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CALIBRATED THIN Π_1^1 σ-IDEALS ARE G_δ

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ABSTRACT. Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a calibrated thin $\mathbf{\Pi_1^1}$ σ-ideal. Then I is $\boldsymbol{G}_\delta$. This solves an open problem, which was posed by Kechris, Louveau and Woodin. Using our result we obtain a new proof of Kaufman's theorem concerning U-sets and U_0-sets.

The study of σ-ideals of compact sets has been motivated by problems in harmonic analysis, namely by problems concerning U-sets and U_0-sets. The theory of σ-ideals of compact sets was developed by Kechris, Louveau and Woodin in [3]. Their results on the structure of σ-ideals of compact sets were used by Debs and Saint-Raymond ([1]) to give a positive answer to the old question of whether every Borel U-set is meager. The main aim of this paper is to give an answer to an open question which was posed in [3], namely to prove the assertion from the title. Using our result we obtain a new proof of Kaufman's theorem concerning U-sets and U_0-sets.

Let E be a compact metric space, and $\mathcal{K}(E)$ be the space of all closed subsets of E with the Hausdorff metric

$$\begin{aligned}\delta(K,L) &= \sup\{\max(\operatorname{dist}(x,K),\operatorname{dist}(y,L)); x\in L,\, y\in K\},\ \text{if } K,\, L\neq\emptyset,\\ &= \operatorname{diam}(E)+1,\ \text{if } K=\emptyset,\, L\neq\emptyset,\ \text{or } K\neq\emptyset,\, L=\emptyset,\\ &= 0,\ \text{if } K=L=\emptyset.\end{aligned}$$

If $A \subset E$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A. A set $I \subset \mathcal{K}(E)$ is called *σ-ideal* if

(i) $K,\, L \in \mathcal{K}(E)$, $K \in I$, $L \subset K$, then $L \in I$,
(ii) $K,\, K_1,\, K_2, \dots \in \mathcal{K}(E), K_n \in I$ for all $n \in \mathbb{N}$ and $K = \bigcup_{n=1}^{+\infty} K_n$, then $K \in I$.

We say that a σ-ideal I is *calibrated* if, whenever $F \in \mathcal{K}(E)$, $F_n \in I$ for every $n \in \mathbb{N}$, $\mathcal{K}(F \setminus \bigcup_{n=1}^{+\infty} F_n) \subset I$, then $F \in I$. A σ-ideal $I \subset \mathcal{K}(E)$ is said to be *thin* if E contains no uncountable family of pairwise disjoint closed sets which are not in I. A σ-ideal I is called *locally non-Borel* if for every closed set $F \notin I$, $I \cap \mathcal{K}(F)$ is not Borel.

Let X be a Polish space and $P \subset X$ be $\mathbf{\Pi_1^1}$. A mapping $\varphi : X \to [0,\omega_1]$ is said to be *$\mathbf{\Pi_1^1}$-rank on P* if

(i) $\forall x \in X \setminus P : \varphi(x) = \omega_1$,
(ii) $\forall x \in P : \varphi(x) < \omega_1$,

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(iii) $\{(x,y) \in X \times X;\, \varphi(x) < \varphi(y)\}$ is $\mathbf{\Pi^1_1}$ in $X \times X$,
(iv) $\{(x,y) \in X \times X;\, x \in P,\, \varphi(x) \leq \varphi(y)\}$ is $\mathbf{\Pi^1_1}$ in $X \times X$.

Remark. In the following, only properties described below in Theorems C, D and E of $\mathbf{\Pi^1_1}$-rank will be important for us.

A subset P of X is called $\mathbf{\Pi^1_1}$-*complete* if it is $\mathbf{\Pi^1_1}$ and, for any Polish space Y and any $\mathbf{\Pi^1_1}$ subset Q of Y, there is a Borel mapping $f : Y \to X$ such that $Q = f^{-1}(P)$. It is easy to see that no $\mathbf{\Pi^1_1}$-complete set is analytic.

The following problem was posed in [3]: Is every calibrated thin $\mathbf{\Pi^1_1}$ σ-ideal $I \subset \mathcal{K}(E)$, where E is a compact metric space, necessarily $\boldsymbol{G}_\delta$?

