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NIELSEN-THURSTON REDUCIBILITY AND RENORMALIZATION

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Abstract. Consider an orientation preserving homeomorphism f of the 2-disk with an infinite set of nested periodic orbits $\{\mathcal{O}_n\}_{n\geq 1}$, such that, for all $m > 1$, the restriction of f to the complement of the first m orbits, from $\mathcal{O}_1$ to $\mathcal{O}_m$, is $m-1$ times reducible in the sense of Nielsen and Thurston. We define combinatorial renormalization operators for such maps, and study the fixed points of these operators. We also recall the corresponding theory for endomorphisms of the interval, and give elements of comparison of the theories in one and two dimensions.

1. Introduction

Thurston's completion [Th] of the work of Nielsen [Ni] on the classification, up to isotopy, of surface homeomorphisms was accomplished at about the same time renormalization group ideas were introduced in the theory of one-dimensional dynamical systems [Fe], [CT], [TC]. Some key concepts from these theories, which were formulated quite independently, are quite similar. In this paper we examine some relationships between the two theories. Our aim is to discuss some combinatorial aspects of renormalization group theory for orientation preserving homeomorphisms of the 2-disk, and compare them with what is known in this context for endomorphisms of the interval. In the following discussion, for a sequence $(a_i)_{i\geq 1}$ of integers greater than or equal to 2, we set

$$q_n = a_1 \cdot a_2 \cdot \cdots \cdot a_n.$$

For f a continuous map of the unit m-dimensional disk $\mathbb{D}^m$ into itself and $(a_i)_{i\geq 1}$ a sequence as above, we say that f is $(a_i)_{i\geq 1}$-*infinitely renormalizable* if there exists a sequence of m-disks

$$\mathbb{D}^m \supset \mathcal{D}_1(f) \supset \mathcal{D}_2(f) \supset \cdots \supset \mathcal{D}_n(f) \dots$$

such that, for each n,

$$f^j(\mathcal{D}_n(f)) \cap \mathcal{D}_n(f) = \emptyset, \qquad \text{for } 0 < j \leq q_n - 1,$$

and

$$f^{q_n}(\mathcal{D}_n(f)) \subset \mathcal{D}_n(f).$$

In such a case, by Brouwer's fixed point Theorem, $\mathcal{D}_n$ contains one point x_n of a periodic orbit $\mathcal{O}_n$ with period q_n. Collections of periodic orbits arising this way have

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well understood special combinatorial structures in the special cases when f is an endomorphism and $m = 1$, and when f is an orientation preserving homeomorphism and $m = 2$. The map which associates $f^{q_n}|_{\mathcal{D}_n(f)}$ to f is called a *renormalization operator* and plays an important role in many problems of smooth and holomorphic dynamics. We shall define renormalization operators in the combinatorial context for the cases of dimensions 1 and 2, and compare their dynamical properties. In particular, we shall see that fixed points of combinatorial renormalization operators, as defined in §3, are much more abundant for endomorphisms in one dimension than for homeomorphisms in two dimensions, despite the fact that permutation groups are finite while braid groups are not.

2. Combinatorics

There are classical ways to associate combinatorial objects to periodic orbits of endomorphisms of the interval I (or $f \in \mathrm{End}(I)$) and orientation preserving homeomorphisms of the 2-disk $\mathbb{D}^2$ (or $f \in \mathrm{Homeo}^+(\mathbb{D}^2)$).

If $x_1 < x_2 < \cdots < x_p$ are the p points of a periodic orbit $\mathcal{O}$ of $f \in \mathrm{End}(I)$, let h be the bijection sending x_i to i. There is a single permutation π, in fact a cycle, in the permutation group $\mathbf{S}_p$ on p elements such that $f|_{\mathcal{O}} = h^{-1} \circ \pi \circ h$. We say that π *represents the action of* f *on* $\mathcal{O}$, and that f *realizes the cycle* π. More general permutations can be used similarly to represent the dynamics of f on finite collections of periodic orbits.

