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DEFORMATIONS OF DIHEDRAL REPRESENTATIONS

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Abstract. G. Burde proved (1990) that the $\mathrm{SU}_2(\mathbb{C})$ representation space of two-bridge knot groups is one-dimensional. The same holds for all torus knot groups.

The aim of this note is to prove the following:
Given a knot $k \subset S^3$ we denote by $\hat{C}_2$ its twofold branched covering space. Assume that there is a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exist representations of the knot group onto the binary dihedral group $D_p \subset \mathrm{SU}_2(\mathbb{C})$ and these representations are smooth points on a one-dimensional curve of representations into $\mathrm{SU}_2(\mathbb{C})$.

Let $k \subset S^3$ be a knot, let $G = \pi_1(S^3 - k)$, and assume $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation, i.e. the only subspaces of $\mathbb{C}^2$ which are invariant under $\varrho(G)$ are $\{0\}$ and $\{\mathbb{C}^2\}$. According to a result of Thurston (see [5, Proposition 3.2.1]) it is possible to deform ϱ non-trivially, i.e. ϱ is contained in a component R of the $\mathrm{SL}_2(\mathbb{C})$ representation space which is at least four-dimensional.

There is no general theorem which allows the deformation of representations $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$. In [6] the authors proved that every non-abelian representation corresponding to a simple root of the Alexander polynomial on the complex unit circle is a limit point of an arc of non-abelian representations.

Given a knot $k \subset S^3$ we denote by $\hat{C}_2$ the twofold branched covering of the pair (S^3, k) and by $\Delta(t)$ the Alexander polynomial of the knot. The aim of this note is to prove the following theorem:

Theorem 1. *Let $k \subset S^3$ be given. Assume there exists a prime number p such that $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then there exists a representation $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$ such that $\mathrm{Im}\,\varrho$ is a binary dihedral group of order $4p$. Moreover, $[\varrho] \in \Re(G)$ is a regular point and there is a neighborhood $U = U([\varrho]) \subset \Re(G)$ which is diffeomorphic to an interval.*

Remark 1. Theorem 1 applies if $H_1(\hat{C}_2)$ is cyclic and non-trivial. For example it is valid for all 2-bridge knots (see [2]).

Remark 2. The converse of Theorem 1 is false. For example, in the case of the knot $k = 9_{35}$, the group $H_1(\hat{C}_2, \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, but all dihedral representations are smooth points on one-dimensional components of $\Re(G)$.

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1. Introduction

Given a finitely generated group G we denote

$$\mathcal{R}(G,\mathfrak{G}) := \{\varrho : G \to \mathfrak{G} \mid \varrho \text{ is a homomorphism}\}$$

where $\mathfrak{G} \in \{\mathrm{SL}_2(\mathbb{C}), \mathrm{SU}_2(\mathbb{C})\}$. The space $\mathcal{R}(G,\mathfrak{G})$ has the structure of a complex (resp. real) affine algebraic set if $\mathfrak{G} = \mathrm{SL}_2(\mathbb{C})$ (resp. $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$). We call two representations $\varrho, \varrho' : G \to \mathfrak{G}$ equivalent ($\varrho \sim \varrho'$) iff they differ by an inner automorphism of $\mathfrak{G}$. Let $\mathcal{A}(G,\mathfrak{G}) := \{\varrho \in \mathcal{R}(G,\mathfrak{G}) \mid \mathrm{Im}(\varrho) \text{ is abelian}\}$.

If $\mathfrak{G} = \mathrm{SU}_2(\mathbb{C})$ we denote by $\mathfrak{R}(G)$ the space of conjugacy classes of non-abelian $\mathrm{SU}_2(\mathbb{C})$ representations, i.e.

$$\mathfrak{R}(G) := (\mathcal{R}(G,\mathrm{SU}_2(\mathbb{C})) \setminus \mathcal{A}(G,\mathrm{SU}_2(\mathbb{C})))/\sim .$$

For each $\varrho \in \mathcal{R}(G,\mathrm{SL}_2(\mathbb{C}))$ there is a character $\chi_\varrho : G \to \mathbb{C}$ given by $\chi_\varrho : g \mapsto \mathrm{tr}\,\varrho(g)$. We have $\chi_\varrho = \chi_{\varrho'}$ if $\varrho \sim \varrho'$. The set $X(G)$ of these characters is a complex affine algebraic set; its ambient coordinates are given by $\{\chi(g_i)\}$ where $\chi \in X(G)$ is a character and $\{g_i\} \subset G$ is finite. Moreover, the map $t : \mathcal{R}(G,\mathrm{SL}_2(\mathbb{C})) \to X(G)$ given by $t : \varrho \mapsto \chi_\varrho$ is polynomial in the ambient coordinates (for details see [5]).

