PREScribing GAUSSian CURVATURE ON $R^2$

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Abstract. We derive a sufficient condition for a radially symmetric function $K(x)$ which is positive somewhere to be a conformal curvature on $R^2$. In particular, we show that every nonnegative radially symmetric continuous function $K(x)$ on $R^2$ is a conformal curvature.

In this paper, we consider the prescribing Gaussian curvature problem. Let $(M,g)$ be a Riemannian manifold of dimension 2 with Gaussian curvature $k$. Given a function $K$ on $M$, one may ask the following question: Can we find a new conformal metric $g_1$ on $M$ (i.e., there exists $u$ on $M$ such that $g_1 = e^{2u}g$) such that $K$ is the Gaussian curvature of $g_1$? This is equivalent to the problem of solving the elliptic equation

$$\Delta u - k + Ke^{2u} = 0$$

on $M$, where $\Delta$ is the Laplacian of $(M,g)$. This problem has been considered by many authors. In case $M$ is compact, we refer to [6] for details and references.

In case $M = R^2$, equation (0) becomes

$$\Delta u + K(x)e^{2u} = 0$$

and this problem is well understood if $K(x)$ is nonpositive; in particular, if $|K(x)|$ decays slower than $|X|^{-2}$ at infinity, then equation (1) has no solution (see [11], [13]). However, if $K(x)$ is positive at some point, the situation is totally different. If $K(x) > 0$ for some $x_0 \in R^2$, R. C. McOwen [10] proved that, for $K(x) = O(r^{-l})$ as $r \to \infty$, equation (1) has a $C^2$ solution, where $l$ is a positive constant. Also, it is not difficult to see that equation (1) has solutions for every positive constant $K(x) = C$.

Since there is no known nonexistence result for $K \geq 0$ on $R^2$, one may propose the following

**Problem 1.** Is it true that every nonnegative function (smooth enough) on $R^2$ is a conformal Gaussian curvature function?

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We shall prove an existence theorem for equation (1) when $K(x)$ is a radially symmetric function. As usual, we set

$$K_-(x) = \min\{K(x), 0\},$$

$$K_+(x) = \max\{K(x), 0\},$$

so $K(x) = K_-(x) + K_+(x)$.

**Theorem 1.** If $K(x) = \tilde{K}(r)$ is a radially symmetric continuous function on $\mathbb{R}^2$, and there exists an $\alpha > 0$ such that

$$\int_0^\infty s^{(1+2\alpha)} \left| \tilde{K}_-(s) \right| ds < \infty,$$

then equation (1) has infinitely many solutions.

**Corollary 2.** If $K(x) \geq 0$ is a radially symmetric continuous function on $\mathbb{R}^2$, then equation (1) has infinitely many solutions.

**Remark 3.** The above theorem seems to suggest a positive answer to Problem (1). This is particularly interesting because in dimensions $n \geq 3$, not every positive function on $\mathbb{R}^n$ is a conformal scalar curvature function. W. M. Ni [12] has shown that a nonnegative function $K(x)$ on $\mathbb{R}^n$ cannot be a conformal scalar curvature function if $K(x)$ satisfies $K(x) \geq C|x|^l$ near $\infty$, where $C > 0$ and $l > 2$ are constants. Moreover, W. Y. Ding and W. M. Ni [3] have shown that there exist smooth radial functions $K(x)$ which are constant at infinity such that the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$$

has no radial solution.

Our proof is based on the following Schauder-Tychonoff fixed point theorem (cf. [1], [4]).

**Theorem (Schauder-Tychonoff).** Let $E$ be a separated locally convex topological vector space, let $A$ be a nonempty closed convex subset of $E$, and let $T$ be a continuous map of $A$ into itself such that $T(A)$ is relatively compact (i.e., $\overline{T(A)}$ is compact) in $E$. Then $T$ admits at least one fixed point.

**Proof of Theorem 1.** Let $K(x) = \tilde{K}(r)$ with $r = |x|$; we try to find a solution $u(r)$ of (1) with $u(0) = \beta$ and $u'(0) = 0$. Then (1) is equivalent to the following integral equation:

$$u(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s)e^{2u(s)} ds. \tag{3}$$

Now we choose $0 < \alpha' < \alpha$ and $\beta$ such that

$$\int_0^e s \log \left( \frac{r}{s} \right) \left| \tilde{K}_-(s) \right| e^{2(\beta+1)} ds < \frac{1}{2}, \tag{4}$$

$$\int_0^e \left| \tilde{K}_-(s) \right| e^{2(\beta+1)} ds < \frac{\alpha'}{2}, \tag{5}$$

$$\int_e^\infty s^{(1+2\alpha')} \left| \tilde{K}_-(s) \right| e^{2(\beta+1)} ds < \frac{\alpha'}{2}, \tag{6}$$

$$\int_e^\infty s^{(1+2\alpha')} \log \left( \frac{e}{s} \right) \left| \tilde{K}_-(s) \right| e^{2(\beta+1)} ds < \frac{1}{2}. \tag{7}$$

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Define the functions \( A_\beta(r) \) and \( B_\beta(r) \) by

\[
\begin{align*}
A_\beta(r) &= (\beta + 1), & \text{if } 0 \leq r \leq e, \\
A_\beta(r) &= (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right), & \text{if } e \leq r,
\end{align*}
\]

(8)

\[
B_\beta(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_+(s)e^{2A_\beta(s)} \, ds.
\]

(9)

Let \( X \) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology, i.e., \( \lim_{n \to \infty} f_n = f \) in \( X \) iff \( f_n \) converges to \( f \) uniformly on any compact subset of \([0, \infty)\).

