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A CHARACTERIZATION OF CANCELLATION IDEALS

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ABSTRACT. An ideal I of a commutative ring R with identity is called a cancellation ideal if whenever $IB = IC$ for ideals B and C of R, then $B = C$. We show that an ideal I is a cancellation ideal if and only if I is locally a regular principal ideal.

Let R be a commutative ring with identity. An ideal I of R is called a *cancellation ideal* if whenever $IB = IC$ for ideals B and C of R, then $B = C$. It is easily seen that I is a cancellation ideal if and only if whenever $IB \subseteq IC$ for ideals B and C of R, then $B \subseteq C$. A good introduction to cancellation ideals may be found in Gilmer [1, Section 6]. As for examples, it is easy to see that a principal ideal (a) is a cancellation ideal if and only if (a) is a regular ideal (i.e., a is not a zero divisor). An invertible ideal is a cancellation ideal. More generally, an ideal that is locally a regular principal ideal is a cancellation ideal. The purpose of this paper is to prove the converse.

Kaplansky [2, Theorem 287] proved that a finitely generated cancellation ideal in a quasi-local domain is principal. We begin with the following lemma which is a modification of Kaplansky's result (see [1, Exercise 7, page 67]). We use essentially the same argument.

Lemma. *Let R be a commutative ring with identity and let I be a cancellation ideal of R. Suppose that $I = (x, y) + A$ where A is an ideal of R containing MI for some maximal ideal M. Then $I = (x) + A$ or $I = (y) + A$.*

Proof. Put $J = \left(x^2 + y^2, xy, xA, yA, A^2\right)$. Then it is easily checked that $IJ = I^3$. Since I is a cancellation ideal, we have $J = I^2$. Thus $x^2 = \lambda\left(x^2 + y^2\right) +$ terms from $\left(xy, xA, yA, A^2\right)$. First, suppose that $\lambda \in M$. Since $\lambda x \in MI \subseteq A$, we have $x^2 \in \left(y^2, xy, xA, yA, A^2\right)$. Let $K = (y) + A$. Then $I^2 = IK$. Since I is a cancellation ideal, we have $I = K$. Next, suppose that $\lambda \notin M$. Then for some $\mu \in R$ and $m \in M$, we have $\mu(-\lambda) = 1 + m$. Now $-\mu\lambda y^2 = \mu(\lambda - 1)x^2 +$ terms from $\left(xy, xA, yA, A^2\right)$. Since $my^2 = (my)\,y \in (MI)\,y \subseteq Ay$, we have $y^2 \in \left(x^2, xy, xA, yA, A^2\right)$. Thus, as in the first case, we get that $I = (x) + A$. □

Theorem. *Let R be a commutative ring with identity. An ideal I of R is a cancellation ideal if and only if I is locally a regular principal ideal.*

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Proof. We have already remarked that an ideal that is locally a regular principal ideal is a cancellation ideal. Conversely, suppose that I is a cancellation ideal. Let M be a maximal ideal of R. We show that I_M is a regular principal ideal. We may assume that $I \subseteq M$. Choose a subset $\{b_\alpha\}_{\alpha\in\Lambda}$ of I so that $\{\overline{b}_\alpha\}_{\alpha\in\Lambda}$ is a basis for the R/M-vector space I/MI. Suppose that $|\Lambda| > 1$. Then for $\alpha_1, \alpha_2 \in \Lambda$ with $\alpha_1 \neq \alpha_2$, we get $I = (b_{\alpha_1}, b_{\alpha_2}) + (\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$. By the lemma, say, $I = (b_{\alpha_1}) + (\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$. But then $\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_2\}\}$ is a R/M-basis for I/MI, a contradiction. Hence $I = (a) + MI$ for some $a \in I$. Let $b \in I$. Then $(b)\, I = (b)\,((a) + MI) = (a)\,(b) + M\,(b)\,I \subseteq (a)\,I + M\,(b)\,I = ((a) + M\,(b))\,I$. Hence $(b) \subseteq (a) + M\,(b)$. Then $b = ra + mb$ for some $m \in M$, so $(1-m)\,b = ra$ and hence since $1-m$ is a unit in R_M, $b \in (a)_M$. Thus $I_M = (a)_M$. Suppose that $ca = 0$ in R_M. Then $(cI)_M = (ca)_M = 0_M$, so $(cI)_M = (cMI)_M$. Since $(cI)_N = (cMI)_N$ for all other maximal ideals N of R, we have $cI = cMI$. Since I is a cancellation ideal, $(c) = (c)\,M$. Thus $c = 0$ in R_M. Hence I_M is regular. □

Corollary 1. *Let R be a commutative ring with identity, S a multiplicatively closed subset of R, and I a cancellation ideal of R. Then I_S is a cancellation ideal in R_S.*

We would like to thank the referee for suggesting the following corollary.

Corollary 2. *Let R be a subring of the integral domain T. If I is a cancellation ideal of R, then IT is a cancellation ideal of T.*

While we have shown that a cancellation ideal I is locally a regular principal ideal, I itself need not be regular. Gilmer [1, Exercise 10, page 456] has given an example of a finitely generated cancellation ideal that is not regular.

References

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