PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 3039–3047
S 0002-9939(97)04195-6

DEFORMATIONS OF DIHEDRAL REPRESENTATIONS

MICHAEL HEUSENER AND ERIC KLASSEN

(Communicated by Ronald Stern)

Abstract. G. Burde proved (1990) that the $\mathrm{SU}_2(\mathbb{C})$ representation space of two-bridge knot groups is one-dimensional. The same holds for all torus knot groups.

The aim of this note is to prove the following:
Given a knot $k \subset S^3$ we denote by $\hat{C}_2$ its twofold branched covering space. Assume that there is a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exist representations of the knot group onto the binary dihedral group $D_p \subset \mathrm{SU}_2(\mathbb{C})$ and these representations are smooth points on a one-dimensional curve of representations into $\mathrm{SU}_2(\mathbb{C})$.

Let $k \subset S^3$ be a knot, let $G = \pi_1(S^3 - k)$, and assume $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation, i.e. the only subspaces of $\mathbb{C}^2$ which are invariant under $\varrho(G)$ are $\{0\}$ and $\{\mathbb{C}^2\}$. According to a result of Thurston (see [5, Proposition 3.2.1]) it is possible to deform ϱ non-trivially, i.e. ϱ is contained in a component R of the $\mathrm{SL}_2(\mathbb{C})$ representation space which is at least four-dimensional.

There is no general theorem which allows the deformation of representations $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$. In [6] the authors proved that every non-abelian representation corresponding to a simple root of the Alexander polynomial on the complex unit circle is a limit point of an arc of non-abelian representations.

Given a knot $k \subset S^3$ we denote by $\hat{C}_2$ the twofold branched covering of the pair (S^3, k) and by $\Delta(t)$ the Alexander polynomial of the knot. The aim of this note is to prove the following theorem:

Theorem 1. *Let $k \subset S^3$ be given. Assume there exists a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exists a representation $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$ such that $\operatorname{Im} \varrho$ is a binary dihedral group of order $4p$. Moreover, $[\varrho] \in \Re(G)$ is a regular point and there is a neighborhood $U = U([\varrho]) \subset \Re(G)$ which is diffeomorphic to an interval.*

Remark 1. Theorem 1 applies if $H_1(\hat{C}_2)$ is cyclic and non-trivial. For example it is valid for all 2-bridge knots (see [2]).

Remark 2. The converse of Theorem 1 is false. For example, in the case of the knot $k = 9_{35}$, the group $H_1(\hat{C}_2, \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, but all dihedral representations are smooth points on one-dimensional components of $\Re(G)$.

Received by the editors September 7, 1993.
1991 *Mathematics Subject Classification.* Primary 57M25, 57M05.
Key words and phrases. Knot groups, group representations, $\mathrm{SU}_2(\mathbb{C})$.
The second author was supported in part by a National Science Foundation Postdoctoral Research Fellowship.

1. Introduction

Given a finitely generated group G we denote

$$\mathcal{R}(G,\mathfrak{G}) := \{\varrho : G \to \mathfrak{G} |\ \varrho \text{ is a homomorphism}\}$$

where $\mathfrak{G} \in \{\mathrm{SL}_2(\mathbb{C}), \mathrm{SU}_2(\mathbb{C})\}$. The space $\mathcal{R}(G,\mathfrak{G})$ has the structure of a complex (resp. real) affine algebraic set if $\mathfrak{G} = \mathrm{SL}_2(\mathbb{C})$ (resp. $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$). We call two representations $\varrho, \varrho' : G \to \mathfrak{G}$ equivalent ($\varrho \sim \varrho'$) iff they differ by an inner automorphism of $\mathfrak{G}$. Let $\mathcal{A}(G,\mathfrak{G}) := \{\varrho \in \mathcal{R}(G,\mathfrak{G}) |\, \mathrm{Im}(\varrho) \text{ is abelian}\}$.

If $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$ we denote by $\Re(G)$ the space of conjugacy classes of non-abelian $\mathrm{SU}_2(\mathbb{C})$ representations, i.e.

$$\Re(G) := (\mathcal{R}(G, \mathrm{SU}_2(\mathbb{C})) \setminus \mathcal{A}(G, \mathrm{SU}_2(\mathbb{C})))/\sim .$$

For each $\varrho \in \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C}))$ there is a character $\chi_\varrho : G \to \mathbb{C}$ given by $\chi_\varrho : g \mapsto \mathrm{tr}\, \varrho(g)$. We have $\chi_\varrho = \chi_{\varrho'}$ if $\varrho \sim \varrho'$. The set $X(G)$ of these characters is a complex affine algebraic set; its ambient coordinates are given by $\{\chi(g_i)\}$ where $\chi \in X(G)$ is a character and $\{g_i\} \subset G$ is finite. Moreover, the map $t : \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C})) \to X(G)$ given by $t : \varrho \mapsto \chi_\varrho$ is polynomial in the ambient coordinates (for details see [5]).