Assume now that the p points $x_1, x_2, \ldots, x_p$ of a periodic orbit $\mathcal{O}$ of $f \in \mathrm{Homeo}^+(\mathbb{D}^2)$ are in the interior of $\mathbb{D}^2$, and that $y_1, y_2, \ldots, y_p$ are the p points of a periodic orbit $\mathcal{O}'$ of $g \in \mathrm{Homeo}^+(\mathbb{D}^2)$. We say $(\mathcal{O}, f)$ and $(\mathcal{O}', g)$ *have the same braid type* if there exists $h \in \mathrm{Homeo}^+(\mathbb{D}^2)$ sending $\mathcal{O}$ onto $\mathcal{O}'$, such that $h^{-1} \circ g \circ h$ is isotopic to f relative to $\mathcal{O}$. It is straightforward to check that "to have the same braid type" is an equivalence relation. The corresponding equivalence classes are called *braid types*: if $(\mathcal{O}, f)$ is a representative of the braid type β, we also say that β *represents the action of* f *on* $\mathbb{D}^2 \backslash \mathcal{O}$, and that f *realizes the braid type* β. More general braid types can be used similarly to represent the dynamics of f on finite collections of periodic orbits. The braid types with n strands form the *braid type set* $\mathbf{BT}_n$ which is obtained from the Artin braid group $\mathbf{B}_n$ by quotienting by the conjugacies and the center: in the case of a single periodic orbit, any representative b in $\mathbf{B}_n$ of a braid type β in $\mathbf{BT}_n$ is mapped to a cycle by the canonical projection τ from $\mathbf{B}_n$ to $\mathbf{S}_n$: we say β is a *cycle* if $\tau(b)$ is a cycle (a property which does not depend on the choice of b).

If b and b' are two representatives in $\mathbf{B}_n$ of $\beta \in \mathbf{BT}_n$, then b^m and b'^m represent the same element of $\mathbf{BT}_n$ for each $m \in \mathbb{Z}$. As a consequence, any $\beta \in \mathbf{BT}_n$ generates a group $\mathbf{G}_\beta$ by iteration.

Remark 1. The set of powers of a permutation form a cyclic subgroup of $\mathbf{S}_n$. On the contrary, assuming β is a cycle, $\mathbf{G}_\beta$ is a cyclic group if and only if β is an *elementary* braid type in the sense of Nielsen-Thurston, *i.e.*, β is the braid type of a periodic orbit of a rigid rotation of the disk. In all other cases when β is a cycle, $\mathbf{G}_\beta$ is isomorphic to $\mathbb{Z}$.

A sequence $(\mathcal{O}_i)_{i\geq 1}$ of periodic orbits of $f \in \mathrm{End}(I)$, with periods q_i, is an $(a_i)_{i\geq 1}$-*cascade of periodic orbits* if for each $j \geq 1$, there are pairwise disjoint intervals $I_{j,k}$ with $0 \leq k \leq q_j - 1$ so that each $I_{j,k}$ contains one point of $\mathcal{O}_j$ and a_{j+1} points of $\mathcal{O}_{j+1}$ which are all mapped to the same $I_{j,l}$. To each cascade of periodic orbits of an endomorphism on the interval corresponds a well-defined sequence $(\pi_i)_{i\geq 1}$ of permutations, where $\pi_i \in \mathbf{S}_{q_i}$ represents the action of f on $\mathcal{O}_i$. We denote by $\mathcal{P}$ the set of such sequences of permutations.

A sequence $(\mathcal{O}_i)_{i\geq 1}$ of periodic orbits of $f \in \mathrm{Homeo}^+(\mathbb{D}^2)$, with periods q_i, is an $(a_i)_{i\geq 1}$-*cascade of periodic orbits* if for each $j \geq 1$, there are pairwise disjoint disks $D_{j,k}$ with $0 \leq k \leq q_j - 1$ so that each $D_{j,k}$ contains one point of $\mathcal{O}_j$ and a_{j+1} points of $\mathcal{O}_{j+1}$, and $\partial D_{j,k}$ is mapped to a curve homotopic to $\partial D_{j,l}$ relative to the points of $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{j+1}$. Let β_l be the braid type of $(\mathcal{O}_l, f)$: then we denote by $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$ the braid type of $(\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_n, f)$.

