PRESCRIBING GAUSSIAN CURVATURE ON $R^2$

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(Communicated by Peter Li)

Abstract. We derive a sufficient condition for a radially symmetric function $K(x)$ which is positive somewhere to be a conformal curvature on $R^2$. In particular, we show that every nonnegative radially symmetric continuous function $K(x)$ on $R^2$ is a conformal curvature.

In this paper, we consider the prescribing Gaussian curvature problem. Let $(M, g)$ be a Riemannian manifold of dimension 2 with Gaussian curvature $k$. Given a function $K$ on $M$, one may ask the following question: Can we find a new conformal metric $g_1$ on $M$ (i.e., there exists $u$ on $M$ such that $g_1 = e^{2u}g$) such that $K$ is the Gaussian curvature of $g_1$? This is equivalent to the problem of solving the elliptic equation

$$\Delta u - k + Ke^{2u} = 0$$

on $M$, where $\Delta$ is the Laplacian of $(M, g)$. This problem has been considered by many authors. In case $M$ is compact, we refer to [6] for details and references.

In case $M = R^2$, equation (0) becomes

$$\Delta u + K(x)e^{2u} = 0$$

and this problem is well understood if $K(x)$ is nonpositive; in particular, if $|K(x)|$ decays slower than $|X|^{-2}$ at infinity, then equation (1) has no solution (see [11], [13]). However, if $K(x)$ is positive at some point, the situation is totally different. If $K(x_0) > 0$ for some $x_0 \in R^2$, R. C. McOwen [10] proved that, for $K(x) = O(r^{-l})$ as $r \to \infty$, equation (1) has a $C^2$ solution, where $l$ is a positive constant. Also, it is not difficult to see that equation (1) has solutions for every positive constant $K(x) = C$.

Since there is no known nonexistence result for $K \geq 0$ on $R^2$, one may propose the following

Problem 1. Is it true that every nonnegative function (smooth enough) on $R^2$ is a conformal Gaussian curvature function?
We shall prove an existence theorem for equation (1) when $K(x)$ is a radially symmetric function. As usual, we set

$$K_-(x) = \min\{K(x), 0\},$$
$$K_+(x) = \max\{K(x), 0\},$$

so $K(x) = K_-(x) + K_+(x)$.

**Theorem 1.** If $K(x) = \tilde{K}(r)$ is a radially symmetric continuous function on $\mathbb{R}^2$, and there exists an $\alpha > 0$ such that

$$\int_0^{\infty} s^{(1+2\alpha)} |\tilde{K}_-(s)| ds < \infty,$$

then equation (1) has infinitely many solutions.

**Corollary 2.** If $K(x) \geq 0$ is a radially symmetric continuous function on $\mathbb{R}^2$, then equation (1) has infinitely many solutions.

**Remark 3.** The above theorem seems to suggest a positive answer to Problem (1). This is particularly interesting because in dimensions $n \geq 3$, not every positive function on $\mathbb{R}^n$ is a conformal scalar curvature function. W. M. Ni [12] has shown that a nonnegative function $K(x)$ on $\mathbb{R}^n$ cannot be a conformal scalar curvature function if $K(x)$ satisfies $K(x) \geq C|x|^l$ near $\infty$, where $C > 0$ and $l > 2$ are constants. Moreover, W. Y. Ding and W. M. Ni [3] have shown that there exist smooth radial functions $K(x)$ which are constant at infinity such that the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$$

has no radial solution.

Our proof is based on the following Schauder-Tychonoff fixed point theorem (cf. [1], [4]).

**Theorem (Schauder-Tychonoff).** Let $E$ be a separated locally convex topological vector space, let $A$ be a nonempty closed convex subset of $E$, and let $T$ be a continuous map of $A$ into itself such that $T(A)$ is relatively compact (i.e., $\overline{T(A)}$ is compact) in $E$. Then $T$ admits at least one fixed point.

**Proof of Theorem 1.** Let $K(x) = \tilde{K}(r)$ with $r = |x|$; we try to find a solution $u(r)$ of (1) with $u(0) = \beta$ and $u'(0) = 0$. Then (1) is equivalent to the following integral equation:

$$u(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s)e^{2u(s)} ds.$$

Now we choose $0 < \alpha' < \alpha$ and $\beta$ such that

$$\int_0^{e} s \log \left( \frac{e}{s} \right) |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{1}{2},$$
$$\int_0^{e} |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{\alpha'}{2},$$
$$\int_{e}^{\infty} s^{(1+2\alpha')} |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{\alpha'}{2},$$
$$\int_{e}^{\infty} s^{(1+2\alpha')} \log \left( \frac{e}{s} \right) |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{1}{2}.$$
Define the functions $A_\beta(r)$ and $B_\beta(r)$ by

$$
\begin{cases}
A_\beta(r) = (\beta + 1), & \text{if } 0 \leq r \leq e, \\
A_\beta(r) = (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right), & \text{if } e \leq r,
\end{cases}
$$

(8)

$$
B_\beta(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}_+(s)e^{2A_\beta(s)} ds.
$$

(9)

Let $X$ denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology, i.e., $\lim_{n \to \infty} f_n = f$ in $X$ iff $f_n$ converges to $f$ uniformly on any compact subset of $[0, \infty)$.

