A CHARACTERIZATION OF CANCELLATION IDEALS

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Abstract. An ideal \( I \) of a commutative ring \( R \) with identity is called a cancellation ideal if whenever \( IB = IC \) for ideals \( B \) and \( C \) of \( R \), then \( B = C \). We show that an ideal \( I \) is a cancellation ideal if and only if \( I \) is locally a regular principal ideal.

Let \( R \) be a commutative ring with identity. An ideal \( I \) of \( R \) is called a cancellation ideal if whenever \( IB = IC \) for ideals \( B \) and \( C \) of \( R \), then \( B = C \). It is easily seen that \( I \) is a cancellation ideal if and only if whenever \( IB \subseteq IC \) for ideals \( B \) and \( C \) of \( R \), then \( B \subseteq C \). A good introduction to cancellation ideals may be found in Gilmer [1, Section 6]. As for examples, it is easy to see that a principal ideal \( (a) \) is a cancellation ideal if and only if \( (a) \) is a regular ideal (i.e., \( a \) is not a zero divisor). An invertible ideal is a cancellation ideal. More generally, an ideal that is locally a regular principal ideal is a cancellation ideal. The purpose of this paper is to prove the converse.

Kaplansky [2, Theorem 287] proved that a finitely generated cancellation ideal in a quasi-local domain is principal. We begin with the following lemma which is a modification of Kaplansky’s result (see [1, Exercise 7, page 67]). We use essentially the same argument.

Lemma. Let \( R \) be a commutative ring with identity and let \( I \) be a cancellation ideal of \( R \). Suppose that \( I = (x, y) + A \) where \( A \) is an ideal of \( R \) containing \( MI \) for some maximal ideal \( M \). Then \( I = (x) + A \) or \( I = (y) + A \).

Proof. Put \( J = (x^2 + y^2, xy, xA, yA, A^2) \). Then it is easily checked that \( IJ = I^2 \). Since \( I \) is a cancellation ideal, we have \( J = I^2 \). Thus \( x^2 = \lambda (x^2 + y^2) + \text{terms from } (xy, xA, yA, A^2) \). First, suppose that \( \lambda \in M \). Since \( \lambda x \in MI \subseteq A \), we have \( x^2 \in (y^2, xy, xA, yA, A^2) \). Let \( K = (y) + A \). Then \( I^2 = IK \). Since \( I \) is a cancellation ideal, we have \( I = K \). Next, suppose that \( \lambda \notin M \). Then for some \( \mu \in R \) and \( m \in M \), we have \( \mu (-\lambda) = 1 + m \). Now \( -\mu \lambda y^2 = \mu (\lambda - 1) x^2 + \text{terms from } (xy, xA, yA, A^2) \). Since \( my^2 = (my) y \in (MI) y \subseteq Ay \), we have \( y^2 \in (x^2, xy, xA, yA, A^2) \). Thus, as in the first case, we get that \( I = (x) + A \). \( \Box \)

Theorem. Let \( R \) be a commutative ring with identity. An ideal \( I \) of \( R \) is a cancellation ideal if and only if \( I \) is locally a regular principal ideal.

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Proof. We have already remarked that an ideal that is locally a regular principal ideal is a cancellation ideal. Conversely, suppose that \( I \) is a cancellation ideal. Let \( M \) be a maximal ideal of \( R \). We show that \( I_M \) is a regular principal ideal. We may assume that \( I \subseteq M \). Choose a subset \( \{ b_\alpha \}_{\alpha \in \Lambda} \) of \( I \) so that \( \{ b_\alpha \}_{\alpha \in \Lambda} \) is a basis for the \( R/M \)-vector space \( I/MI \). Suppose that \( |\Lambda| > 1 \). Then for \( \alpha_1, \alpha_2 \in \Lambda \) with \( \alpha_1 \neq \alpha_2 \), we get \( I = (b_{\alpha_1}, b_{\alpha_2}) + ((b_\alpha | \alpha \in \Lambda - \{\alpha_1, \alpha_2\}) + MI \). By the lemma, say, \( I = (a) + ((b_\alpha | \alpha \in \Lambda - \{\alpha_2\}) + MI \). But then \( (b_\alpha | \alpha \in \Lambda - \{\alpha_2\}) \) is a \( R/M \)-basis for \( I/MI \), a contradiction. Hence \( I = (a) + MI \) for some \( a \in I \). Let \( b \in I \). Then \( (b) I = (b) ((a) + MI) = (a)(b) + M(b) I \subseteq (a) I + M(b) I = ((a) + M(b)) I \). Hence \( (b) \subseteq (a) + M(b) \). Then \( b = ra + mb \) for some \( m \in M \), so \( (1 - m)b = ra \) and hence since \( 1 - m \) is a unit in \( R_M \), \( b \in (a)_M \). Thus \( I_M = (a)_M \). Suppose that \( ca = 0 \) in \( R_M \). Then \( (cI)_M = (ca)_M = 0_M \), so \( (cI)_M = (cMI)_M \). Since \( (cI)_N = (cMI)_N \) for all other maximal ideals \( N \) of \( R \), we have \( cI = cMI \). Since \( I \) is a cancellation ideal, \( (c) = (c)M \). Thus \( c = 0 \) in \( R_M \). Hence \( I_M \) is regular.

Corollary 1. Let \( R \) be a commutative ring with identity, \( S \) a multiplicatively closed subset of \( R \), and \( I \) a cancellation ideal of \( R \). Then \( IS \) is a cancellation ideal in \( R_S \).

We would like to thank the referee for suggesting the following corollary.

Corollary 2. Let \( R \) be a subring of the integral domain \( T \). If \( I \) is a cancellation ideal of \( R \), then \( IT \) is a cancellation ideal of \( T \).

While we have shown that a cancellation ideal \( I \) is locally a regular principal ideal, \( I \) itself need not be regular. Gilmer \cite[Exercise 10, page 456]{Gilmer1992} has given an example of a finitely generated cancellation ideal that is not regular.

References