

HYPERSPACES AND CONES

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ABSTRACT. We characterize locally connected continua X for which its hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension and is homeomorphic to the cone of a continuum Z .

INTRODUCTION

During the history of hyperspaces several people have studied those continua X for which its hyperspace of subcontinua $\mathcal{C}(X)$ is homeomorphic to its cone $\mathcal{K}(X)$; see for example [D-R], [I], [N1], [N2, chapter VIII], [R1], and [R2]. In his book [N2], Nadler also considers the case when the hyperspace $\mathcal{C}(X)$ of a continuum X is homeomorphic to the cone $\mathcal{K}(Z)$ of a continuum Z . Nadler ([N2, p. 333]) says that he can prove that “if X is a locally connected continuum for which its hyperspace $\mathcal{C}(X)$ has finite dimension and is homeomorphic to the cone of a continuum Z , then X and Z must be arcs or circles”. However, we prove the following result:

Theorem 3. *If X is a simple n -od, then $\mathcal{C}(X)$ is homeomorphic to the cone $\mathcal{K}(Z)$ of a continuum Z .*

We also prove the following theorem:

Theorem 4. *Let X be a locally connected continuum whose hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension. If $\mathcal{C}(X)$ is homeomorphic to the cone, $\mathcal{K}(Z)$, over a continuum Z , then X is an arc, a simple closed curve, or a simple n -od.*

Definitions. If Z is a topological space and $A \subset Z$, then the closure of A in Z is denoted by $\text{Cl}_Z(A)$, or by $\text{Cl}(A)$ if there is no confusion, its interior by $\text{Int}_Z(A)$, and its boundary by $\partial_Z(A)$. If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball around A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$; we will write $\mathcal{V}_\varepsilon(y)$ for $\mathcal{V}_\varepsilon(\{y\})$. If A is a subset of Y , then $\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$. The set of positive integers is denoted by \mathbb{N} .

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a continuum X is a continuum contained in X . By a *graph* we mean a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end-points. By a *segment* of a graph X we shall always mean one of those arcs or a simple closed curve. The end-points of the segments of X are called *vertices* of X . Given a point $x \in X$ and

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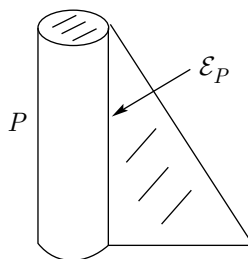


FIGURE 1

a natural number n , the *order* of X at x , denoted $\text{ord}_x X$, is n provided that for every $\varepsilon > 0$, there exists an open set U of X containing x with $\text{diam}(U) < \varepsilon$ such that $\partial_X(U)$ consists of exactly n points ([N3, Chapter 9]). For each point $v \in X$, we have either $\text{ord}_v X = 1$ if v is an end-point of X , or $\text{ord}_v X \geq 2$ otherwise. If $\text{ord}_v X \geq 3$, then v is called a *ramification point* of X . By a *simple n -od* ($n \geq 3$) we mean a graph X with only one ramification point and exactly n segments without circles. A simple 3-od will be called a *simple triod*.

Given a continuum X , the *hyperspace of nonempty subcontinua* of X is:

$$\mathcal{C}(X) = \{A \subset X \mid A \text{ is a continuum}\}.$$

It is known that $\mathcal{C}(X)$ is a metric space with the Hausdorff metric ([N2, (0.1)]), and in fact, $\mathcal{C}(X)$ is a continuum ([N2, (1.13)]).

Given a continuum X , the *cone over X* is the decomposition space of the upper semicontinuous decomposition $(X \times [0, 1])/X \times \{1\}$. The cone over X will be denoted by $\mathcal{K}(X)$, its base $X \times \{0\}$ by $\mathcal{B}(X)$, and its vertex $X \times \{1\} \in \mathcal{K}(X)$ by v . The symbol π will denote the projection $\pi: \mathcal{K}(X) \setminus \{v\} \rightarrow \mathcal{B}(X)$ given by $\pi((x, t)) = (x, 0)$.

Duda in [D] made a very interesting study of the hyperspaces of graphs, showing that $\mathcal{C}(X)$ is a finite-dimensional locally connected continuum if and only if X is a graph, and that $\mathcal{C}(X)$ is a polyhedron if and only if X is a graph.

From now on, we are going to assume that X is a locally connected continuum for which $\dim(\mathcal{C}(X)) < \infty$, hence X is a graph (see [D, 1.1]).

We are going to use Duda's notation of [D], in particular we are going to assume that the graph X satisfies the following conditions:

- (α): *Each segment of X has length equal to 1 and the distance between any two of its points is equal to the length of the shortest arc joining them.*
- (β): *Each vertex of X is either an end-point or a ramification point of X .*

Given a graph X , $\mathcal{D}_{\mathcal{C}(X)} = \text{Cl}_{\mathcal{C}(X)}\{C \in \mathcal{C}(X) \mid \dim_C \mathcal{C}(X) = 2\}$, that is, $\mathcal{D}_{\mathcal{C}(X)}$ is the closure of the 2-dimensional part of the polyhedron $\mathcal{C}(X)$.

