

## A CHARACTERIZATION OF CANCELLATION IDEALS

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(Communicated by Wolmer V. Vasconcelos)

**ABSTRACT.** An ideal  $I$  of a commutative ring  $R$  with identity is called a cancellation ideal if whenever  $IB = IC$  for ideals  $B$  and  $C$  of  $R$ , then  $B = C$ . We show that an ideal  $I$  is a cancellation ideal if and only if  $I$  is locally a regular principal ideal.

Let  $R$  be a commutative ring with identity. An ideal  $I$  of  $R$  is called a *cancellation ideal* if whenever  $IB = IC$  for ideals  $B$  and  $C$  of  $R$ , then  $B = C$ . It is easily seen that  $I$  is a cancellation ideal if and only if whenever  $IB \subseteq IC$  for ideals  $B$  and  $C$  of  $R$ , then  $B \subseteq C$ . A good introduction to cancellation ideals may be found in Gilmer [1, Section 6]. As for examples, it is easy to see that a principal ideal  $(a)$  is a cancellation ideal if and only if  $(a)$  is a regular ideal (i.e.,  $a$  is not a zero divisor). An invertible ideal is a cancellation ideal. More generally, an ideal that is locally a regular principal ideal is a cancellation ideal. The purpose of this paper is to prove the converse.

Kaplansky [2, Theorem 287] proved that a finitely generated cancellation ideal in a quasi-local domain is principal. We begin with the following lemma which is a modification of Kaplansky's result (see [1, Exercise 7, page 67]). We use essentially the same argument.

**Lemma.** *Let  $R$  be a commutative ring with identity and let  $I$  be a cancellation ideal of  $R$ . Suppose that  $I = (x, y) + A$  where  $A$  is an ideal of  $R$  containing  $MI$  for some maximal ideal  $M$ . Then  $I = (x) + A$  or  $I = (y) + A$ .*

*Proof.* Put  $J = (x^2 + y^2, xy, xA, yA, A^2)$ . Then it is easily checked that  $IJ = I^3$ . Since  $I$  is a cancellation ideal, we have  $J = I^2$ . Thus  $x^2 = \lambda(x^2 + y^2) +$  terms from  $(xy, xA, yA, A^2)$ . First, suppose that  $\lambda \in M$ . Since  $\lambda x \in MI \subseteq A$ , we have  $x^2 \in (y^2, xy, xA, yA, A^2)$ . Let  $K = (y) + A$ . Then  $I^2 = IK$ . Since  $I$  is a cancellation ideal, we have  $I = K$ . Next, suppose that  $\lambda \notin M$ . Then for some  $\mu \in R$  and  $m \in M$ , we have  $\mu(-\lambda) = 1 + m$ . Now  $-\mu\lambda y^2 = \mu(\lambda - 1)x^2 +$  terms from  $(xy, xA, yA, A^2)$ . Since  $my^2 = (my)y \in (MI)y \subseteq Ay$ , we have  $y^2 \in (x^2, xy, xA, yA, A^2)$ . Thus, as in the first case, we get that  $I = (x) + A$ .  $\square$

**Theorem.** *Let  $R$  be a commutative ring with identity. An ideal  $I$  of  $R$  is a cancellation ideal if and only if  $I$  is locally a regular principal ideal.*

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Received by the editors May 16, 1996.

1991 *Mathematics Subject Classification.* Primary 13A15.

*Key words and phrases.* Cancellation ideal.

M. Roitman thanks the University of Iowa for its hospitality.

*Proof.* We have already remarked that an ideal that is locally a regular principal ideal is a cancellation ideal. Conversely, suppose that  $I$  is a cancellation ideal. Let  $M$  be a maximal ideal of  $R$ . We show that  $I_M$  is a regular principal ideal. We may assume that  $I \subseteq M$ . Choose a subset  $\{b_\alpha\}_{\alpha \in \Lambda}$  of  $I$  so that  $\{\bar{b}_\alpha\}_{\alpha \in \Lambda}$  is a basis for the  $R/M$ -vector space  $I/MI$ . Suppose that  $|\Lambda| > 1$ . Then for  $\alpha_1, \alpha_2 \in \Lambda$  with  $\alpha_1 \neq \alpha_2$ , we get  $I = (b_{\alpha_1}, b_{\alpha_2}) + (\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$ . By the lemma, say,  $I = (b_{\alpha_1}) + (\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$ . But then  $\{b_\alpha \mid \alpha \in \Lambda - \{\alpha_2\}\}$  is a  $R/M$ -basis for  $I/MI$ , a contradiction. Hence  $I = (a) + MI$  for some  $a \in I$ . Let  $b \in I$ . Then  $(b)I = (b)((a) + MI) = (a)(b) + M(b)I \subseteq (a)I + M(b)I = ((a) + M(b))I$ . Hence  $(b) \subseteq (a) + M(b)$ . Then  $b = ra + mb$  for some  $m \in M$ , so  $(1 - m)b = ra$  and hence since  $1 - m$  is a unit in  $R_M$ ,  $b \in (a)_M$ . Thus  $I_M = (a)_M$ . Suppose that  $ca = 0$  in  $R_M$ . Then  $(cI)_M = (ca)_M = 0_M$ , so  $(cI)_M = (cMI)_M$ . Since  $(cI)_N = (cMI)_N$  for all other maximal ideals  $N$  of  $R$ , we have  $cI = cMI$ . Since  $I$  is a cancellation ideal,  $(c) = (c)M$ . Thus  $c = 0$  in  $R_M$ . Hence  $I_M$  is regular.  $\square$

**Corollary 1.** *Let  $R$  be a commutative ring with identity,  $S$  a multiplicatively closed subset of  $R$ , and  $I$  a cancellation ideal of  $R$ . Then  $I_S$  is a cancellation ideal in  $R_S$ .*

We would like to thank the referee for suggesting the following corollary.

**Corollary 2.** *Let  $R$  be a subring of the integral domain  $T$ . If  $I$  is a cancellation ideal of  $R$ , then  $IT$  is a cancellation ideal of  $T$ .*

While we have shown that a cancellation ideal  $I$  is locally a regular principal ideal,  $I$  itself need not be regular. Gilmer [1, Exercise 10, page 456] has given an example of a finitely generated cancellation ideal that is not regular.

#### REFERENCES

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