2. Real points in $X(G)$

Because $\mathrm{SU}_2(\mathbb{C})$ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, there is an obvious map

$$t : \mathcal{R}(G, \mathrm{SU}_2(\mathbb{C})) \to X(G)$$

which associates to each representation the corresponding character. This map induces an injection $\hat{t} : \Re(G) \to X(G)$ (see [8]).

Definition 1. We call a representation $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ real iff $\mathrm{Im}(\varrho) \subset \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{Im}(\varrho) \subset \mathrm{SU}_2(\mathbb{C})$.

If $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is conjugate to a real representation, then $\chi_\varrho : G \to \mathbb{C}$ is a real-valued function. The converse is true for irreducible representations:

Lemma 1. *Let $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. If $\chi_\varrho(G) \subset \mathbb{R}$, then ϱ is equivalent to a real representation.*

Proof. Choose $g \in G$ such that $\varrho(g) \neq \pm\mathbf{1}$. By [5, Lemma 1.5.1] there exists $h \in G$ such that the restriction of ϱ to the subgroup generated by g and h is irreducible and such that $\chi_\varrho(h) \neq \pm 2$.

By conjugation we assume that $\varrho(h) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ where $\lambda + \lambda^{-1} \in \mathbb{R}$ and $\lambda \neq \pm 1$. Moreover, we have $\varrho(g) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a + d \in \mathbb{R}$ and $cb \neq 0$.

If $|\lambda + \lambda^{-1}| > 2$ we have $\lambda \in \mathbb{R}$. Under the assumptions of the lemma we obtain $\chi_\varrho(hg) = \lambda a + \lambda^{-1} d \in \mathbb{R}$. This together with $a + d \in \mathbb{R}$ implies that $a, d \in \mathbb{R}$.

Let $X_\mu \in \mathrm{SL}_2(\mathbb{C})$ be the matrix $X_\mu = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right)$ where $\mu \in \mathbb{C}^*$. It is easy to see that

$$X_\mu \varrho(g) X_\mu^{-1} = \begin{pmatrix} a & \mu^2 b \\ \mu^{-2} c & d \end{pmatrix}.$$

By choosing $\mu^2 = b^{-1}$ we obtain $\varrho \sim \varrho'$ where

$$\varrho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \varrho'(g) = \begin{pmatrix} a & 1 \\ c' & d \end{pmatrix}$$

with $c' \neq 0$. But $\varrho'(g) \in \mathrm{SL}_2(\mathbb{C})$ and therefore $c' \in \mathbb{R}$.

Now, let $f \in G$ be given and assume $\varrho'(f) = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Again, $\alpha + \delta \in \mathbb{R}$ and $\chi_{\varrho'}(fh) \in \mathbb{R}$ which gives $\alpha, \delta \in \mathbb{R}$. Consider

$$\varrho'(fg) = \begin{pmatrix} \alpha a + \beta c' & * \\ * & \gamma + d\delta \end{pmatrix}.$$

By use of $\chi_{\varrho'}(fg) \in \mathbb{R}$ and $\chi_{\varrho'}(hfg) \in \mathbb{R}$ we obtain $\alpha a + \beta c' \in \mathbb{R}$ and $\gamma + d\delta \in \mathbb{R}$. As a result we get $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$; remember $a, d, c' \in \mathbb{R}$ and $c' \neq 0$. Therefore, we have $\varrho'(G) \subset \mathrm{SL}_2(\mathbb{R})$.

If $|\lambda + \lambda^{-1}| < 2$ we have $\pm 1 \neq \lambda \in S^1 \subset \mathbb{C}$, i.e. $\varrho'(h) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right)$. Again we have $\varrho(g) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $cb \neq 0$ and $a + d \in \mathbb{R}$. Therefore, $d = \bar{a} + h$ where $h \in \mathbb{R}$. Moreover, $\chi_\varrho(gh) = a\lambda + d\bar{\lambda} = a\lambda + \bar{a}\bar{\lambda} + \bar{\lambda}h \in \mathbb{R}$ and as a result we obtain $h = 0$ and $d = \bar{a}$ (see [9]).