Now consider the set

\[ Y = \{ u \in X | B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty) \}. \]

It is easy to see that \( Y \) is a closed convex subset of \( X \). Let \( T \) be the mapping

\[
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_+(s)e^{2u(s)} \, ds.
\]

(10)

We shall prove that \( T \) is a continuous mapping from \( Y \) into itself such that \( TY \) is relatively compact.

First, we verify that \( TY \subset Y \). Assume \( u \in Y \). Hence we have

\[
B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty).
\]

(11)

It is easy to see that \( Tu \) is continuous. Now for \( 0 \leq r \leq e \) we have

\[
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_-(s)e^{2u(s)} \, ds - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_+(s)e^{2u(s)} \, ds
\]

\[
\leq \beta - \int_0^e s \log \left( \frac{e}{s} \right) \tilde{K}_-(s)e^{2(\beta+1)} \, ds
\]

\[
\leq (\beta + 1) = A_\beta(r).
\]

For \( e \leq r \), we have

\[
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_-(s)e^{2u(s)} \, ds - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}_+(s)e^{2u(s)} \, ds
\]

\[
\leq \beta - \log \left( \frac{e}{r} \right) \int_0^e s \tilde{K}_-(s)e^{2(\beta+1)} \, ds - \int_0^e s \log \left( \frac{e}{s} \right) \tilde{K}_-(s)e^{2(\beta+1)} \, ds
\]

\[
- \log \left( \frac{e}{r} \right) \int_e^\infty s^{(1+2\alpha')} \tilde{K}_-(s)e^{2(\beta+1)} \, ds
\]

\[
- \int_e^\infty s^{(1+2\alpha')} \log \left( \frac{e}{s} \right) \tilde{K}_-(s)e^{2(\beta+1)} \, ds
\]

\[
\leq \beta + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2} + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2}
\]

\[
= (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right) = A_\beta(r).
\]
On the other hand, since $u(r) \in Y$, we have
\[
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \bar{K}^{-}(s)e^{2u(s)} \, ds - \int_0^r s \log \left( \frac{r}{s} \right) \bar{K}^{+}(s)e^{2u(s)} \, ds
\geq \beta - \int_0^r s \log \left( \frac{r}{s} \right) e^{2\beta}(s) \, ds
= B_\beta(r).
\]
This verifies that $TY \subset Y$.

To show that $T$ is continuous in $Y$, let $\{u_m\}_{m=1}^\infty \subset Y$ be a sequence converging to $u \in Y$ in the space $X$. Then $u_m$ converges to $u$ uniformly on any compact interval of $[0, \infty)$. Now
\[
|Tu_m(r) - Tu(r)| \leq \int_0^r s \log \left( \frac{r}{s} \right) |\bar{K}(s)| |e^{2u_m(s)} - e^{2u(s)}| \, ds,
\]
but
\[
s \log \left( \frac{r}{s} \right) |\bar{K}(s)| |e^{2u_m(s)} - e^{2u(s)}| \leq s \log \left( \frac{r}{s} \right) |\bar{K}(s)||e^{2A_\beta(s)} - e^{2B_\beta(s)}| \leq s \log \left( \frac{r}{s} \right) |\bar{K}(s)|e^{2A_\beta(s)}
\]
and $s \log \left( \frac{r}{s} \right) |\bar{K}(s)|e^{2A_\beta(s)}$ is integrable on any compact interval of $[0, \infty)$. Hence from (12) and the uniform convergence of $u_m$ to $u$ on any compact interval, we conclude that $Tu_m$ converges to $Tu$ uniformly on any compact interval, which implies that $Tu_m$ converges to $Tu$ in $X$. This verifies that $T$ is continuous in $Y$.

We can easily compute that
\[
|\langle Tu \rangle'(r)| = \left| \int_0^r \left( \frac{s}{r} \right) \bar{K}(s)e^{2u(s)} \, ds \right| \leq \int_0^r \left( \frac{s}{r} \right) |\bar{K}(s)|e^{2A_\beta(s)} \, ds.
\]
Hence, on any compact interval of $[0, \infty)$, $TY$ is uniformly bounded and equiconv-}

tinuous. This proves that $TY$ is relatively compact in $Y$. So by the Schauder-

Tychonoff fixed point theorem, $T$ has a fixed point $u$ in $Y$. This $u$ is a solution of (2) and hence a solution of (1). We notice that, if (3) has a solution for some $\beta$, then it has a solution for all $\beta_1 \leq \beta$. This completes the proof of Theorem 1.

In the $n \geq 3$ dimension case, if we also assume that $K(x) = \bar{K}(r)$ is radially symmetric in (1), and we want to find a radially symmetric solution $u(r)$ such that $u(0) = \beta$ and $u'(0) = 0$, then (1) is equivalent to
\[
u(r) = \beta - \frac{1}{n-2} \int_0^r \left( 1 - \frac{s}{r} \right)^{n-2} \bar{K}(s)e^{2u(s)} \, ds.
\]
In this situation we can show the following

**Theorem 4.** If $K(x) = \bar{K}(r)$ is a radially symmetric continuous function on $R^n$, $n \geq 3$, such that
\[
\int_0^\infty s|\bar{K}^{-}(s)| \, ds < \infty,
\]
then the equation
\[
\Delta u + K(x)e^{2u} = 0
\]
has infinitely many solutions.
Proof. The argument is essentially the same as in the proof of Theorem 1. We leave the details to the readers.

References

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