PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 3119–3123
S 0002-9939(97)04150-6

PRESCRIBING GAUSSIAN CURVATURE ON R^2

SANXING WU

(Communicated by Peter Li)

ABSTRACT. We derive a sufficient condition for a radially symmetric function $K(x)$ which is positive somewhere to be a conformal curvature on R^2. In particular, we show that every nonnegative radially symmetric continuous function $K(x)$ on R^2 is a conformal curvature.

In this paper, we consider the prescribing Gaussian curvature problem. Let (M, g) be a Riemannian manifold of dimension 2 with Gaussian curvature k. Given a function K on M, one may ask the following question: Can we find a new conformal metric g_1 on M (i.e., there exists u on M such that $g_1 = e^{2u}g$) such that K is the Gaussian curvature of g_1? This is equivalent to the problem of solving the elliptic equation

$$\Delta u - k + Ke^{2u} = 0 \tag{0}$$

on M, where Δ is the Laplacian of (M, g). This problem has been considered by many authors. In case M is compact, we refer to [6] for details and references.

In case $M = R^2$, equation (0) becomes

$$\Delta u + K(x)e^{2u} = 0 \tag{1}$$

and this problem is well understood if $K(x)$ is nonpositive; in particular, if $|K(x)|$ decays slower than $|X|^{-2}$ at infinity, then equation (1) has no solution (see [11], [13]). However, if $K(x)$ is positive at some point, the situation is totally different. If $K(x_0) > 0$ for some $x_0 \in R^2$, R. C. McOwen [10] proved that, for $K(x) = O(r^{-l})$ as $r \to \infty$, equation (1) has a C^2 solution, where l is a positive constant. Also, it is not difficult to see that equation (1) has solutions for every positive constant $K(x) = C$.

Since there is no known nonexistence result for $K \geq 0$ on R^2, one may propose the following

Problem 1. Is it true that every nonnegative function (smooth enough) on R^2 is a conformal Gaussian curvature function?

Received by the editors May 10, 1996.
1991 *Mathematics Subject Classification.* Primary 58G30; Secondary 53C21.
Key words and phrases. Prescribing Gaussian curvature, semilinear elliptic PDE, integral equation.

We shall prove an existence theorem for equation (1) when $K(x)$ is a radially symmetric function. As usual, we set

$$K_-(x) = \min\{K(x), 0\},$$
$$K_+(x) = \max\{K(x), 0\},$$

so $K(x) = K_-(x) + K_+(x)$.

Theorem 1. *If $K(x) = \widetilde{K}(r)$ is a radially symmetric continuous function on R^2, and there exists an $\alpha > 0$ such that*

$$\int_0^\infty s^{(1+2\alpha)} |\widetilde{K}_-(s)|\, ds < \infty, \tag{2}$$

then equation (1) *has infinitely many solutions.*

Corollary 2. *If $K(x) \geq 0$ is a radially symmetric continuous function on R^2, then equation* (1) *has infinitely many solutions.*

Remark 3. The above theorem seems to suggest a positive answer to Problem (1). This is particularly interesting because in dimensions $n \geq 3$, not every positive function on R^n is a conformal scalar curvature function. W. M. Ni [12] has shown that a nonnegative function $K(x)$ on R^n cannot be a conformal scalar curvature function if $K(x)$ satisfies $K(x) \geq C|x|^l$ near ∞, where $C > 0$ and $l > 2$ are constants. Moreover, W. Y. Ding and W. M. Ni [3] have shown that there exist smooth radial functions $K(x)$ which are constant at infinity such that the equation

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0$$

has no radial solution.

Our proof is based on the following Schauder-Tychonoff fixed point theorem (cf. [1], [4]).

Theorem (Schauder-Tychonoff)**.** *Let E be a separated locally convex topological vector space, let A be a nonempty closed convex subset of E, and let T be a continuous map of A into itself such that $T(A)$ is relatively compact (i.e., $\overline{T(A)}$ is compact) in E. Then T admits at least one fixed point.*

Proof of Theorem 1. Let $K(x) = \widetilde{K}(r)$ with $r = |x|$; we try to find a solution $u(r)$ of (1) with $u(0) = \beta$ and $u'(0) = 0$. Then (1) is equivalent to the following integral equation:

$$u(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}(s) e^{2u(s)}\, ds. \tag{3}$$