A partial answer was given in [5]. We will prove:

Theorem 1. *Let E be a compact metric space and $I \subset \mathcal{K}(E)$ be a calibrated thin $\mathbf{\Pi^1_1}$ σ-ideal. Then I is a $\boldsymbol{G}_\delta$ subset of $\mathcal{K}(E)$.*

We will use the following theorems.

Theorem A ([2], p. 133). *Let E be a compact metric space. Let P, B be two disjoint subsets of E with P in $\mathbf{\Sigma^1_1}$. If there is no $\boldsymbol{F}_\sigma$ set C separating P from B (i.e. $P \subset C$, $B \cap C = \emptyset$), then there is a homeomorphic copy F of $2^{\mathbb{N}}$ with $F \subset P \cup B$, and $F \cap B$ is countable dense in F.*

Theorem B ([2], p. 132). *Let E be a compact metric space. Then every $\mathbf{\Pi^1_1}$ σ-ideal $I \subset \mathcal{K}(E)$ is either $\boldsymbol{G}_\delta$ or else $\mathbf{\Pi^1_1}$-complete.*

Theorem C ([2], p. 144). *Any $\mathbf{\Pi^1_1}$ subset of a Polish space admits a $\mathbf{\Pi^1_1}$-rank.*

Theorem D ([2], p. 144). *Let X be a Polish space, $P \subset X$ be a $\mathbf{\Pi^1_1}$ set and φ be a $\mathbf{\Pi^1_1}$-rank on P. Then for every ordinal $\alpha < \omega_1$ the set $\{x \in X; \varphi(x) < \alpha\}$ is Borel.*

Theorem E ([2], p. 148). *Let X be a Polish space, $P \subset X$ be a $\mathbf{\Pi^1_1}$ set and φ be a $\mathbf{\Pi^1_1}$-rank on P. If $Q \subset P$ is $\mathbf{\Sigma^1_1}$, then φ is bounded on Q, i.e.* $\sup\{\varphi(x);\, x \in Q\} < \omega_1$.

Let E be a compact metric space, $K \in \mathcal{K}(E)$, $r > 0$. Then a ball in the space $\mathcal{K}(E)$ with the center K and with the radius $r > 0$ is denoted by $B(K,r)$.

We start with an easy observation.

Lemma 2. *Let E be a compact metric space, K, $L \in \mathcal{K}(E)$, $K \neq L$, $K \cap L = \emptyset$. Then there exists $\varepsilon > 0$ such that $\overline{B(K,\varepsilon)} \cap \overline{B(L,\varepsilon)} = \emptyset$ and if $K' \in \overline{B(K,\varepsilon)}$, $L' \in \overline{B(L,\varepsilon)}$, then $K' \cap L' = \emptyset$.*

Proof. If $K = \emptyset$ or $L = \emptyset$, then we put $\varepsilon = \frac{1}{4}(\operatorname{diam}(E) + 1)$ and we are done. Otherwise $\varepsilon = \frac{1}{4}\operatorname{dist}(K,L)$. If $K' \in \overline{B(K,\varepsilon)}$, $L' \in \overline{B(L,\varepsilon)}$, then $\operatorname{dist}(K',L') \geq \frac{1}{2}\varepsilon > 0$ and the proof is complete. □

Theorem 3. *Let E be a compact metric space, $I \subset \mathcal{K}(E)$, $I \neq \mathcal{K}(E)$ be a calibrated locally non-Borel $\mathbf{\Pi^1_1}$ σ-ideal. Then there exists a family of pairwise disjoint elements from $\mathcal{K}(E) \setminus I$ with the cardinality of the continuum.*

Proof. There exists an unbounded $\mathbf{\Pi^1_1}$-rank φ on I (see Theorems B, C and D). We will construct a transfinite sequence $(K_\alpha)_{\alpha<\omega_1}$ such that

(i) $K_\alpha \in I$,
(ii) $K_\alpha \cap K_\beta = \emptyset$ for every α, $\beta < \omega_1$, $\alpha \neq \beta$,

(iii) $\varphi(K_\alpha) > \alpha$.