Now consider the set

$$
Y = \{ u \in X | B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty) \}.
$$

It is easy to see that $Y$ is a closed convex subset of $X$. Let $T$ be the mapping

$$
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}(s)e^{2u(s)} ds.
$$

(10)

We shall prove that $T$ is a continuous mapping from $Y$ into itself such that $TY$ is relatively compact.

First, we verify that $TY \subset Y$. Assume $u \in Y$. Hence we have

$$
B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty).
$$

(11)

It is easy to see that $Tu$ is continuous. Now for $0 \leq r \leq e$ we have

$$
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}_-(s)e^{2u(s)} ds - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}_+(s)e^{2u(s)} ds \\
\leq \beta - \int_0^e s \log \left( \frac{r}{s} \right) \overline{K}_-(s)e^{2(\beta+1)} ds \\
\leq (\beta + 1) = A_\beta(r).
$$

For $e \leq r$, we have

$$
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}_-(s)e^{2u(s)} ds - \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}_+(s)e^{2u(s)} ds \\
\leq \beta - \log \left( \frac{r}{e} \right) \int_0^e s \overline{K}_-(s)e^{2(\beta+1)} ds - \int_0^e s \log \left( \frac{r}{s} \right) \overline{K}_-(s)e^{2(\beta+1)} ds \\
- \log \left( \frac{r}{e} \right) \int_e^\infty s^{(1+2\alpha')} \overline{K}_-(s)e^{2(\beta+1)} ds \\
- \int_e^\infty s^{(1+2\alpha')} \log \left( \frac{r}{s} \right) \overline{K}_-(s)e^{2(\beta+1)} ds \\
\leq \beta + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2} + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2} \\
= (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right) = A_\beta(r).
$$
On the other hand, since \( u(r) \in Y \), we have

\[
(Tu)(r) = \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) e^{2u(s)} ds - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) e^{2u(s)} ds
\]

\[
\geq \beta - \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) e^{2A_\beta(s)} ds
\]

\[
= B_\beta(r).
\]

This verifies that \( TY \subset Y \).

To show that \( T \) is continuous in \( Y \), let \( \{u_m\}_{m=1}^\infty \subset Y \) be a sequence converging to \( u \in Y \) in the space \( X \). Then \( u_m \) converges to \( u \) uniformly on any compact interval of \([0, \infty)\). Now

\[
|Tu_m(r) - Tu(r)| \leq \int_0^r \frac{s}{r} \log \left( \frac{r}{s} \right) |\tilde{K}(s)||e^{2u_m(s)} - e^{2u(s)}| ds,
\]

but

\[
\frac{s}{r} \log \left( \frac{r}{s} \right) |e^{2u_m(s)} - e^{2u(s)}| \leq \frac{s}{r} \log \left( \frac{r}{s} \right) (e^{2A_\beta(s)} - e^{2B_\beta(s)})
\]

\[
\leq \frac{s}{r} \log \left( \frac{r}{s} \right) e^{2A_\beta(s)}
\]

and \( \frac{s}{r} \log \left( \frac{r}{s} \right) e^{2A_\beta(s)} \) is integrable on any compact interval of \([0, \infty)\). Hence from (12) and the uniform convergence of \( u_m \) to \( u \) on any compact interval, we conclude that \( Tu_m \) converges to \( Tu \) uniformly on any compact interval, which implies that \( Tu_m \) converges to \( Tu \) in \( X \). This verifies that \( T \) is continuous in \( Y \).

We can easily compute that

\[
|\langle Tu \rangle'(r)| = \int_0^r \frac{s}{r} \tilde{K}(s) e^{2u(s)} ds \leq \int_0^r \frac{s}{r} \tilde{K}(s) e^{2A_\beta(s)} ds.
\]

Hence, on any compact interval of \([0, \infty)\), \( TY \) is uniformly bounded and equicontinuous. This proves that \( TY \) is relatively compact in \( Y \). So by the Schauder-Tychonoff fixed point theorem, \( T \) has a fixed point \( u \) in \( Y \). This \( u \) is a solution of (2) and hence a solution of (1). We notice that, if (3) has a solution for some \( \beta \), then it has a solution for all \( \beta \leq \beta \). This completes the proof of Theorem 1.

In the \( n \geq 3 \) dimension case, if we also assume that \( K(x) = \tilde{K}(r) \) is radially symmetric in (1), and we want to find a radially symmetric solution \( u(r) \) such that \( u(0) = \beta \) and \( u'(0) = 0 \), then (1) is equivalent to

\[
u(r) = \beta - \frac{1}{n-2} \int_0^r s \left( 1 - \left( \frac{s}{r} \right)^{n-2} \right) \tilde{K}(s) e^{2u(s)} ds.
\]

In this situation we can show the following

**Theorem 4.** If \( K(x) = \tilde{K}(r) \) is a radially symmetric continuous function on \( R^n \), \( n \geq 3 \), such that

\[
\int_0^\infty \frac{s}{r} |\tilde{K}(s)| ds < \infty,
\]

then the equation

\[
\Delta u + K(x)e^{2u} = 0
\]

has infinitely many solutions.
Proof. The argument is essentially the same as in the proof of Theorem 1. We leave the details to the readers.

REFERENCES

12. ______, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry, Indiana Univ. Math. J. 31 (1982), 495–529. MR 84e:35049

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