To each cascade of periodic orbits of an orientation preserving homeomorphism of the 2-disk corresponds a well-defined sequence $(\beta_i)_{i\geq 1}$ of braid types, where $\beta_i \in \mathbf{BT}_{q_i}$ represents the action of f on $\mathcal{O}_i$. We denote by $\mathcal{B}$ the set of such sequences of braid types (for a different approach to cascades, see [GGH]). By Remark 1, elements of $\mathcal{B}$ generate isomorphic copies of $\mathbb{Z}$ by iteration. Notice also that if $f : \mathbb{D}^2 \to \mathbb{D}^2$ has an $(a_i)_{i\geq 1}$-cascade of periodic orbits, then, for all $m > 0$, the restriction of f to the complement of the first m orbits, from $\mathcal{O}_1$ to $\mathcal{O}_m$, is $m-1$ times reducible in the sense of Nielsen and Thurston [Ni], [Th].

Remark 2. Let $(\mathcal{O}_i)_{i\geq 1}$ be a cascade of a periodic orbit of $f \in \mathrm{Homeo}^+(\mathbb{D}^2)$, and let $(\beta_i)_{i\geq 1}$ stand for the corresponding sequence of braid types. The braid type β_2 is determined by $\beta_2^{a_1}|_{D_{2,j}}$. Thus the braid type $\beta_1 \cup \beta_2$ is determined by the choice of a representative b_1 of β_1 in $\mathbf{B}_{a_1}$, $\beta_2^{a_1}|_{D_{2,j}}$, and an integer $k_2 \in \mathbb{Z}$ which represents an element of the center of $\mathbf{B}_{a_2}$. Similarly the braid type $\beta_1 \cup \beta_2 \cup \beta_3$ is determined by b_1, $\beta_2^{a_1}|_{D_{2,j}}$, $\beta_3^{q_2}|_{D_{3,l}}$, and a pair of integers $(k_2, k_3) \in \mathbb{Z}$. If we introduce an integer k_1 which represents an element of the center of $\mathbf{B}_{a_1}$ by which we conjugate b_1 to another representative of β_1 in $\mathbf{B}_{a_1}$, (k_2, k_3) has to be replaced by $(k_2 + k_1 a_2, k_3 + k_1 a_2 a_3)$. Otherwise speaking, the meaningful index k_i is not an integer, but a residue class modulo $\frac{q_i}{a_1}$. This allows us to sometimes use the sequence $((\beta_i^{q_{i-1}}, k_i))_{i\geq 1}$ with $k_1 = 0$ instead of the sequence $(\beta_i)_{i\geq 1}$ to represent a cascade of periodic orbits.

Clearly, we have

Proposition 1. *An endomorphism of the interval or an orientation preserving homeomorphism of the 2-disk which is $(a_i)_{i\geq 1}$-infinitely renormalizable has an $(a_i)_{i\geq 1}$-cascade of periodic orbits.*

To facilitate the discussion, it is useful to extend the definition of cascades to include cases when the π_i's or the β_i's are not necessarily cycles: we just say *cascade* in the general case, instead of "cascade of periodic orbits". When the π_i's or the β_i's are cycles for each $i \geq 1$, the collection $(\pi_i)_{i\geq 1}$ or $(\beta_i)_{i\geq 1}$ is called *minimal*.

3. Combinatorial renormalization

Let $(\pi_i)_{i\geq 1}$ be the collection of permutations which represents the action of $f \in \mathrm{End}(I)$ on an $(a_i)_{i\geq 1}$-cascade of periodic orbits $(\mathcal{O}_i)_{i\geq 1}$. The *combinatorial renormalization operators* $\mathbf{R}_{1,a_1,j} \colon \mathcal{P} \to \mathcal{P}$, $0 \leq j \leq a_1 - 1$, are defined by

$$\mathbf{R}_{1,a_1,j} \colon (\pi_i)_{i\geq 1} \mapsto (\pi_{i,j})_{i\geq 1}$$

where $\pi_{i,j}$ is the restriction to the points in $I_{i+1,j}$ of the permutation $\pi_{i+1}^{a_1}$. This definition extends readily to more general cascades.