Choose $\mu \in \mathbb{C}^*$ such that $|\mu^2 b| = |\mu^{-2} c|$. Let $\varrho'(f) := X_\mu \varrho(f) X_\mu^{-1}$ for all $f \in G$. We obtain:

$$\varrho \sim \varrho' \quad \text{such that} \quad \varrho'(h) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right) \quad \text{and} \quad \varrho'(g) = \left(\begin{smallmatrix} a & \beta \\ \gamma & \bar{a} \end{smallmatrix}\right)$$

where $\beta\gamma \neq 0$ and $|\beta| = |\gamma|$. The fact that $\varrho'(g) \in \mathrm{SL}_2(\mathbb{C})$ ($\beta\gamma = 1 - a\bar{a}$) implies $\gamma = -\bar{\beta}$ or $\gamma = \bar{\beta}$. Analogous to the case $|\chi_\varrho(h)| > 2$ it is easy to see that $\varrho'(G) \subset \mathrm{SU}_2(\mathbb{C})$ if $\gamma = -\bar{\beta}$ and

$$\varrho'(G) \subset \mathcal{S} := \left\{ \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \,\middle|\, a\bar{a} - b\bar{b} = 1 \right\} \quad \text{if } \gamma = \bar{\beta}.$$

Let $X \in \mathrm{SL}_2(\mathbb{C})$ be defined as $X := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$. We get

$$\mathrm{SL}_2(\mathbb{R}) = X\mathcal{S}X^{-1}$$

and therefore ϱ' is conjugate to a representation $\varrho'' : G \to \mathrm{SL}_2(\mathbb{R})$. $\square$

Again, let $\mathfrak{G} \in \{\mathrm{SL}_2(\mathbb{C}), \mathrm{SU}_2(\mathbb{C})\}$, and let $\mathfrak{g}$ be its Lie algebra. The Lie group $\mathfrak{G}$ acts on itself by conjugation and the differential of that action at the unit element $\mathbf{1} \in \mathfrak{G}$ defines the adjoint representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{Aut}\,\mathfrak{g}$. In other words, given $A \in \mathfrak{G}$ we have a map $c_A : \mathfrak{G} \to \mathfrak{G}$ defined by $c_A : B \mapsto ABA^{-1}$ and $\mathrm{Ad}(A)(X) := d_{\mathbf{1}}(c_A)(X)$ for all $X \in \mathfrak{g}$.

Therefore, given a representation $\varrho : G \to \mathfrak{G}$ the Lie algebra turns into a G-module via $\mathrm{Ad} \circ \varrho$, i.e.

$$gX := \mathrm{Ad}(\varrho(g))(X) \quad \text{for all } g \in G \text{ and } X \in \mathfrak{g}.$$

We denote by $\mathfrak{g}_\varrho$ the G-module $\mathfrak{g}$ via $\mathrm{Ad} \circ \varrho$. Let $B^1(G, \mathfrak{g}_\varrho)$ (resp. $Z^1(G, \mathfrak{g}_\varrho)$) (resp. $H^1(G, \mathfrak{g}_\varrho)$) be the coboundaries (resp. cocycles) (resp. first cohomology group) of G with coefficients in $\mathfrak{g}_\varrho$.

Let $T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ be the scheme-theoretic Zariski tangent space to $\mathcal{R}(G, \mathfrak{G})$ at a representation ϱ (see [8] and [10]). Following A. Weil (see [11]) $T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ is isomorphic to the space of group 1-cocycles, i.e.

$$T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G})) \cong Z^1(G, \mathfrak{g}_\varrho).$$

Let

$$\mathfrak{G}_\varrho := \{\varrho' \in \mathcal{R}(G, \mathfrak{G}) | \exists A \in \mathfrak{G} \text{ such that } \forall\, g \in G,\ \varrho'(g) = A\varrho(g)A^{-1}\}$$

denote the orbit of ϱ under the action of $\mathfrak{G}$. We obtain (see [11]) $T_\varrho^{\mathrm{Zar}}(\mathfrak{G}_\varrho) \cong B^1(G, \mathfrak{g}_\varrho)$ and therefore we can make the identification:

$$T_{[\varrho]}^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G})/\sim) \cong H^1(G, \mathfrak{g}_\varrho). \tag{1}$$

We denote by $\mathfrak{sl}$ (resp. $\mathfrak{su}$) the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ (resp. $\mathrm{SU}_2(\mathbb{C})$). In fact we obtain the complex 3-dimensional Lie algebra $\mathfrak{sl}$ by tensoring the real 3-dimensional Lie algebra $\mathfrak{su}$ with $\mathbb{C}$, i.e. there is an isomorphism

$$\mathfrak{su} \otimes \mathbb{C} \cong \mathfrak{sl} \quad \text{given by} \quad X \otimes a \mapsto Xa. \tag{2}$$