Choose $K_0 \in I$ such that $\varphi(K_0) > 0$. Suppose that for $\lambda < \omega_1$ we have constructed K_α, $\alpha < \lambda$ fulfilling (i), (ii) and (iii). There exists $T \in \mathcal{K}(E) \setminus I$ such that $T \cap \bigcup_{\alpha<\lambda} K_\alpha = \emptyset$; otherwise the calibration of I gives $E \in I$ and it implies $I = \mathcal{K}(E)$, a contradiction. If

$$\sup\{\varphi(K);\, K \in I, K \subset T\} < \xi < \omega_1,$$

then

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(E);\, \varphi(K) < \xi\} \subset I.$$

We obtain

$$I \cap \mathcal{K}(T) \subset \{K \in \mathcal{K}(T);\, \varphi(K) < \xi\} \subset I \cap \mathcal{K}(T).$$

This and Theorem D imply that $I \cap \mathcal{K}(T)$ is a Borel set. It is a contradiction with locally non-Borelness of I. Thus we have

$$\sup\{\varphi(K);\, K \in I, K \subset T\} = \omega_1.$$

We choose $K_\lambda \in I \cap \mathcal{K}(T)$ such that $\varphi(K_\lambda) > \lambda$. It finishes the construction of $(K_\alpha)_{\alpha<\omega_1}$. Denote $B = \{K_\alpha;\, \alpha < \omega_1\}$. Let C be a Borel set such that $B \subset C \subset I$. Then we have

$$\sup\{\varphi(K);\, K \in C\} \geq \sup\{\varphi(K);\, K \in B\} = \omega_1.$$

It is a contradiction, since φ must be bounded on each Borel subset of I (Theorem E). Now Theorem A gives that there exists a homeomorphic copy F of $2^{\mathbb{N}}$ such that $F \subset B \cup \mathcal{K}(E) \setminus I$ and $Q = F \cap B$ is countable and dense in F.

Following [2], p. 111, we denote the set of all finite sequences from $\{0,1\}$ by $\mathrm{Seq}\{0,1\}$ and if $\iota \in \mathrm{Seq}\{0,1\}$, then $\iota\hat{\ }0$ ($\iota\hat{\ }1$, respectively) stands for the concatenation of the sequences ι and (0) (ι and (1), respectively).

Now for every $\iota \in \mathrm{Seq}\{0,1\}$ we construct $P_\iota \in \mathcal{K}(E)$, $\varepsilon_\iota > 0$ such that

(i) $P_\iota \in Q$,
(ii) for every $K_0 \in \overline{B(P_{\iota\hat{\ }0}, \varepsilon_{\iota\hat{\ }0})}$, $K_1 \in \overline{B(P_{\iota\hat{\ }1}, \varepsilon_{\iota\hat{\ }1})}$ we have $K_0 \cap K_1 = \emptyset$,
(iii) $\varepsilon_\iota < 2^{-|\iota|}$ ($|\iota|$ stands for the length of ι),
(iv) $\overline{B(P_{\iota\hat{\ }0}, \varepsilon_{\iota\hat{\ }0})} \cap \overline{B(P_{\iota\hat{\ }1}, \varepsilon_{\iota\hat{\ }1})} = \emptyset$,
(v) $\overline{B(P_{\iota\hat{\ }0}, \varepsilon_{\iota\hat{\ }0})} \cup \overline{B(P_{\iota\hat{\ }1}, \varepsilon_{\iota\hat{\ }1})} \subset B(P_\iota, \varepsilon_\iota)$.

Choose $P_\emptyset \in Q$, $\varepsilon_\emptyset \in (0,1)$. Now suppose that P_ι, ε_ι were defined. Since F has no isolated point and Q is dense in F, we can choose $P_{\iota\hat{\ }0}$, $P_{\iota\hat{\ }1} \in Q \cap B(P_\iota, \varepsilon_\iota)$ such that $P_{\iota\hat{\ }0} \neq P_{\iota\hat{\ }1}$. As $Q \subset B$ we have $P_{\iota\hat{\ }0} \cap P_{\iota\hat{\ }1} = \emptyset$. According to Lemma 2 there exists $\varepsilon \in (0, 2^{-|\iota|-1})$ such that the conditions (ii) and (iv) are fulfilled for $P_{\iota\hat{\ }0}$, $P_{\iota\hat{\ }1}$ and $\varepsilon_{\iota\hat{\ }0} = \varepsilon_{\iota\hat{\ }1} = \varepsilon$. If ε is sufficiently small, then the condition (v) is also fulfilled. Put

$$W = \bigcap_{n=1}^{+\infty} \bigcup_{|\iota|=n} B(P_\iota, \varepsilon_\iota).$$