2. Real points in $X(G)$

Because $\mathrm{SU}_2(\mathbb{C})$ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, there is an obvious map

$$t : \mathcal{R}(G,\mathrm{SU}_2(\mathbb{C})) \to X(G)$$

which associates to each representation the corresponding character. This map induces an injection $\hat{t} : \mathfrak{R}(G) \to X(G)$ (see [8]).

Definition 1. We call a representation $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ real iff $\mathrm{Im}(\varrho) \subset \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{Im}(\varrho) \subset \mathrm{SU}_2(\mathbb{C})$.

If $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is conjugate to a real representation, then $\chi_\varrho : G \to \mathbb{C}$ is a real-valued function. The converse is true for irreducible representations:

Lemma 1. *Let $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. If $\chi_\varrho(G) \subset \mathbb{R}$, then ϱ is equivalent to a real representation.*

Proof. Choose $g \in G$ such that $\varrho(g) \neq \pm\mathbf{1}$. By [5, Lemma 1.5.1] there exists $h \in G$ such that the restriction of ϱ to the subgroup generated by g and h is irreducible and such that $\chi_\varrho(h) \neq \pm 2$.

By conjugation we assume that $\varrho(h) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ where $\lambda + \lambda^{-1} \in \mathbb{R}$ and $\lambda \neq \pm 1$. Moreover, we have $\varrho(g) = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ where $a + d \in \mathbb{R}$ and $cb \neq 0$.

If $|\lambda + \lambda^{-1}| > 2$ we have $\lambda \in \mathbb{R}$. Under the assumptions of the lemma we obtain $\chi_\varrho(hg) = \lambda a + \lambda^{-1} d \in \mathbb{R}$. This together with $a + d \in \mathbb{R}$ implies that $a, d \in \mathbb{R}$.

Let $X_\mu \in \mathrm{SL}_2(\mathbb{C})$ be the matrix $X_\mu = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1}\end{smallmatrix}\right)$ where $\mu \in \mathbb{C}^*$. It is easy to see that

$$X_\mu \varrho(g) X_\mu^{-1} = \begin{pmatrix} a & \mu^2 b \\ \mu^{-2} c & d \end{pmatrix}.$$

By choosing $\mu^2 = b^{-1}$ we obtain $\varrho \sim \varrho'$ where

$$\varrho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \varrho'(g) = \begin{pmatrix} a & 1 \\ c' & d \end{pmatrix}$$

with $c' \neq 0$. But $\varrho'(g) \in \mathrm{SL}_2(\mathbb{C})$ and therefore $c' \in \mathbb{R}$.

Now, let $f \in G$ be given and assume $\varrho'(f) = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Again, $\alpha + \delta \in \mathbb{R}$ and $\chi_{\varrho'}(fh) \in \mathbb{R}$ which gives $\alpha, \delta \in \mathbb{R}$. Consider

$$\varrho'(fg) = \begin{pmatrix} \alpha a + \beta c' & * \\ * & \gamma + d\delta \end{pmatrix}.$$

By use of $\chi_{\varrho'}(fg) \in \mathbb{R}$ and $\chi_{\varrho'}(hfg) \in \mathbb{R}$ we obtain $\alpha a + \beta c' \in \mathbb{R}$ and $\gamma + d\delta \in \mathbb{R}$. As a result we get $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$; remember $a, d, c' \in \mathbb{R}$ and $c' \neq 0$. Therefore, we have $\varrho'(G) \subset \mathrm{SL}_2(\mathbb{R})$.

If $|\lambda + \lambda^{-1}| < 2$ we have $\pm 1 \neq \lambda \in S^1 \subset \mathbb{C}$, i.e. $\varrho'(h) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \bar\lambda\end{smallmatrix}\right)$. Again we have $\varrho(g) = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ where $cb \neq 0$ and $a + d \in \mathbb{R}$. Therefore, $d = \bar a + h$ where $h \in \mathbb{R}$. Moreover, $\chi_\varrho(gh) = a\lambda + d\bar\lambda = a\lambda + \bar a\bar\lambda + \bar\lambda h \in \mathbb{R}$ and as a result we obtain $h = 0$ and $d = \bar a$ (see [9]).