Now, given $\varrho : G \to \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{SL}_2(\mathbb{C})$ the algebras $\mathfrak{su}$ and $\mathfrak{sl}$ turn into G-modules via ϱ and from (2) we obtain

$$H^1(G, \mathfrak{su}_\varrho) \otimes \mathbb{C} \cong H^1(G, \mathfrak{sl}_\varrho). \tag{3}$$

Let $k \subset S^3$ be given and let $G := \pi_1(C)$ where C is the complement of a regular neighborhood of the knot k. Let $\varrho : G \to \mathrm{SU}_2(\mathbb{C}) \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ be a non-abelian representation. The representation $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is irreducible and $\varrho(\mathrm{Im}(\pi_1(\partial C) \rightarrowtail \pi_1(C))) \not\subset \{\pm \mathbf{1}\}$. By a theorem of Thurston (see [5, Proposition 3.2.1]) $t(\varrho) \in X(G)$ is contained in a component $X_0 \subset X(G)$ of complex dimension at least one.

Lemma 2. *Let $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$ be non-abelian and assume that $\dim_\mathbb{R} H^1(G, \mathfrak{su}_\varrho) = 1$. Then $t(\varrho) \in X(G)$ is a non-singular point and it is contained in a one-dimensional component of $X(G)$.*

Proof. Let X_0, where $\dim_\mathbb{C} X_0 \geq 1$, be an algebraic component of $X(G)$ which contains $t(\varrho)$. Using (1) and (3) we get

$$\begin{aligned} 1 \leq \dim_\mathbb{C} X_0 \leq \dim_\mathbb{C} T^{\mathrm{Zar}}_{t(\varrho)}(X_0) &\leq \dim_\mathbb{C} T^{\mathrm{Zar}}_{t(\varrho)}(X(G)) \\ &= \dim_\mathbb{C} H^1(G, \mathfrak{sl}_\varrho) = \dim_\mathbb{R} H^1(G, \mathfrak{su}_\varrho) = 1. \end{aligned} \tag{4}$$

This results in $1 = \dim_\mathbb{C} X_0 = \dim T^{\mathrm{Zar}}_{t(\varrho)}(X(G))$ and therefore $t(\varrho) \in X(G)$ is a regular point. □

Let τ be the mapping that associates with every point $x \in \mathbb{C}^N$ the point $\tau(x)$ with complex conjugate coordinates. Let $\mathcal{C} \in \mathbb{C}^N$ be a complex algebraic curve; we denote by $\mathcal{C}_\mathbb{R}$ the set of real points of $\mathcal{C}$, i.e. $\mathcal{C}_\mathbb{R} := \{x \in \mathcal{C} | \tau(x) = x\}$.

Lemma 3. *Let $\mathcal{C} \subset \mathbb{C}^N$ be an affine algebraic curve which is invariant under complex conjugation. Let $x \in \mathcal{C}_\mathbb{R} \subset \mathcal{C}$ be a regular point of $\mathcal{C}$. Then x has a neighborhood in $\mathcal{C}_\mathbb{R}$ which is diffeomorphic to $\mathbb{R}$.*

Proof. Let $\tau : \mathcal{C}_r \to \mathcal{C}_r$ denote complex conjugation, where $\mathcal{C}_r \subset \mathcal{C}$ is the manifold of regular points of $\mathcal{C}$. Then τ generates a smooth action of $\mathbb{Z}_2$ on the manifold $\mathcal{C}_r$, of which x is a fixed point.

It is a standard result that x has an invariant neighborhood U on which this action is smoothly equivalent to the linear action of $\mathbb{Z}_2$ on $T_x(\mathcal{C}_r)$ generated by the derivative $d_x(\tau)$. (To prove this, one first chooses a $\mathbb{Z}_2$-invariant Riemannian metric and then uses the exponential map at x to relate the two $\mathbb{Z}_2$-actions.)