Now we choose $0 < \alpha' < \alpha$ and β such that

$$\int_0^e s \log\left(\frac{e}{s}\right) \left|\widetilde{K}_-(s)\right| e^{2(\beta+1)}\, ds < \frac{1}{2}, \tag{4}$$

$$\int_0^e s \left|\widetilde{K}_-(s)\right| e^{2(\beta+1)}\, ds < \frac{\alpha'}{2}, \tag{5}$$

$$\int_e^\infty s^{(1+2\alpha')} \left|\widetilde{K}_-(s)\right| e^{2(\beta+1)}\, ds < \frac{\alpha'}{2}, \tag{6}$$

$$\int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) \left|\widetilde{K}_-(s)\right| e^{2(\beta+1)}\, ds < \frac{1}{2}. \tag{7}$$

Define the functions $A_\beta(r)$ and $B_\beta(r)$ by

$$\begin{cases} A_\beta(r) = (\beta+1), & \text{if } 0 \le r \le e, \\ A_\beta(r) = (\beta+1) + \alpha' \log\left(\frac{r}{e}\right), & \text{if } e \le r, \end{cases} \tag{8}$$

$$B_\beta(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}_+(s) e^{2A_\beta(s)}\, ds. \tag{9}$$

Let X denote the locally convex space of all continuous functions on $[0,\infty)$ with the usual topology, *i.e.*, $\lim_{n\to\infty} f_n = f$ in X iff f_n converges to f uniformly on any compact subset of $[0,\infty)$.

Now consider the set

$$Y = \{u \in X | B_\beta(r) \le u(r) \le A_\beta(r), \quad r \in [0,\infty)\}.$$

It is easy to see that Y is a closed convex subset of X. Let T be the mapping

$$(Tu)(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}(s) e^{2u(s)}\, ds. \tag{10}$$

We shall prove that T is a continuous mapping from Y into itself such that TY is relatively compact.

First, we verify that $TY \subset Y$. Assume $u \in Y$. Hence we have

$$B_\beta(r) \le u(r) \le A_\beta(r), \qquad r \in [0,\infty). \tag{11}$$

It is easy to see that Tu is continuous. Now for $0 \le r \le e$ we have

$$\begin{aligned} (Tu)(r) &= \beta - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}_-(s) e^{2u(s)}\, ds - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}_+(s) e^{2u(s)}\, ds \\ &\le \beta - \int_0^e s \log\left(\frac{e}{s}\right) \widetilde{K}_-(s) e^{2(\beta+1)}\, ds \\ &\le (\beta+1) = A_\beta(r). \end{aligned}$$

For $e \le r$, we have

$$\begin{aligned} (Tu)(r) &= \beta - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}_-(s) e^{2u(s)}\, ds - \int_0^r s \log\left(\frac{r}{s}\right) \widetilde{K}_+(s) e^{2u(s)}\, ds \\ &\le \beta - \log\left(\frac{r}{e}\right) \int_0^e s\widetilde{K}_-(s) e^{2(\beta+1)}\, ds - \int_0^e s \log\left(\frac{e}{s}\right) \widetilde{K}_-(s) e^{2(\beta+1)}\, ds \\ &\quad - \log\left(\frac{r}{e}\right) \int_e^\infty s^{(1+2\alpha')} \widetilde{K}_-(s) e^{2(\beta+1)}\, ds \\ &\quad - \int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) \widetilde{K}_-(s) e^{2(\beta+1)}\, ds \\ &\le \beta + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} \\ &= (\beta+1) + \alpha' \log\left(\frac{r}{e}\right) = A_\beta(r). \end{aligned}$$

On the other hand, since $u(r) \in Y$, we have

$$\begin{aligned}(Tu)(r) &= \beta - \int_0^r s\log\left(\frac{r}{s}\right)\widetilde{K}_-(s)e^{2u(s)}\,ds - \int_0^r s\log\left(\frac{r}{s}\right)\widetilde{K}_+(s)e^{2u(s)}\,ds \\ &\geq \beta - \int_0^r s\log\left(\frac{r}{s}\right)\widetilde{K}_+(s)e^{2A_\beta(s)}\,ds \\ &= B_\beta(r).\end{aligned}$$

This verifies that $TY \subset Y$.

To show that T is continuous in Y, let $\{u_m\}_{m=1}^\infty \subset Y$ be a sequence converging to $u \in Y$ in the space X. Then u_m converges to u uniformly on any compact interval of $[0,\infty)$. Now

$$|Tu_m(r) - Tu(r)| \leq \int_0^r s\log\left(\frac{r}{s}\right)|\widetilde{K}(s)|\,|e^{2u_m(s)} - e^{2u(s)}|\,ds, \tag{12}$$

but

$$\begin{aligned}s\log\left(\frac{r}{s}\right)\left|\widetilde{K}(s)\right|\,|e^{2u_m(s)} - e^{2u(s)}| &\leq s\log\left(\frac{r}{s}\right)|\widetilde{K}(s)|(e^{2A_\beta(s)} - e^{2B_\beta(s)}) \\ &\leq s\log\left(\frac{r}{s}\right)|\widetilde{K}(s)|e^{2A_\beta(s)}\end{aligned}$$

and $s\log(\frac{r}{s})|\widetilde{K}(s)|e^{2A_\beta(s)}$ is integrable on any compact interval of $[0,\infty)$. Hence from (12) and the uniform convergence of u_m to u on any compact interval, we conclude that Tu_m converges to Tu uniformly on any compact interval, which implies that Tu_m converges to Tu in X. This verifies that T is continuous in Y.

