PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 3069–3073
S 0002-9939(97)04175-0

HYPERSPACES AND CONES

SERGIO MACÍAS

(Communicated by James West)

ABSTRACT. We characterize locally connected continua X for which its hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension and is homeomorphic to the cone of a continuum Z.

INTRODUCTION

During the history of hyperspaces several people have studied those continua X for which its hyperspace of subcontinua $\mathcal{C}(X)$ is homeomorphic to its cone $\mathcal{K}(X)$; see for example [D-R], [I], [N1], [N2, chapter VIII], [R1], and [R2]. In his book [N2], Nadler also considers the case when the hyperspace $\mathcal{C}(X)$ of a continuum X is homeomorphic to the cone $\mathcal{K}(Z)$ of a continuum Z. Nadler ([N2, p. 333]) says that he can prove that "if X is a locally connected continuum for which its hyperspace $\mathcal{C}(X)$ has finite dimension and is homeomorphic to the cone of a continuum Z, then X and Z must be arcs or circles". However, we prove the following result:

Theorem 3. *If X is a simple n-od, then $\mathcal{C}(X)$ is homeomorphic to the cone $\mathcal{K}(Z)$ of a continuum Z.*

We also prove the following theorem:

Theorem 4. *Let X be a locally connected continuum whose hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension. If $\mathcal{C}(X)$ is homeomorphic to the cone, $\mathcal{K}(Z)$, over a continuum Z, then X is an arc, a simple closed curve, or a simple n-od.*

Definitions. If Z is a topological space and $A \subset Z$, then the closure of A in Z is denoted by $\mathrm{Cl}_Z(A)$, or by $\mathrm{Cl}(A)$ if there is no confusion, its interior by $\mathrm{Int}_Z(A)$, and its boundary by $\partial_Z(A)$. If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball around A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$; we will write $\mathcal{V}_\varepsilon(y)$ for $\mathcal{V}_\varepsilon(\{y\})$. If A is a subset of Y, then $\mathrm{diam}(A) = \sup\{d(a_1, a_2) | a_1, a_2 \in A\}$. The set of positive integers is denoted by $\mathbb{N}$.

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a continuum X is a continuum contained in X. By a *graph* we mean a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end-points. By a *segment* of a graph X we shall always mean one of those arcs or a simple closed curve. The end-points of the segments of X are called *vertices* of X. Given a point $x \in X$ and

Received by the editors November 19, 1995.
1991 *Mathematics Subject Classification.* Primary 54B20.
Key words and phrases. Hyperspace, continuum, cone.

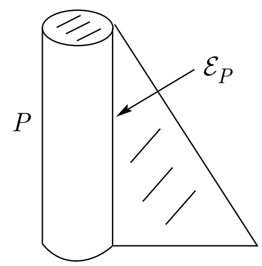

FIGURE 1

a natural number n, the *order* of X at x, denoted $\operatorname{ord}_x X$, is n provided that for every $\varepsilon > 0$, there exists an open set U of X containing x with $\operatorname{diam}(U) < \varepsilon$ such that $\partial_X(U)$ consists of exactly n points ([N3, Chapter 9]). For each point $v \in X$, we have either $\operatorname{ord}_v X = 1$ if v is an end-point of X, or $\operatorname{ord}_v X \geq 2$ otherwise. If $\operatorname{ord}_v X \geq 3$, then v is called a *ramification point* of X. By a *simple n-od* ($n \geq 3$) we mean a graph X with only one ramification point and exactly n segments without circles. A simple 3-od will be called a *simple triod*.

Given a continuum X, the *hyperspace of nonempty subcontinua of* X is:

$$\mathcal{C}(X) = \{A \subset X | A \text{ is a continuum}\}.$$

It is known that $\mathcal{C}(X)$ is a metric space with the Hausdorff metric ([N2, (0.1)]), and in fact, $\mathcal{C}(X)$ is a continuum ([N2, (1.13)]).

Given a continuum X, the *cone over* X is the decomposition space of the upper semicontinuous decomposition $(X \times [0,1])/X \times \{1\}$. The cone over X will be denoted by $\mathcal{K}(X)$, its base $X \times \{0\}$ by $\mathcal{B}(X)$, and its vertex $X \times \{1\} \in \mathcal{K}(X)$ by v. The symbol π will denote the projection $\pi \colon \mathcal{K}(X) \setminus \{v\} \to \mathcal{B}(X)$ given by $\pi((x,t)) = (x,0)$.

Duda in [D] made a very interesting study of the hyperspaces of graphs, showing that $\mathcal{C}(X)$ is a finite-dimensional locally connected continuum if and only if X is a graph, and that $\mathcal{C}(X)$ is a polyhedron if and only if X is a graph.

From now on, we are going to assume that X is a locally connected continuum for which $\dim(\mathcal{C}(X)) < \infty$, hence X is a graph (see [D, 1.1]).