It remains only to verify that if $L \subset \mathbb{C}^N$ (in our case, $L = T_x(\mathcal{C}_r)$) is a one-dimensional complex linear subspace invariant under τ, then the points on L which are fixed by τ consist precisely of a real line through the origin. We leave this easy exercise to the reader. □

Let $\varrho_0 : G \to \mathrm{SU}_2(\mathbb{C})$ be a non-abelian representation. Assume that $t(\varrho_0) \in X(G)$ is a regular point which is contained in a one-dimensional irreducible component X_0 of $X(G)$. Since $X(G)$ is defined over $\mathbb{Q}$ (see [4, 2.3]) it is invariant under complex conjugation. Therefore, Lemma 3 implies that there is a smooth arc

$\chi_s : \mathbb{R} \to X_0$, $s \in \mathbb{R}$, of characters such that $\chi_0 = t(\varrho_0)$ and $\chi_s(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Moreover, there exists $\varepsilon > 0$ such that if $|s| < \varepsilon$, the character χ_s corresponds to an irreducible representation.

Now, let $E \subset \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C}))$ be the set of all irreducible representations. The restriction $t|_E : E \to t(E) \subset X(G)$ is a fiber bundle with fiber $\mathrm{PSL}_2(\mathbb{C})$. Hence any map $\mathbb{R} \to t(E) \subset X(G)$ can be lifted (since $\mathbb{R}$ is contractible).

This implies that there is a smooth arc $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, such that $t(\varrho_s) = \chi_s$ for all $s \in \mathbb{R}$. In order to get an arc $\varrho_s : G \to \mathrm{SU}_2(\mathbb{C})$ we need the following lemma.

Lemma 4. *Let $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, be a smooth arc of representations such that $\varrho_0 : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) is irreducible and $\chi_{\varrho_s}(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Then there exists an $\varepsilon > 0$ and a smooth arc $A_s \in \mathrm{SL}_2(\mathbb{C})$ such that for each s, $|s| < \varepsilon$, $\varrho' : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) where $\varrho'(g) = A_s \varrho(g) A_s^{-1}$.*

Proof. As in the proof of Lemma 1 we choose $g, h \in G$ such that $\varrho_0(g) \neq \pm\mathbf{1}$, $\chi_{\varrho_0}(h) \neq \pm 2$ and ϱ_0 restricted to the subgroup generated by g and h is irreducible. Each of the conditions above is an open condition, i.e. we can assume that there is an $\varepsilon > 0$ such that the $\varrho_s(g) \neq \pm\mathbf{1}$, $\chi_{\varrho_s}(h) \neq \pm 2$ and ϱ_s restricted to the subgroup generated by g and h is irreducible.

For that reason, there is a smooth arc of matrices $A_s \in \mathrm{SL}_2(\mathbb{C})$ such that

$$A_s \varrho_s(h) A_s^{-1} = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix} \quad \text{and} \quad A_s \varrho_s(g) A_s^{-1} = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$$

where $\lambda_s \neq \pm 1$ and $b_s c_s \neq 0$ for every $|s| < \varepsilon$.

By following the proof of Lemma 1 we can find a smooth arc $\varrho'_s : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) such that $\varrho_s \sim \varrho'_s$ for all $|s| < \varepsilon$. $\square$

Proposition 1. *Let G be a knot group and $\varrho_0 : G \to \mathrm{SU}_2(\mathbb{C})$ a non-abelian representation. If $\dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1$, then a neighborhood of $[\varrho_0]$ in $\Re(G)$ is a one-dimensional manifold, i.e. there is a neighborhood of $[\varrho_0] \in \Re(G)$ which is diffeomorphic to an open interval.*

Proof. By Lemma 2, $t(\varrho_0)$ is a regular point of a one-dimensional component X_0 of $X(G)$. Lemma 3 implies that there is a smooth arc of non-conjugate representations $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$ through ϱ_0 such that $\chi_{\varrho_s}(G) \subset \mathbb{R}$, and by Lemma 4 we may assume that $\varrho_s : G \to \mathrm{SU}_2(\mathbb{C})$. Therefore, we get

$$1 = \dim_{\mathbb{R}}(X_0)_{\mathbb{R}} \leq \dim_{\mathbb{R}} T^{\mathrm{Zar}}_{[\varrho_0]}(\Re(G)) = \dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1.$$

Consequently there is neighborhood $U = U(t(\varrho_0)) \subset X_0$ such that $(X_0)_{\mathbb{R}} \cap U = \hat{t}(\Re(G)) \cap U$. So $\hat{t}$ is a diffeomorphism from a neighborhood of $[\varrho_0]$ in $\Re(G)$ to a smooth real arc in X_0. $\square$

3. Deforming binary dihedral representations

During this section it is more convenient to work with unit quaternions. Therefore, we identify $\mathrm{SU}_2(\mathbb{C})$ with S^3 (see [8]). The Lie algebra of S^3 is the set $\mathbb{E}$ of pure quaternions and S^3 acts via Ad, i.e. $\mathrm{Ad}(q)X = qXq^{-1}$ for $q \in S^3$ and $X \in \mathbb{E}$.