Let $(\beta_i)_{i\geq 1}$ be the collection of braid types which represents the action of $f \in \text{Homeo}^+(\mathbb{D}^2)$ on the complement in $\mathbb{D}^2$ of an $(a_i)_{i\geq 1}$-cascade of periodic orbits $(\mathcal{O}_i)_{i\geq 1}$. The *combinatorial renormalization operators* $\mathbf{R}_{2,a_1,j}\colon \mathcal{B} \to \mathcal{B}$, $0 \leq j \leq a_1 - 1$, are defined by

$$\mathbf{R}_{2,a_1,j}\colon (\beta_i)_{i\geq 1} \mapsto (\beta_{i,j})_{i\geq 1}$$

where $\beta_{i,j}$ is the restriction to the points in $D_{i+1,j}$ of the braid type $\beta_{i+1}^{a_1}$. This definition extends readily to more general cascades.

Remark 3. Like the permutation groups $\mathbf{S}_n$ for $n > 2$, the Artin braid groups $\mathbf{B}_n$ for $n > 2$ are not commutative. However, for any finite sequence $\{b_1, b_2, \ldots, b_m\}$ of elements of $\mathbf{B}_n$, the products $b_1 \cdot b_2 \cdot \cdots \cdot b_m$, $b_2 \cdot b_3 \cdot \cdots \cdot b_1, \ldots, b_m \cdot b_1 \cdot \cdots \cdot b_{m-1}$ all represent the same element of $\mathbf{BT}_n$. As a consequence, $\beta_2^{a_1}$ splits into a_1 identical braid types with a_2 strands if β_1 is a cycle.

From this remark we easily get the following result:

Proposition 2. *For any $a_1 > 1$, there are cascades (such that π_1 is a cycle), whose images under the $\mathbf{R}_{1,a_1,j}$'s for $0 \leq j \leq a_1 - 1$ are all distinct. On the contrary, the $\mathbf{R}_{2,a_1,j}$'s for $0 \leq j \leq a_1 - 1$ are all the same if $(\beta_i)_{i\geq 1} = (\beta_1')^m_{i\geq 1}$ for some $m \in \mathbb{Z}$, where β_1' is a cycle.*

In the sequel, we will sometimes use the notation $\mathbf{R}_{2,a_1}$ to designate any of the $\mathbf{R}_{2,a_1,j}$'s when $(\beta_i)_{i\geq 1} = (\beta_1')^m_{i\geq 1}$ for some $m \in \mathbb{Z}$, where β_1' is a cycle, as in Proposition 2.

4. Dynamics of renormalization

For $k \in \{1, 2\}$, the product

$$\mathbf{R}_{k,a_n,j} \circ \cdots \circ \mathbf{R}_{k,a_2,j} \circ \mathbf{R}_{k,a_1,j}$$

acts (naturally) only on $(b_i)_{i\geq 1}$-cascades with $b_l = a_l$ for $l \geq n$, so that the combinatorial renormalization operators with $k \in \{1, 2\}$ form a pseudo-semigroup.

Given a family of collections of permutations $\{(\pi_{\alpha,i})_{i\geq 1}\}_{\alpha\in A}$, we say $(\pi_i)_{i\geq 0}$ is in the *closure* of this family if for each $m > 0$, there exists an α such that

$$\pi_i = \pi_{\alpha,i} \quad \text{for } 1 \leq i \leq m.$$

A similar definition can also be formulated for braid types.

One can then discuss the dynamics of the pseudo-semigroups of combinatorial renormalization operators. For this dynamics, it is plain that periodic points are necessarily labeled by periodic a_i sequences (for $i > 0$), and that fixed points all correspond to cases when all a_i's with $i \geq 1$ are equal to a_1. We will compare the structure of the fixed points sets in dimensions 1 and 2.