Clearly $W \subset F$ and W is a homeomorphic copy of $2^{\mathbb{N}}$. If L_0, $L_1 \in W$, $L_0 \neq L_1$, then there exist ι, $\tau \in \mathrm{Seq}\{0,1\}$ such that $|\iota| = |\tau|$, $\iota \neq \tau$, $L_0 \in B(P_\iota, \varepsilon_\iota)$, $L_1 \in B(P_\tau, \varepsilon_\tau)$. The property (ii) implies that $L_0 \cap L_1 = \emptyset$. Thus the elements of W form a family of pairwise disjoint sets. This implies that $W \setminus Q$ is a family of pairwise disjoint sets from $\mathcal{K}(E) \setminus I$ with the cardinality of the continuum and we are done. □

We introduce the notion of a I-thin set to exhibit an interesting corollary of Theorem 3.

Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a σ-ideal. A set $A \subset E$ is *I-thin* if there is no uncountable family $\Phi \subset \mathcal{K}(A)$ of pairwise disjoint sets which are not in I.

Corollary 4. *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a calibrated locally non-Borel $\mathbf{\Pi_1^1}$ σ-ideal. Then $K \in \mathcal{K}(E)$ is I-thin if and only if $K \in I$.*

Since σ-ideals U (of closed sets of uniqueness in the unit circle $\mathbb{T}$) and U_0 (of closed sets of extended uniqueness in the unit circle $\mathbb{T}$) (cf. [2]) fulfill the conditions in Theorem 3 we have the following corollary.

Corollary 5 (Kaufman, see [2], p. 235). *Let $K \in \mathcal{K}(\mathbb{T}) \setminus U$. Then we can find a family $\{K_x\}_{x \in 2^{\mathbb{N}}}$ of pairwise disjoint sets from $\mathcal{K}(\mathbb{T}) \setminus U$ contained in K. Similarly replace U by U_0.*

Corollary 4 and Corollary 5 follow from Theorem 3 and the following easy observation.

Observation 6. *Let E be a metric compact space, $I \subset \mathcal{K}(E)$ be a calibrated locally non-Borel $\mathbf{\Pi_1^1}$ σ-ideal and $F \in \mathcal{K}(E) \setminus I$. Then the σ-ideal $I \cap \mathcal{K}(F)$ is a calibrated locally non-Borel $\mathbf{\Pi_1^1}$ σ-ideal in the space $\mathcal{K}(F)$.*

Lemma 7. *Let E be a compact metric space and $K \in \mathcal{K}(E)$. Then*

$$f_K : L \mapsto L \cap K$$

is a Borel mapping from $\mathcal{K}(E)$ to $\mathcal{K}(E)$.

Proof. It is well-known that $\mathcal{K}(E)$ is a compact metric space and that the topology induced by the Hausdorff metric and the topology generated by the sets of the form $\{L \in \mathcal{K}(E);\, L \subset V\}$, $\{L \in \mathcal{K}(E);\, L \cap V \neq \emptyset\}$ for V open in E coincide on $\mathcal{K}(E)$ (cf. [2], p. 117, [4]). Therefore it is sufficient to prove that the sets

$$X = \{L \in \mathcal{K}(E);\, L \cap K \cap G \neq \emptyset\}, \qquad Y = \{L \in \mathcal{K}(E);\, L \cap K \subset G\}$$

are Borel for every G open in E. There exist closed sets F_n, $n \in \mathbb{N}$, such that $K \cap G = \bigcup_{n=1}^{+\infty} F_n$. We have

$$\{L \in \mathcal{K}(E);\, L \cap K \cap G \neq \emptyset\} = \bigcup_{n=1}^{+\infty} \{L \in \mathcal{K}(E);\, L \cap F_n \neq \emptyset\}$$

and therefore X is Borel. The set Y is also Borel, since

$$\{L \in \mathcal{K}(E);\, L \cap K \subset G\} = \mathcal{K}(E) \setminus \{L \in \mathcal{K}(E);\, L \cap K \cap (E \setminus G) \neq \emptyset\}. \quad \square$$