The binary dihedral group $N \subset S^3$ is given by $N = S^1_A \cup S^1_B$ where $S^1_A := \{a + b\boldsymbol{i} | a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$ and $S^1_B := \{c\boldsymbol{j} + d\boldsymbol{k} | c, d \in \mathbb{R},\ c^2 + d^2 = 1\}$. The set $S^1_A \subset N$ is a subgroup of index 2 and hence normal (see [8]). Each element of S^1_B can be expressed as

$$e^{\boldsymbol{i}\theta}\boldsymbol{j} = \cos\theta\boldsymbol{j} + \sin\theta\boldsymbol{k}.$$

Let

$$G = \langle S_1, \dots, S_n | S_{j_l}^{\epsilon_l} S_l S_{j_l}^{-\epsilon_l} = S_{l+1},\ l = 1, \dots, n-1 \rangle, \quad \epsilon_l \in \{\pm 1\}, \tag{5}$$

be a Wirtinger presentation of G.

Every non-abelian representation $\varrho : G \to N \subset S^3$ is given by

$$\varrho : S_l \mapsto e^{\boldsymbol{i}\theta_l}\boldsymbol{j}, \quad 1 \le l \le n, \tag{6}$$

where the $\zeta_l := e^{\boldsymbol{i}\theta_l}$ satisfy $\zeta_{j_l}^2 = \zeta_l \zeta_{l+1}$.

Remark 3. It is implicit from [8] and [3] that all binary dihedral representations of a knot group have finite image. Futhermore if there exists a binary dihedral representation, it is easy to see there exists one of order $4p$ for some $p \in \mathbb{N}$ prime. By [3, Chapter 13] there exists a binary dihedral representation of order $4p$ if and only if $H_1(\hat{C}_2, \mathbb{Z}_p) \neq 0$ which is equivalent to the condition $p|\Delta(-1)$.

We would like to use Proposition 1 in order to deform binary dihedral representations of order $4p$ where p is a prime number. Let $\varrho : G \to N$ be such a binary dihedral representation. By Proposition 1 it is enough to prove that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$.

Remark 4. Following an argument given by Hodgson (for details see [7]) we have $\dim H^1(G, \mathfrak{su}_\varrho) \geq 1$. Namely, let C be the complement of k in S^3; we have $H^i(G, \mathfrak{su}_\varrho) \cong H^i(C, \mathfrak{su}_\varrho) =: H^i_\varrho(C)$ for $i = 0, 1$. The long exact sequence in cohomology gives:

$$H^1_\varrho(C) \xrightarrow{\alpha} H^1_\varrho(\partial C) \xrightarrow{\beta} H^2_\varrho(C, \partial C).$$

We have $\dim H^1_\varrho(\partial C) = \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{su}_\varrho) = 2$ and $\dim \operatorname{Im}(\beta) + \dim \ker(\beta) = \dim H^1_\varrho(\partial C)$. Moreover, α and β are dual maps via Poincaré duality. Hence, $\dim \operatorname{Im}(\alpha) = \dim \operatorname{Im}(\beta)$ and by $\operatorname{Im}(\alpha) = \ker(\beta)$ we obtain

$$2 \dim \operatorname{Im}(\alpha) = 2 \Rightarrow \dim H^1_\varrho(C) \geq 1.$$

Remark 5. The fact that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$ if $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ is implicitly contained in [3, Proposition 14.10]. Nevertheless, to make this paper more complete we include the proof here.

Since ϱ is non-abelian it is irreducible and the orbit $\mathrm{SU}_2(\mathbb{C})_\varrho$ is 3-dimensional. By definition of the coboundary operator δ^0 (see [1]) we get $\dim B^1(G, \mathfrak{su}_\varrho) = 3$ because $\operatorname{Ad}(\varrho(G)) \subset \operatorname{Aut} \mathfrak{su}$ has no non-trivial fixed vector. Ergo,

$$\dim H^1(G, \mathfrak{su}_\varrho) = 1 \Leftrightarrow \dim Z^1(G, \mathfrak{su}_\varrho) = 4.$$