We are going to use Duda's notation of [D], in particular we are going to assume that the graph X satisfies the following conditions:

(α)**:** *Each segment of X has length equal to* 1 *and the distance between any two of its points is equal to the length of the shortest arc joining them.*

(β)**:** *Each vertex of X is either an end-point or a ramification point of X.*

Given a graph X, $\mathcal{D}_{\mathcal{C}(X)} = \operatorname{Cl}_{\mathcal{C}(X)}\{C \in \mathcal{C}(X) | \dim_C \mathcal{C}(X) = 2\}$, that is, $\mathcal{D}_{\mathcal{C}(X)}$ is the closure of the 2-dimensional part of the polyhedron $\mathcal{C}(X)$.

Given an arbitrary polyhedron P, by $\mathcal{E}_P$ we denote the closure (in P) of the subset of P consisting of all points $p \in P$ for which there exists a closed neighborhood V homeomorphic to a 2-dimensional ball and such that p lies on the boundary of that ball (see Figure 1).

Duda showed the following two results:

Theorem 1 ([D, 8.2])**.** *If X is a graph, then $\mathcal{D}_{\mathcal{C}(X)}$ is homeomorphic to X with a* 2*-dimensional ball attached to each segment A of X along an arc lying on the*

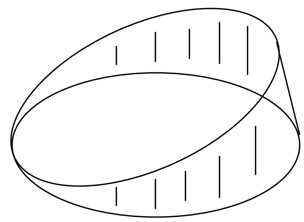

FIGURE 2

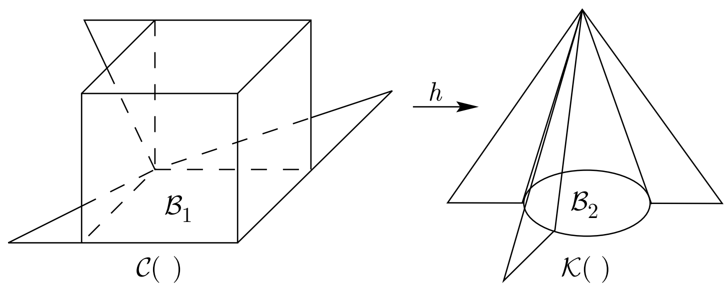

FIGURE 3

boundary of that ball in such a way that any two distinct balls are disjoint outside X.

Note that $\mathcal{D}_{\mathcal{C}(X)}$ consists of two types of "*discs*" B: on one hand B is just a triangle; on the other hand B is a loop (a segment for which both end-points coincide) with a 2-dimensional ball attached along that loop (see Figure 2).

Theorem 2 ([D, 8.4])**.** *If X is a graph containing a ramification point, then $\mathcal{E}_{\mathcal{C}(X)}$ is homeomorphic to X.*

THE CHARACTERIZATION

Theorem 3. *If X is a simple n-od, then $\mathcal{C}(X)$ is homeomorphic to the cone of a continuum Z.*

Proof. Let $\mathcal{B}^{n-1}$ be the closed unit $(n-1)$-ball in $\mathbb{R}^{n-1}$ with center at the origin; let $A_1, \dots, A_n$ be the straight line segments, which are mutually disjoint, from points $p_1, \dots, p_n$ on the surface $\mathcal{S}^{n-2}$ of $\mathcal{B}^{n-1}$ to points $q_1, \dots, q_n$ in $\mathbb{R}^{n-1} \backslash \mathcal{B}_{n-1}$. Then let $Z = \mathcal{B}^{n-1} \cup (\bigcup_{k=1}^{n} A_n)$.

Observe that $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ have the same structure: each of them is the union of a convex n-ball and n 2-balls, which intersect on the surface of the n-ball in an n-od that is made up of n convex arcs (see Figure 3). Denote the n-balls in $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and denote the n-ods in $\mathcal{C}(X)$ and $\mathcal{K}(Z)$ just mentioned by N_1 and N_2. Then, there is a homeomorphism, h, from $\mathcal{B}_1$ onto $\mathcal{B}_2$ such that $h(N_1) = N_2$. Clearly, h can be extended to a homeomorphism from $\mathcal{C}(X)$ onto $\mathcal{K}(Z)$. Q.E.D.

Theorem 4. *Let X be a locally connected continuum whose hyperspace of subcontinua, $\mathcal{C}(X)$, has finite dimension. If $\mathcal{C}(X)$ is homeomorphic to the cone, $\mathcal{K}(Z)$, over a continuum Z, then X is an arc, a simple closed curve, or a simple n-od.*