Proof of Theorem 1. If $E \in I$, then we are done. Suppose that $E \notin I$. Let $\mathcal{F} \subset \mathcal{K}(E)$ be a maximal system (with respect to the inclusion) of pairwise disjoint elements from $\mathcal{K}(E) \setminus I$ such that $I \cap \mathcal{K}(F)$ is $\boldsymbol{G}_\delta$ in $\mathcal{K}(E)$, whenever $F \in \mathcal{F}$. The system $\mathcal{F}$ is countable since I is thin. Put

$$I^\star = \{K \in \mathcal{K}(E);\ \text{for every } F \in \mathcal{F} \text{ we have } K \cap F \in I\}.$$

We claim $I = I^\star$. Clearly $I \subset I^\star$. Suppose that $K \in I^\star$. Let $L \in \mathcal{K}(E)$, $L \cap \bigcup\{F;\, F \in \mathcal{F}\} = \emptyset$. If L is an element from $\mathcal{K}(E) \setminus I$, then the σ-ideal $\mathcal{K}(L) \cap I$ must be locally non-Borel in the space $\mathcal{K}(L)$; otherwise we obtain a contradiction with the maximality of $\mathcal{F}$. Using Observation 6 and Theorem 3 we obtain that the

σ-ideal $\mathcal{K}(L) \cap I$ contains an uncountable family of pairwise disjoint closed sets which are not in I. This contradicts the thinness of I. Thus we can conclude

$$\mathcal{K}(K \setminus \bigcup\{F;\, F \in \mathcal{F}\}) \subset I.$$

Since $K \setminus \bigcup\{F;\, F \in \mathcal{F}\} = K \setminus \bigcup\{F \cap K;\, F \in \mathcal{F}\}$, $\mathcal{F}$ is countable and I is calibrated, we obtain $K \in I$. Using Lemma 7 and the following fact

$$I = I^\star = \bigcap\{f_F^{-1}(I);\, F \in \mathcal{F}\} = \bigcap\{f_F^{-1}(I \cap \mathcal{K}(F));\, F \in \mathcal{F}\}$$

we can conclude that I is Borel, therefore $\boldsymbol{G_\delta}$ (Theorem B) and the proof is complete. ◻

Remark. Let us note that the σ-ideal of countable compact subsets of the interval $[0,1]$ is calibrated $\mathbf{\Pi_1^1}$ and is not $\boldsymbol{G_\delta}$. This means that thinness cannot be omitted in our theorem. We will show that the same holds for calibration. We will give only a sketch of the proof. Let $E = [0,1]$ and let $(V_n)_{n=1}^{+\infty}$ be a sequence of non-empty open sets forming an open base of E. Let $(F_n)_{n=1}^{+\infty}$ be a sequence of pairwise disjoint closed subsets of E satisfying the following conditions for every $n \in \mathbb{N}$:

(i) $F_n \subset V_n$,
(ii) F_n has positive Lebesgue measure.

(Of course F_n's are nowhere dense.) Put

$$B = \{K \in \mathcal{K}(E); K \text{ has null Lebesgue measure}\} \cup \bigcup_{n=1}^{+\infty} \mathcal{K}(F_n).$$

The set B is Borel and hereditary (i.e. if $K, L \in \mathcal{K}(E)$, $K \in B$, $L \subset K$, then $L \in B$). Put

$$I = B_\sigma = \{K \in \mathcal{K}(E); K \text{ can be covered by countably many elements from } B\}.$$

Following [2] (pp. 197–198) we can consider the Cantor-Bendixson rank rk_B associated with the B-derivative. We will use the following theorem.

Theorem F ([2], p. 202). *Let E be a compact metric space, B be a Borel hereditary subset of $\mathcal{K}(E)$ consisting of nowhere dense sets and $I = B_\sigma$. If every non-empty open subset of E contains $K \in I$ with $rk_B(K) > 1$, then I is $\mathbf{\Pi_1^1}$-complete.*

It is not difficult to check that B fulfills the conditions in the above theorem. It shows that I is $\mathbf{\Pi_1^1}$ and non-$\boldsymbol{G_\delta}$. The σ-ideal I is thin since I contains thin σ-ideal of closed sets with Lebesgue null measure.

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