Choose $\mu \in \mathbb{C}^*$ such that $|\mu^2 b| = |\mu^{-2} c|$. Let $\varrho'(f) := X_\mu \varrho(f) X_\mu^{-1}$ for all $f \in G$. We obtain:

$$\varrho \sim \varrho' \quad \text{such that} \quad \varrho'(h) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \bar\lambda\end{smallmatrix}\right) \quad \text{and} \quad \varrho'(g) = \left(\begin{smallmatrix}a & \beta\\ \gamma & \bar a\end{smallmatrix}\right)$$

where $\beta\gamma \neq 0$ and $|\beta| = |\gamma|$. The fact that $\varrho'(g) \in \mathrm{SL}_2(\mathbb{C})$ ($\beta\gamma = 1 - a\bar a$) implies $\gamma = -\bar\beta$ or $\gamma = \bar\beta$. Analogous to the case $|\chi_\varrho(h)| > 2$ it is easy to see that $\varrho'(G) \subset \mathrm{SU}_2(\mathbb{C})$ if $\gamma = -\bar\beta$ and

$$\varrho'(G) \subset \mathcal{S} := \left\{ \left(\begin{smallmatrix}a & b\\ \bar b & \bar a\end{smallmatrix}\right) \,\middle|\, a\bar a - b\bar b = 1 \right\} \quad \text{if } \gamma = \bar\beta.$$

Let $X \in \mathrm{SL}_2(\mathbb{C})$ be defined as $X := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & i\\ i & 1\end{smallmatrix}\right)$. We get

$$\mathrm{SL}_2(\mathbb{R}) = X\mathcal{S}X^{-1}$$

and therefore ϱ' is conjugate to a representation $\varrho'' : G \to \mathrm{SL}_2(\mathbb{R})$. □

Again, let $\mathfrak{G} \in \{\mathrm{SL}_2(\mathbb{C}), \mathrm{SU}_2(\mathbb{C})\}$, and let $\mathfrak{g}$ be its Lie algebra. The Lie group $\mathfrak{G}$ acts on itself by conjugation and the differential of that action at the unit element $\mathbf{1} \in \mathfrak{G}$ defines the adjoint representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{Aut}\,\mathfrak{g}$. In other words, given $A \in \mathfrak{G}$ we have a map $c_A : \mathfrak{G} \to \mathfrak{G}$ defined by $c_A : B \mapsto ABA^{-1}$ and $\mathrm{Ad}(A)(X) := d_{\mathbf{1}}(c_A)(X)$ for all $X \in \mathfrak{g}$.

Therefore, given a representation $\varrho : G \to \mathfrak{G}$ the Lie algebra turns into a G-module via $\mathrm{Ad} \circ \varrho$, i.e.

$$gX := \mathrm{Ad}(\varrho(g))(X) \quad \text{for all } g \in G \text{ and } X \in \mathfrak{g}.$$

We denote by $\mathfrak{g}_\varrho$ the G-module $\mathfrak{g}$ via $\mathrm{Ad} \circ \varrho$. Let $B^1(G, \mathfrak{g}_\varrho)$ (resp. $Z^1(G, \mathfrak{g}_\varrho)$) (resp. $H^1(G, \mathfrak{g}_\varrho)$) be the coboundaries (resp. cocycles) (resp. first cohomology group) of G with coefficients in $\mathfrak{g}_\varrho$.

Let $T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ be the scheme-theoretic Zariski tangent space to $\mathcal{R}(G, \mathfrak{G})$ at a representation ϱ (see [8] and [10]). Following A. Weil (see [11]) $T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G}))$ is isomorphic to the space of group 1-cocycles, i.e.

$$T_\varrho^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G})) \cong Z^1(G, \mathfrak{g}_\varrho).$$

Let

$$\mathfrak{G}_\varrho := \{\varrho' \in \mathcal{R}(G, \mathfrak{G}) | \exists A \in \mathfrak{G} \text{ such that } \forall\, g \in G,\ \varrho'(g) = A\varrho(g)A^{-1}\}$$

denote the orbit of ϱ under the action of $\mathfrak{G}$. We obtain (see [11]) $T_\varrho^{\mathrm{Zar}}(\mathfrak{G}_\varrho) \cong B^1(G, \mathfrak{g}_\varrho)$ and therefore we can make the identification:

$$T_{[\varrho]}^{\mathrm{Zar}}(\mathcal{R}(G, \mathfrak{G})/\sim) \cong H^1(G, \mathfrak{g}_\varrho). \tag{1}$$