Remark 6. According to [8, Proposition 18] we are able to calculate $Z^1(G, \mathfrak{su}_\varrho)$ as follows. Specifically, each relator

$$S_{j_i}^{\epsilon_i} S_i S_{j_i}^{-\epsilon_i} = S_{i+1}$$

gives the equations:

$$(\mathbf{1} - \operatorname{Ad} \circ \varrho(S_{i+1})) X_{j_i} + \operatorname{Ad} \circ \varrho(S_{j_i}) X_i - X_{i+1} = 0 \quad \text{if } \epsilon_{j_i} = 1, \tag{7}$$

$$(\mathbf{1} - \operatorname{Ad} \circ \varrho(S_i)) X_{j_i} + \operatorname{Ad} \circ \varrho(S_{j_i}) X_{i+1} - X_i = 0 \quad \text{if } \epsilon_{j_i} = -1. \tag{8}$$

Altogether, we obtain a system of linear equations over $\mathbb{E}$ and

$$Z^1(G, \mathfrak{su}_\varrho) \cong \{(X_1, \dots X_n) \in (\mathbb{E})^n | (X_1, \dots, X_n) \text{ is a solution of the system}\}.$$

By choosing the basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of $\mathbb{E}$ we obtain that

$$\mathrm{Ad}(\boldsymbol{j}) \quad \text{acts as} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathrm{Ad}(e^{\theta \boldsymbol{i}}) \quad \text{acts as} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

By setting $X_i = x_i \boldsymbol{i} + y_i \boldsymbol{j} + z_i \boldsymbol{k}$ and defining

$$u_i := \begin{pmatrix} \cos 2\theta_i & -\sin 2\theta_i \\ \sin 2\theta_i & \cos 2\theta_i \end{pmatrix} \quad \text{and} \quad t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we obtain from (7) and (8) the following system of equations:

$$2x_{j_i} - x_i - x_{i+1} = 0, \quad 1 \le i \le n-1, \tag{9}$$

and

$$(\mathbf{1} - u_{i+1} t) \begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i} t \begin{pmatrix} y_i \\ z_i \end{pmatrix} - \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = 0 \quad \text{if } \epsilon_i = 1, \tag{10}$$

$$(\mathbf{1} - u_i t) \begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i} t \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} - \begin{pmatrix} y_i \\ z_i \end{pmatrix} = 0 \quad \text{if } \epsilon_i = -1. \tag{11}$$

Let $J(t)$ be the $n \times (n-1)$ Jacobian obtained from the Wirtinger presentation (5) of G (see [3, Chapter 9]). A presentation matrix $A(t)$ for the Alexander module is obtained from any $(n-1) \times (n-1)$ minor of $J(t)$. Equation (9) is equivalent to

$$J(-1) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

This system of equations has only the solution $x_i = x \in \mathbb{R}$ for all $1 \le i \le n$ because $|\det A(-1)| = |\Delta(-1)|$ is an odd integer.

In order to solve the equations (10) and (11) we introduce the following complex notation:

$$b_j := y_j + z_j \boldsymbol{i}; \quad t : \mathbb{C} \to \mathbb{C},\ t : z \mapsto \bar{z} \quad \text{and} \quad u_i : \mathbb{C} \to \mathbb{C},\ u_i : z \mapsto \zeta_i^2 z,$$

where $\zeta_i = e^{\boldsymbol{i}\theta_i}$. Then equation (10) is equivalent to

$$b_{j_i} - \zeta_{i+1}^2 \bar{b}_{j_i} + \zeta_{j_i}^2 \bar{b}_i - b_{i+1} = 0 \quad \text{if } \epsilon_i = 1 \tag{12}$$

and equation (11) transforms into

$$b_{j_i} - \zeta_i^2 \bar{b}_{j_i} + \zeta_{j_i}^2 \bar{b}_{i+1} - b_i = 0 \quad \text{if } \epsilon_i = -1. \tag{13}$$

Define new variables t_j and s_j in $\mathbb{R}$ by $b_j = \zeta_j(t_j + s_j \boldsymbol{i})$. Substituting in (12) (and (13)) yields

$$(\zeta_{j_i} + \zeta_{i+1}^2 \bar{\zeta}_{j_i}) s_{j_i} \boldsymbol{i} - \zeta_{j_i}^2 \bar{\zeta}_i s_i \boldsymbol{i} - \zeta_{i+1} s_{i+1} \boldsymbol{i} + (\zeta_{j_i} - \zeta_{i+1}^2 \bar{\zeta}_{j_i}) t_{j_i} + \zeta_{j_i}^2 \bar{\zeta}_i t_i - \zeta_{i+1} t_{i+1} = 0$$

if $\epsilon_{j_i} = 1$ and an analogous equation if $\epsilon_{j_i} = -1$. Now, multiplication of this equation with $-\bar{\zeta}_{i+1} \boldsymbol{i}$ if $\epsilon_{j_i} = 1$ (and multiplication of the analogous equation with $-\bar{\zeta}_i \boldsymbol{i}$ if $\epsilon_{j_i} = -1$) gives

$$(\bar{\zeta}_{j_i} \zeta_i + \bar{\zeta}_i \zeta_{j_i}) s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i} t_{j_i} (\bar{\zeta}_{j_i} \zeta_i - \zeta_{j_i} \bar{\zeta}_i) \boldsymbol{i} - \epsilon_{j_i} (t_i - t_{i+1}) \boldsymbol{i} = 0$$