Proposition 3. *For any $a_1 > 1$, any cyclic permutation π on a_1 elements is the π_1 of a minimal fixed point $(\pi_i)_{i\geq 1}$ of all $\mathbf{R}_{1,a_1,j}$'s, $j \in \{0, 1, \ldots, a_1 - 1\}$. For any $a_1 > 1$, any braid type β with a_1 strands is the β_1 of a fixed point $(\beta_i)_{i\geq 1}$ of $\mathbf{R}_{2,a_1}$.*

Proof. In the 1-dimensional case, we just set $\pi_1 = \pi$, and for $i > 1$:

$$\pi_i = \pi_i'' \circ \pi_i',$$

where $\pi_i'(k) = (k + a_1) \mod a_1^i$ and where π_i'' is the identity on all $k > a_1$ and acts like π on the first a_1 elements.

The 2-dimensional proof is then obtained by changing notations and using Proposition 2. □

Theorem 1. *For any $a_1 > 1$ and any $j \in \{0, 1, \ldots, a_1 - 1\}$, any cyclic permutation π on a_1 elements is the π_1 of uncountably many fixed points $(\pi_i)_{i\geq 1}$ of $\mathbf{R}_{1,a_1,j}$. For any $a_1 > 1$ and any braid type β with a_1 strands is the β_1 of countably many fixed points $(\beta_i)_{i\geq 1}$ of $\mathbf{R}_{2,a_1}$.*

Proof. In the 1-dimensional case, we set $\pi_1 = \pi$. For $i > 1$, we choose arbitrarily any set $\sigma_{i,1}$, $\sigma_{i,2}, \ldots, \sigma_{i,a_1^i-2}$ of elements of $\mathbf{S}_{a_1}$, and then σ_{i,a_1^i-1} such that

$$(*) \qquad \sigma_{i,a_1^i-1} \circ \sigma_{i,a_1^i-2} \circ \cdots \circ \sigma_{i,1} = \pi.$$

Consider now the tree X imbedded in $\mathbb{R}^2$ with *vertices at level* m at (m, n) where $m \in \mathbb{N}$ and $n \in \{1, 2, \ldots, m^n\}$ and edges joining each vertex (m, n) to the a_1 vertices $(m + 1, a_1(n - 1) + k)$ with $1 \leq k \leq a_1$; we say $m + 1$ is the *level* of these edges. Define an automorphism A_1 of X by letting π act on the first level edges of X. Clearly A_1 induces π on the first level vertices (all higher level edges and vertices are moved around accordingly). After letting A_1 act on X, let each $\sigma_{1,j}$ act on the second level edges emanating from the vertex $(1, j)$ of $\pi(X)$. This defines a new automorphism, say A_2, of X and we define π_2 as the permutation on the second level vertices induced by A_2. More generally, assuming π_i and the automorphism A_i of X have been defined, we construct A_{i+1} by letting each $\sigma_{i,j}$ act on the $(i+1)^{\text{st}}$ level edges emanating from the vertex (i, j) of $A_i(X)$, and define π_{i+1} as the permutation induced by A_{i+1} on the $(i + 1)^{\text{st}}$ level vertices of X. The sequence of permutations corresponding to any cascade can be constructed in a similar way and minimality is equivalent to first impose that π_1 is a cycle and then replace equation $(*)$ by the condition that the left hand side of $(*)$ is a cycle for each i. The fixed point property is clearly equivalent to impose $(*)$ for each i. Because there are at least two choices of factors in the left hand side of $(*)$ for each i, this construction shows that for each cycle π_1, there are uncountably many fixed points of $\mathbf{R}_{1,a_1,j}$ starting with $\pi_1 = \pi$. In the 2-dimensional case, assume β is a cycle and set $\beta_1 = \beta$ (the non-cyclic case is easily deduced). As a consequence of Remark 2, by the fixed point property under $\mathbf{R}_{2,a_1}$, β_2 is completely determined by β and k_2. The choice of β and k_2 further determines the next β_j's. More precisely, with the notations of Remark 2, using the definition of a braid type, we have

$$\mathbf{R}_{2,a_1}[((\beta_i^{q_{i-1}}, k_i))_{i\geq 1}] = ((\beta_i^{q_{i-1}}, k_i))_{i\geq 2} = \left(\left(\beta_i^{q_i}, k_i - \frac{q_{i-1}}{a_1} k_2\right)\right)_{i\geq 2}.$$

