in \mathcal{F}\}$$

we can conclude that  $I$  is Borel, therefore  $\mathbf{G}_\delta$  (Theorem B) and the proof is complete.  $\square$

*Remark.* Let us note that the  $\sigma$ -ideal of countable compact subsets of the interval  $[0, 1]$  is calibrated  $\Pi_1^1$  and is not  $\mathbf{G}_\delta$ . This means that thinness cannot be omitted in our theorem. We will show that the same holds for calibration. We will give only a sketch of the proof. Let  $E = [0, 1]$  and let  $(V_n)_{n=1}^{+\infty}$  be a sequence of non-empty open sets forming an open base of  $E$ . Let  $(F_n)_{n=1}^{+\infty}$  be a sequence of pairwise disjoint closed subsets of  $E$  satisfying the following conditions for every  $n \in \mathbb{N}$ :

- (i)  $F_n \subset V_n$ ,
- (ii)  $F_n$  has positive Lebesgue measure.

(Of course  $F_n$ 's are nowhere dense.) Put

$$B = \{K \in \mathcal{K}(E); K \text{ has null Lebesgue measure}\} \cup \bigcup_{n=1}^{+\infty} \mathcal{K}(F_n).$$

The set  $B$  is Borel and hereditary (i.e. if  $K, L \in \mathcal{K}(E)$ ,  $K \in B$ ,  $L \subset K$ , then  $L \in B$ ). Put

$$I = B_\sigma = \{K \in \mathcal{K}(E); K \text{ can be covered by countably many elements from } B\}.$$

Following [2] (pp. 197–198) we can consider the Cantor-Bendixson rank  $\text{rk}_B$  associated with the  $B$ -derivative. We will use the following theorem.

**Theorem F** ([2], p. 202). *Let  $E$  be a compact metric space,  $B$  be a Borel hereditary subset of  $\mathcal{K}(E)$  consisting of nowhere dense sets and  $I = B_\sigma$ . If every non-empty open subset of  $E$  contains  $K \in I$  with  $\text{rk}_B(K) > 1$ , then  $I$  is  $\Pi_1^1$ -complete.*

It is not difficult to check that  $B$  fulfills the conditions in the above theorem. It shows that  $I$  is  $\Pi_1^1$  and non- $\mathbf{G}_\delta$ . The  $\sigma$ -ideal  $I$  is thin since  $I$  contains thin  $\sigma$ -ideal of closed sets with Lebesgue null measure.

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