We can easily compute that

$$|(Tu)'(r)| = |\int_0^r \left(\frac{s}{r}\right)\widetilde{K}(s)e^{2u(s)}\,ds| \leq \int_0^r \left(\frac{s}{r}\right)|\widetilde{K}(s)|e^{2A_\beta(s)}\,ds.$$

Hence, on any compact interval of $[0,\infty)$, TY is uniformly bounded and equicontinuous. This proves that TY is relatively compact in Y. So by the Schauder-Tychonoff fixed point theorem, T has a fixed point u in Y. This u is a solution of (2) and hence a solution of (1). We notice that, if (3) has a solution for some β, then it has a solution for all $\beta_1 \leq \beta$. This completes the proof of Theorem 1. □

In the $n \geq 3$ dimension case, if we also assume that $K(x) = \widetilde{K}(r)$ is radially symmetric in (1), and we want to find a radially symmetric solution $u(r)$ such that $u(0) = \beta$ and $u'(0) = 0$, then (1) is equivalent to

$$u(r) = \beta - \frac{1}{n-2}\int_0^r s\left(1 - \left(\frac{s}{r}\right)^{n-2}\right)\widetilde{K}(s)e^{2u(s)}\,ds. \tag{13}$$

In this situation we can show the following

Theorem 4. *If $K(x) = \widetilde{K}(r)$ is a radially symmetric continuous function on R^n, $n \geq 3$, such that*

$$\int_0^\infty s|\widetilde{K}_-(s)|\,ds < \infty, \tag{14}$$

then the equation

$$\Delta u + K(x)e^{2u} = 0 \tag{15}$$

has infinitely many solutions.

Proof. The argument is essentially the same as in the proof of Theorem 1. We leave the details to the readers. $\square$

References

1. K. S. Cheng and J. T. Lin, *On the elliptic equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$*, Trans. Amer. Math. Soc. **304** (1987), 639–668. MR **88j:**35076
2. K. S. Cheng and W. M. Ni, *On the structure of the conformal Gaussian curvature equation on R^2*, Duke Math. J. **62** (1991), 721–737. MR **92f:**35061
3. W. Y. Ding and W. M. Ni, *On the elliptic equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ and related topics*, Duke Math. J. **52** (1985), 485–506. MR **86k:**35040
4. R. E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, New York, 1965.
5. D. Hulin and M. Troyanov, *Prescribing curvature on open surfaces*, Math. Ann. **293** (1992), 277–315. MR **93d:**53047
6. J. Kazdan and F. W. Warner, *Curvature Functions for Compact 2-manifolds*, Ann. Math. **99** (1974), 14–47. MR **49:**7949
7. M. Kalka and D. G. Yang, *On conformal deformation of nonpositive curvature on noncompact surfaces*, Duke Math. J., **72** (1993), 405–430. MR **94i:**53040
8. ———, *On nonpositive curvature functions on noncompact surfaces of finite topological type*, Indiana Univ. Math. J. **43** (1994), 775–804. MR **95j:**53060
9. R. C. McOwen, *On the equation $\Delta u + K(x)e^{2u} = f$ and prescribing negative curvature on* $\mathbf{R}^2$, J. Math. Anal. Appl. **103** (1984), 365–370. MR **86c:**58755
10. ———, *Conformal Metrics in R^2 with Prescribed Gaussian Curvature and Positive Total Curvature*, Indiana Univ. Math. J. **34** (1985), 97–104. MR **86h:**53008
11. W. M. Ni, *On the elliptic equation $\Delta u + Ke^{2u} = 0$ and conformal metrics with prescribed Gaussian curvature*, Invent. Math. **66** (1982), 343–352. MR **89g:**58107
12. ———, *On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), 495–529. MR **84e:**35049
13. D. H. Sattinger, *Conformal metrics in R^2 with prescribed Gaussian curvature*, Indiana Univ. Math. J. **22** (1972), 1–4. MR **46:**4437

Department of Applied Mathematics, 100083, Beijing University of Aeronautics and Astronautics, Beijing, People's Republic of China