Hence fixed points must satisfy the relations:

$$\beta_2^{q_1} = \beta_3^{q_2} = \beta_4^{q_3} = \cdots = \beta_1$$

and

$$k_3 - a_2 k_2 = k_2, k_4 - a_3 a_2 k_2 = k_3, k_5 - a_4 a_3 a_2 k_2 = k_4, \ldots .$$

All fixed points starting with $\beta_1 = \beta$ are obtained this way, and this construction clearly yields fixed points, hence there are countably many of them. □

Proposition 4. *If $(\pi_i)_{i\geq 1}$ is a fixed point of $\mathbf{R}_{1,a_1,j}$, so are all $(\pi_i^m)_{i\geq 1}$'s for all $m \in \mathbf{Z}$. If $(\beta_i)_{i\geq 1}$ is a fixed point of $\mathbf{R}_{2,a_1}$, so is $(\beta_i^m)_{i\geq 1}$ for any $m \in \mathbb{Z}$. The collection $(\pi_i^m)_{i\geq 1}$ or $(\beta_i^m)_{i\geq 1}$ is minimal if and only if m is coprime with a_1 and π_1 or β_1 is a cycle.*

Proof. In the 1-dimensional case, the fixed point property reads

$$\mathbf{R}_{1,a_1,j}(\pi_i) = \pi_{i+1}^{a_1}|_{I_{i+1,j}} = \pi_i \quad \text{for } i \geq 1.$$

It follows that

$$\mathbf{R}_{1,a_1,j}(\pi_i^m) = (\pi_{i+1}^m)^{a_1}|_{I_{i+1,j}} = (\pi_{i+1}^{a_1})^m|_{I_{i+1,j}} = \pi_i^m \quad \text{for } i \geq 1.$$

The minimality statement follows from the fact that when m is not coprime with a_1, π_1^m is not a cycle, while when m is coprime with a_1, it is also coprime with all a_1^i's, so that all π_i^m's are cycles.

The 2-dimensional proof is then obtained by changing notations. □

Theorem 2. *For each cycle π, any $j \in \{0, 1, \ldots, a_1 - 1\}$, and any fixed point $(\pi_i)_{i\geq 1}$ of $\mathbf{R}_{1,a_1,j}$, there are uncountably many both of minimal and non-minimal fixed points of $\mathbf{R}_{1,a_1,j}$ such that $\pi_1 = \pi$, in the closure of the set of all powers of $(\pi_i)_{i\geq 1}$.*

Proof. We shall restrict to minimal fixed points, the non-minimal case being provable in a similar way. Let $(\pi_i)_{i\geq 1}$ stand for a fixed point of $\mathbf{R}_{1,a_1,j}$. Then $(\pi_i^m)_{i\geq 1}$ is also a fixed point of $\mathbf{R}_{1,a_1,j}$ by Proposition 3. Now, π_i^m is a cycle if and only if m is coprime with a_1^i. We denote by $m_1, m_2, \ldots, m_s$ the integers coprime with a_1 in $\{1, 2, \ldots, a_1 - 1\}$, then by m_{n_1,n_2} the integers of the form $n_2 a_1 + m_{n_1}$ in $\{1, 2, \ldots, a_1^2\}$, and we define inductively the integers $m_{n_1,n_2,\ldots,n_k,n_{k+1}} = n_{k+1} a_1^k + m_{n_1,n_2,\ldots,n_k}$ in $\{1, 2, \ldots, a_1^{k+1}\}$. It is clear that each $(\pi_i^{m_{n_1,n_2,\ldots,n_i}})_{i\geq 1}$ is a minimal fixed point $\mathbf{R}_{1,a_1,j}$ in the closure of the set of all powers of $(\pi_i)_{i\geq 1}$, and that there are uncountably many distinct such objects. □

Another proof of Theorem 2 follows from considering conjugacy of the dynamics described by $(\pi_i)_{i\geq 1}$ to some adding machine: see, *e.g.*, section I.4 in [BORT] and references therein.