(use the equation $\zeta_{j_i}^2 = \zeta_i\zeta_{i+1}$). Note that $\bar{\zeta}_{j_i}\zeta_i + \bar{\zeta}_i\zeta_{j_i} \in \mathbb{R}$ and $(\bar{\zeta}_{j_i}\zeta_i - \zeta_{j_i}\bar{\zeta}_i)\boldsymbol{i} \in \mathbb{R}$. The imaginary part yields $t_i =: t \in \mathbb{R}$ for $1 \leq i \leq n$ and from the real part we obtain the following system of linear equations over $\mathbb{R}$:

$$\alpha(j_i, i)s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i}\beta(j_i, i)t = 0 \tag{14}$$

where $\alpha(j_i, i) := (\bar{\zeta}_{j_i}\zeta_i + \bar{\zeta}_i\zeta_{j_i})$ and $\beta(j_i, i) := (\bar{\zeta}_{j_i}\zeta_i - \bar{\zeta}_i\zeta_{j_i})\boldsymbol{i}$ are real numbers.

We summarize the discussion above in the following proposition:

Proposition 2. *Let $k \subset S^3$ be a knot and G its group. Assume there is a non-abelian representation $\varrho : G \to N$ given by* (6). (*By Remark 3, this assumption is equivalent to* $|\Delta(-1)| \neq 1$.)

The point $[\varrho] \in \Re(G)$ has a neighborhood in $\Re(G)$ which is diffeomorphic to an open interval if the system (14) *has a three-dimensional solution space* (*over* $\mathbb{R}$).

We are ready to give a proof of Theorem 1.

Proof of Theorem 1. (This part of the proof uses exactly the argument given in [3, Proposition 14.10].) We would like to give a lower bound for the rank of the linear system

$$\alpha(j_i, i)s_{j_i} - s_i - s_{i+1} = 0. \tag{15}$$

Since a lower bound for the rank of (15) is also a lower bound for the rank of (14) we will then be done.

The system (15) is a linear system of equations with coefficients in $\mathbb{Z}(\zeta)$ where ζ is a primitive p-th root of unity and $\mathbb{Z}(\zeta) \subset \mathbb{Q}(\zeta)$ denotes the ring of algebraic integers. There is a homomorphism

$$\psi_p : \mathbb{Z}(\zeta) \to \mathbb{Z}_p,$$

given by $\psi_p : \zeta \mapsto [1]_p$ and $\psi_p : n \mapsto [n]_p$ where $[n]_p$ denotes the coset of n modulo p. Now, (15) maps under ψ_p onto the system

$$2s_{j_i} - s_i - s_{i+1} = 0$$

over $\mathbb{Z}_p$ and the coefficient matrix of this system is the matrix $A(-1)$ where the coefficients are reduced mod p. Now, $A(-1)$ is an $(n-1) \times (n-1)$ presentation matrix for the first homology group of the double branched cover of (S^3, k) (see [3]). So its rank (mod$p$) is just $(n-1) - \dim_{\mathbb{Z}_p} H_1(\hat{C}_2, \mathbb{Z}_p)$.

But $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ by the assumption. Therefore, we have $\mathrm{rk}_{\mathbb{Z}_p} A(-1) = n-2$ and hence we obtain that the rank of (14) is at least $n-2$. ☐

Acknowledgment

The first author would like to thank the University of Toronto where he was a guest of the Department of Mathematics during the winter term 1992. Moreover he would like to thank the SUNY at Binghamton, University of Tennessee, and Florida State University for their hospitality and support during the spring term 1993.

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Uni–GH–Siegen, Fachbereich Mathematik, Hölderlinstrasse 3, 57068 Siegen, Germany
E-mail address: `heusener@hrz.uni-siegen.d400.de`

Department of Mathematics, Florida State University, Tallahassee, Florida 32306
E-mail address: `klassen@math.fsu.edu`