Given an arbitrary polyhedron P , by \mathcal{E}_P we denote the closure (in P) of the subset of P consisting of all points $p \in P$ for which there exists a closed neighborhood V homeomorphic to a 2-dimensional ball and such that p lies on the boundary of that ball (see Figure 1).

Duda showed the following two results:

Theorem 1 ([D, 8.2]). *If X is a graph, then $\mathcal{D}_{\mathcal{C}(X)}$ is homeomorphic to X with a 2-dimensional ball attached to each segment A of X along an arc lying on the*

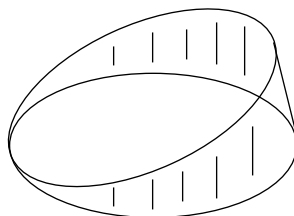


FIGURE 2

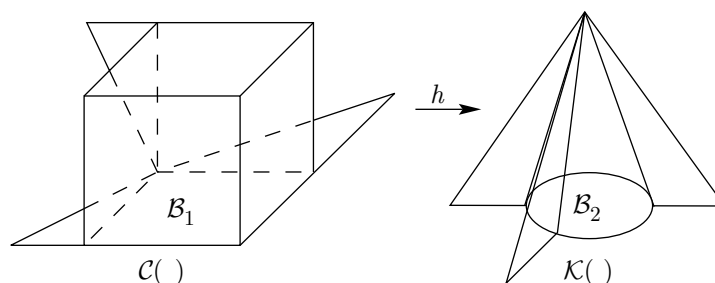


FIGURE 3

boundary of that ball in such a way that any two distinct balls are disjoint outside X .

Note that $\mathcal{D}_{C(X)}$ consists of two types of “discs” B : on one hand B is just a triangle; on the other hand B is a loop (a segment for which both end-points coincide) with a 2-dimensional ball attached along that loop (see Figure 2).

Theorem 2 ([D, 8.4]). *If X is a graph containing a ramification point, then $\mathcal{E}_{C(X)}$ is homeomorphic to X .*

THE CHARACTERIZATION

Theorem 3. *If X is a simple n -od, then $\mathcal{C}(X)$ is homeomorphic to the cone of a continuum Z .*

Proof. Let \mathcal{B}^{n-1} be the closed unit $(n-1)$ -ball in \mathbb{R}^{n-1} with center at the origin; let A_1, \dots, A_n be the straight line segments, which are mutually disjoint, from points p_1, \dots, p_n on the surface \mathcal{S}^{n-2} of \mathcal{B}^{n-1} to points q_1, \dots, q_n in $\mathbb{R}^{n-1} \setminus \mathcal{B}^{n-1}$. Then let $Z = \mathcal{B}^{n-1} \cup (\bigcup_{k=1}^n A_n)$.

Observe that $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ have the same structure: each of them is the union of a convex n -ball and n 2-balls, which intersect on the surface of the n -ball in an n -od that is made up of n convex arcs (see Figure 3). Denote the n -balls in $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ by \mathcal{B}_1 and \mathcal{B}_2 , respectively, and denote the n -ods in $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ just mentioned by N_1 and N_2 . Then, there is a homeomorphism, h , from \mathcal{B}_1 onto \mathcal{B}_2 such that $h(N_1) = N_2$. Clearly, h can be extended to a homeomorphism from $\mathcal{C}(X)$ onto $\mathcal{K}(Z)$. Q.E.D.

Theorem 4. *Let X be a locally connected continuum whose hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension. If $\mathcal{C}(X)$ is homeomorphic to the cone, $\mathcal{K}(Z)$, over a continuum Z , then X is an arc, a simple closed curve, or a simple n -od.*