As a consequence of Remark 1, it is only in the elementary braid type case that one can hope for a counterpart to Theorem 2.

The fixed points with β_1 an elementary braid type are completely determined by the couple (α, k_2), where $\alpha = \frac{2p\pi}{a_1} \in [0, 2\pi[$ is the angle of the rigid rotation associated to β_1 and the integer k_2 is defined in Remark 2 (see also [GSuT]). We then have:

Theorem 3. *Let $(\beta_i)_{i\geq 1}$ be a fixed point of $\mathbf{R}_{2,a_1}$ with β_1 an elementary braid type. Then the set of powers of $(\beta_i)_{i\geq 1}$ contains all fixed points of $\mathbf{R}_{2,a_1}$ whose first braid type β_1' is an elementary braid type with a_1 strands, if and only if $\alpha = \frac{2\pi}{a_1}$ and $k = 1$ or $\alpha = \frac{2(a_1-1)\pi}{a_1}$ and $k = a_1 - 2$. The two fixed points $(\gamma_{i,a_1})_{i\geq 1}$ and $(\delta_{i,a_1})_{i\geq 1}$ so defined for each $a_1 > 1$ are such that for each i, $\gamma_{i,a_1} = (\delta_{i,a_1})^{-1}$. Thus $(\gamma_{i,a_1})_{i\geq 1}$ and $(\delta_{i,a_1})_{i\geq 1}$ define a unique fixed point of $\mathbf{R}_{2,a_1}$ up to inverse.*

Proof. From the computation in the proof of Theorem 1, for any β_1, the fixed point $(\beta_i)_{i\geq 1}$ is characterized by the pair $(\frac{2p\pi}{a_1}, k_2)$. From the same computation, for the n^{th} iterate the corresponding pair reads $(u, v) = (\{\frac{2np\pi}{a_1}\}, n(\frac{p}{a} + k_2) - [\frac{np}{a}]a)$, and it just remains to adjust p and k_2 so that when n varies through $\mathbb{Z}$, $(\frac{a_1 u}{2\pi}, v)$ varies through $\{1, 2, \ldots, a_1 - 1) \times \mathbb{Z}$. We leave the details to the reader. □

Remark 4. In the case when $a_1 = 2$, the special fixed point singled out by Theorem 3 is, up to inverse, the cascade used by Bowen and Franks to construct the first example of a C^1 Kupka-Smale diffeomorphism of the 2-sphere with no sink nor source [BF], as well as the cascade used by Franks and Young to construct the first example of a C^2 diffeomorphism with the same properties [FY] (see also [GT]).

In our discussion, there is no 1-dimensional counterpart for these "nicest" fixed points, as described in the following result, but see also Section I.6 and Theorem II.6.10 in [BORT] where one looks at simultaneous fixed points of a_1 combinatorial renormalization and skewed renormalization operators.

Theorem 4. *For each cycle π and any $j \in \{0, 1, \ldots, a_1 - 1\}$, there is no fixed point $(\pi_i)_{i\geq 1}$ of $\mathbf{R}_{1,a_1,j}$ such that the closure of the set of its powers contains all fixed points of $\mathbf{R}_{1,a_1,j}$ with $\pi_1 = \pi$.*

Proof. From the proof of Theorem 1, the number of distinct permutations one can get as π_i for a fixed point of $\mathbf{R}_{1,a_1,j}$ is $(a_1!)^{a_1^i - 1}$, while, by powers of a given fixed point, one gets at most a_1^i choices for π_i. □