Proof. Suppose $\mathcal{C}(X) \underset{\text{top}}{=} \mathcal{K}(Z)$, and let B be one of the 2-dimensional discs of $\mathcal{D}_{\mathcal{C}(X)}$, B could be a loop with a 2-dimensional ball attached along that loop. Let us observe that $B = \mathrm{Cl}_{\mathcal{C}(X)}(\mathrm{Int}_{\mathcal{C}(X)}(B))$. Let $(z,t) \in \mathrm{Int}_{\mathcal{C}(X)}(B)$; then there exist an $\varepsilon > 0$ and an open set U of $\mathcal{B}(Z)$ such that $U \times (t-\varepsilon, t+\varepsilon) \subset \mathrm{Int}_{\mathcal{C}(X)}(B)$, thus, $U \times (t-\varepsilon, t+\varepsilon)$ is a 2-dimensional open set of $\mathcal{C}(X)$, hence $U \times (0,1)$ is a 2-dimensional open set of $\mathcal{C}(X)$ containing $\{(z,0)\} \times (0,1)$. Since $\{(z,0)\} \times (0,1)$ is connected and the 2-dimensional part of $\mathcal{C}(X)$ is contained in $\mathcal{D}_{\mathcal{C}(X)}$, by Theorem 1, we have that $\{(z,0)\} \times (0,1)$ is contained in B, otherwise $\{(z,0)\} \times (0,1)$ would contain a ramification point (z,s) of $\mathcal{E}_{\mathcal{C}(X)}$, but the dimension of $\mathcal{C}(X)$ at the point (z,s) is bigger than two, a contradiction, hence $\{(z,0)\} \times (0,1)$ is contained in B, in fact $\{(z,0)\} \times (0,1) \subset \mathrm{Int}_{\mathcal{C}(X)}(B)$. Observe that this implies that $v \in B$. Let us note that v is a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$, otherwise $\mathcal{D}_{\mathcal{C}(X)}$ would consist of one disc, which would imply that X would contain only one segment, that is, X would be an arc or a circle, contrary to our hypothesis.

Now let $\mathcal{A} = \pi(\mathrm{Int}_{\mathcal{C}(X)}(B))$; then $\mathcal{A}$ is an open, connected and locally arcwise connected subset of $\mathcal{B}(Z)$.

Note that $\pi(B\backslash\{v\}) = \mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$. To prove this, let $(z,0) \in \pi(B\backslash\{v\})$; then there exists $t \geq 0$ such that $(z,t) \in B\backslash\{v\}$. Since $\mathrm{Int}_{\mathcal{C}(X)}(B)$ is dense, there is a sequence $\{(z_n,t_n)\}_{n=1}^{\infty} \subset \mathrm{Int}_{\mathcal{C}(X)}(B)$ converging to (z,t), hence the sequence $\{(z_n,0)\}_{n=1}^{\infty} \subset \mathcal{A}$ and converges to $(z,0)$. Thus $(z,0) \in \mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$. To see the other inclusion, let $(z,0) \in \mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A})$; then there exists a sequence $\{(z_n,0)\}_{n=1}^{\infty}$ contained in $\mathcal{A}$ and converging to $(z,0)$. Let $0 < t < 1$, hence $\{(z_n,t)\}_{n=1}^{\infty}$ is contained in $\mathrm{Int}_{\mathcal{C}(X)}(B)$, thus $\{(z_n,t)\}_{n=1}^{\infty}$ converges to $(z,t) \in B\backslash\{v\}$ and $\pi((z,t)) = (z,0)$. Therefore $\mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A}) = \pi(B\backslash\{v\})$, and $B = \mathcal{K}(\mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A}))$.

We will show that $\mathcal{E}_{\mathcal{C}(X)}$ contains only one ramification point. Suppose $\mathcal{E}_{\mathcal{C}(X)}$ has more than one ramification point. Let v_1 be a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$ different from v. Since v_1 is a ramification point of $\mathcal{E}_{\mathcal{C}(X)}$, there are at least two 2-dimensional discs B_1 and B_2 of $\mathcal{D}_{\mathcal{C}(X)}$ such that $v_1 \in B_1 \cap B_2$ (condition (β) in the Introduction). Then $\pi(v_1) \in \mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A}_1) \cap \mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A}_2)$, where $\mathcal{A}_l = \pi(\mathrm{Int}_{\mathcal{C}(X)}(B_l))$, $l \in \{1,2\}$. Hence, we would have that $\pi(v_1) \times [0,1) \cup \{v\} \subset B_1 \cap B_2$, but this contradicts the fact that $B_1 \cap B_2$ contains at most two points, Theorem 1.

Finally we will prove $\mathcal{E}_{\mathcal{C}(X)}$ does not contain loops. Suppose $\mathcal{E}_{\mathcal{C}(X)}$ has a loop A; then the disc B of $\mathcal{D}_{\mathcal{C}(X)}$ containing A would not be contractible. On the other hand, since $B = \mathcal{K}(\mathrm{Cl}_{\mathcal{B}(Z)}(\mathcal{A}))$, we would have that B is contractible, a contradiction. Therefore $\mathcal{E}_{\mathcal{C}(X)}$ does not contain loops. Hence $\mathcal{E}_{\mathcal{C}(X)}$ is a graph with one ramification point and without loops, thus $\mathcal{E}_{\mathcal{C}(X)}$ is an n-od. Since X is homeomorphic to $\mathcal{E}_{\mathcal{C}(X)}$, Theorem 2, we have that X is an n-od. Q.E.D.

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Instituto de Matemáticas, Circuito Exterior, Ciudad Universitaria, México, D.F., C.P. 04510, México

E-mail address: `macias@servidor.unam.mx`