Proof. Suppose $\mathcal{C}(X) = \mathcal{K}(Z)$, and let B be one of the 2-dimensional discs of $\mathcal{D}_{\mathcal{C}(X)}$, B could be a loop with a 2-dimensional ball attached along that loop. Let us observe that $B = \text{Cl}_{\mathcal{C}(X)}(\text{Int}_{\mathcal{C}(X)}(B))$. Let $(z, t) \in \text{Int}_{\mathcal{C}(X)}(B)$; then there exist an $\varepsilon > 0$ and an open set U of $\mathcal{B}(Z)$ such that $U \times (t - \varepsilon, t + \varepsilon) \subset \text{Int}_{\mathcal{C}(X)}(B)$, thus, $U \times (t - \varepsilon, t + \varepsilon)$ is a 2-dimensional open set of $\mathcal{C}(X)$, hence $U \times (0, 1)$ is a 2-dimensional open set of $\mathcal{C}(X)$ containing $\{(z, 0)\} \times (0, 1)$. Since $\{(z, 0)\} \times (0, 1)$ is connected and the 2-dimensional part of $\mathcal{C}(X)$ is contained in $\mathcal{D}_{\mathcal{C}(X)}$, by Theorem 1, we have that $\{(z, 0)\} \times (0, 1)$ is contained in B , otherwise $\{(z, 0)\} \times (0, 1)$ would contain a ramification point (z, s) of $\mathcal{E}_{\mathcal{C}(X)}$, but the dimension of $\mathcal{C}(X)$ at the point (z, s) is bigger than two, a contradiction, hence $\{(z, 0)\} \times (0, 1)$ is contained in B , in fact $\{(z, 0)\} \times (0, 1) \subset \text{Int}_{\mathcal{C}(X)}(B)$. Observe that this implies that $v \in B$. Let us note that v is a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$, otherwise $\mathcal{D}_{\mathcal{C}(X)}$ would consist of one disc, which would imply that X would contain only one segment, that is, X would be an arc or a circle, contrary to our hypothesis.

Now let $\mathcal{A} = \pi(\text{Int}_{\mathcal{C}(X)}(B))$; then \mathcal{A} is an open, connected and locally arcwise connected subset of $\mathcal{B}(Z)$.

Note that $\pi(B \setminus \{v\}) = \text{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$. To prove this, let $(z, 0) \in \pi(B \setminus \{v\})$; then there exists $t \geq 0$ such that $(z, t) \in B \setminus \{v\}$. Since $\text{Int}_{\mathcal{C}(X)}(B)$ is dense, there is a sequence $\{(z_n, t_n)\}_{n=1}^{\infty} \subset \text{Int}_{\mathcal{C}(X)}(B)$ converging to (z, t) , hence the sequence $\{(z_n, 0)\}_{n=1}^{\infty} \subset \mathcal{A}$ and converges to $(z, 0)$. Thus $(z, 0) \in \text{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$. To see the other inclusion, let $(z, 0) \in \text{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$; then there exists a sequence $\{(z_n, 0)\}_{n=1}^{\infty}$ contained in \mathcal{A} and converging to $(z, 0)$. Let $0 < t < 1$, hence $\{(z_n, t)\}_{n=1}^{\infty}$ is contained in $\text{Int}_{\mathcal{C}(X)}(B)$, thus $\{(z_n, t)\}_{n=1}^{\infty}$ converges to $(z, t) \in B \setminus \{v\}$ and $\pi((z, t)) = (z, 0)$. Therefore $\text{Cl}_{\mathcal{B}(Z)}(\mathcal{A}) = \pi(B \setminus \{v\})$, and $B = \mathcal{K}(\text{Cl}_{\mathcal{B}(Z)}(\mathcal{A}))$.

We will show that $\mathcal{E}_{\mathcal{C}(X)}$ contains only one ramification point. Suppose $\mathcal{E}_{\mathcal{C}(X)}$ has more than one ramification point. Let v_1 be a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$ different from v . Since v_1 is a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$, there are at least two 2-dimensional discs B_1 and B_2 of $\mathcal{D}_{\mathcal{C}(X)}$ such that $v_1 \in B_1 \cap B_2$ (condition (β) in the Introduction). Then $\pi(v_1) \in \text{Cl}_{\mathcal{B}(Z)}(\mathcal{A}_1) \cap \text{Cl}_{\mathcal{B}(Z)}(\mathcal{A}_2)$, where $\mathcal{A}_l = \pi(\text{Int}_{\mathcal{C}(X)}(B_l))$, $l \in \{1, 2\}$. Hence, we would have that $\pi(v_1) \times [0, 1] \cup \{v\} \subset B_1 \cap B_2$, but this contradicts the fact that $B_1 \cap B_2$ contains at most two points, Theorem 1.

Finally we will prove $\mathcal{E}_{\mathcal{C}(X)}$ does not contain loops. Suppose $\mathcal{E}_{\mathcal{C}(X)}$ has a loop A ; then the disc B of $\mathcal{D}_{\mathcal{C}(X)}$ containing A would not be contractible. On the other hand, since $B = \mathcal{K}(\text{Cl}_{\mathcal{B}(Z)}(\mathcal{A}))$, we would have that B is contractible, a contradiction. Therefore $\mathcal{E}_{\mathcal{C}(X)}$ does not contain loops. Hence $\mathcal{E}_{\mathcal{C}(X)}$ is a graph with one ramification point and without loops, thus $\mathcal{E}_{\mathcal{C}(X)}$ is an n -od. Since X is homeomorphic to $\mathcal{E}_{\mathcal{C}(X)}$, Theorem 2, we have that X is an n -od. Q.E.D.

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