5. Skewed combinatorial renormalization

Let $(\pi_i)_{i\geq 1}$ be the collection of permutations which represents the action of an endomorphism $f\colon I \to I$ on an $(a_i)_{i\geq 1}$-cascade of periodic orbits $(\mathcal{O}_i)_{i\geq 1}$. The *skewed combinatorial renormalization operators* $\mathbf{SR}_{1,a_1,j}\colon \mathcal{P} \to \mathcal{P}$, $0 \leq j \leq a_1 - 1$, are defined by

$$\mathbf{SR}_{1,a_1,j}\colon (\pi_i)_{i\geq 1} \mapsto (\pi_{i,j})_{i\geq 1}$$

where $\pi_{i,j}$ is obtained, from the restriction to the points in $I_{i+1,j}$ of the permutation $\pi_{i+1}^{a_1}$, by conjugacy by $\Delta_{\frac{q_{i+1}}{a_1}}$, where $\Delta_n(i) = n + 1 - i$. This definition extends easily to more general cascades.

Let $(\beta_i)_{i\geq 1}$ be the collection of braid types which represents the action of a homeomorphism $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ on the complement in $\mathbb{D}^2$ of an $(a_i)_{i\geq 1}$-cascade of periodic orbits $(\mathcal{O}_i)_{i\geq 1}$. The *skewed combinatorial renormalization operators* $\mathbf{SR}_{2,a_1,j}\colon \mathcal{B} \to \mathcal{B}$, $0 \leq j \leq a_1 - 1$, are defined by

$$\mathbf{SR}_{2,a_1,j}\colon (\beta_i)_{i\geq 1} \mapsto (\beta_{i,j})_{i\geq 1}$$

where $\beta_{i,j}$ is obtained as follows:

– take a homeomorphism h realizing the restriction $\beta_{i,j}$ of the braid type $\beta_{i+1}^{a_1}$ to the points $\mathcal{O}'$ of $(\mathcal{O}_i)_{i\geq 1} \cap D_{i+1,j}$,

–conjugate h by an orientation reversing homeomorphism k to get a homeomorphism h', and set $\mathcal{O}'' = k(\mathcal{O}')$,

–$\beta_{i,j}$ is then the braid type of $(\mathcal{O}'', h')$.

If β is the braid type of $(\mathcal{O}', h)$, we denote by $-\beta$ the braid type of $(\mathcal{O}'', h')$. The definition of the skewed combinatorial renormalization operator extends readily to more general cascades.

Propositions 2, 3, and 4, and Theorems 1, 2, and 4 have natural counterparts for skewed combinatorial renormalization. Hence, in particular, the symbols $\mathbf{SR}_{2,a_1}$ make sense.

The fixed points of $\mathbf{SR}_{2,a_1}$ must be of the form $((\beta, 0), (-\beta, k_2), (\beta, k_3), \ldots)$. When β_1 is an elementary braid type, the fixed points are again completely determined by the couple (α, k_2), described before Theorem 3. We then have:

Theorem 5. *Let $(\beta_i)_{i\geq 1}$ be a fixed point of $\mathbf{SR}_{2,a_1}$ with β_1 an elementary braid type. Then the set of powers of $(\beta_i)_{i\geq 1}$ contains all fixed points of $\mathbf{SR}_{2,a_1}$ whose first braid type β_1' is an elementary braid type with a_1 strands, if and only if $\alpha = \frac{2\pi}{a_1}$ and $k = 1$ or $\alpha = \frac{2(a_1-1)\pi}{a_1}$ and $k = a_1$. The two fixed points $(\lambda_{i,a_1})_{i\geq 1}$ and $(\mu_{i,a_1})_{i\geq 1}$ so* *defined for each* $a_1 > 1$ *are such that for each* i, $\lambda_{i,a_1} = (\mu_{i,a_1})^{-1}$. *Thus* $(\lambda_{i,a_1})_{i\geq 1}$ *and* $(\mu_{i,a_1})_{i\geq 1}$ *define a unique fixed point of* $\mathbf{SR}_{2,a_1}$ *up to inverse.*

Remark 5. In the case when $a_1 = 2$, the special fixed point singled out by Theorem 3 is, up to inverse, the cascade used in [GStT] to construct the first example of a C^∞ Kupka-Smale diffeomorphism of the 2-sphere with no sink nor source.

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