We denote by $\mathfrak{sl}$ (resp. $\mathfrak{su}$) the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ (resp. $\mathrm{SU}_2(\mathbb{C})$). In fact we obtain the complex 3-dimensional Lie algebra $\mathfrak{sl}$ by tensoring the real 3-dimensional Lie algebra $\mathfrak{su}$ with $\mathbb{C}$, i.e. there is an isomorphism

$$\mathfrak{su} \otimes \mathbb{C} \cong \mathfrak{sl} \quad \text{given by} \quad X \otimes a \mapsto Xa. \tag{2}$$

Now, given $\varrho : G \to \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{SL}_2(\mathbb{C})$ the algebras $\mathfrak{su}$ and $\mathfrak{sl}$ turn into G-modules via ϱ and from (2) we obtain

$$H^1(G, \mathfrak{su}_\varrho) \otimes \mathbb{C} \cong H^1(G, \mathfrak{sl}_\varrho). \tag{3}$$

Let $k \subset S^3$ be given and let $G := \pi_1(C)$ where C is the complement of a regular neighborhood of the knot k. Let $\varrho : G \to \mathrm{SU}_2(\mathbb{C}) \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ be a non-abelian representation. The representation $\varrho : G \to \mathrm{SL}_2(\mathbb{C})$ is irreducible and $\varrho(\mathrm{Im}(\pi_1(\partial C) \rightarrowtail \pi_1(C))) \not\subset \{\pm\mathbf{1}\}$. By a theorem of Thurston (see [5, Proposition 3.2.1]) $t(\varrho) \in X(G)$ is contained in a component $X_0 \subset X(G)$ of complex dimension at least one.

Lemma 2. *Let $\varrho : G \to \mathrm{SU}_2(\mathbb{C})$ be non-abelian and assume that $\dim_\mathbb{R} H^1(G, \mathfrak{su}_\varrho) = 1$. Then $t(\varrho) \in X(G)$ is a non-singular point and it is contained in a one-dimensional component of $X(G)$.*

Proof. Let X_0, where $\dim_\mathbb{C} X_0 \geq 1$, be an algebraic component of $X(G)$ which contains $t(\varrho)$. Using (1) and (3) we get

$$\begin{aligned} 1 \leq \dim_\mathbb{C} X_0 \leq \dim_\mathbb{C} T^{\mathrm{Zar}}_{t(\varrho)}(X_0) &\leq \dim_\mathbb{C} T^{\mathrm{Zar}}_{t(\varrho)}(X(G)) \\ &= \dim_\mathbb{C} H^1(G, \mathfrak{sl}_\varrho) = \dim_\mathbb{R} H^1(G, \mathfrak{su}_\varrho) = 1. \end{aligned} \tag{4}$$

This results in $1 = \dim_\mathbb{C} X_0 = \dim T^{\mathrm{Zar}}_{t(\varrho)}(X(G))$ and therefore $t(\varrho) \in X(G)$ is a regular point. □

Let τ be the mapping that associates with every point $x \in \mathbb{C}^N$ the point $\tau(x)$ with complex conjugate coordinates. Let $\mathcal{C} \in \mathbb{C}^N$ be a complex algebraic curve; we denote by $\mathcal{C}_\mathbb{R}$ the set of real points of $\mathcal{C}$, i.e. $\mathcal{C}_\mathbb{R} := \{x \in \mathcal{C} | \tau(x) = x\}$.

Lemma 3. *Let $\mathcal{C} \subset \mathbb{C}^N$ be an affine algebraic curve which is invariant under complex conjugation. Let $x \in \mathcal{C}_\mathbb{R} \subset \mathcal{C}$ be a regular point of $\mathcal{C}$. Then x has a neighborhood in $\mathcal{C}_\mathbb{R}$ which is diffeomorphic to $\mathbb{R}$.*

Proof. Let $\tau : \mathcal{C}_r \to \mathcal{C}_r$ denote complex conjugation, where $\mathcal{C}_r \subset \mathcal{C}$ is the manifold of regular points of $\mathcal{C}$. Then τ generates a smooth action of $\mathbb{Z}_2$ on the manifold $\mathcal{C}_r$, of which x is a fixed point.

It is a standard result that x has an invariant neighborhood U on which this action is smoothly equivalent to the linear action of $\mathbb{Z}_2$ on $T_x(\mathcal{C}_r)$ generated by the derivative $d_x(\tau)$. (To prove this, one first chooses a $\mathbb{Z}_2$-invariant Riemannian metric and then uses the exponential map at x to relate the two $\mathbb{Z}_2$-actions.)

It remains only to verify that if $L \subset \mathbb{C}^N$ (in our case, $L = T_x(\mathcal{C}_r)$) is a one-dimensional complex linear subspace invariant under τ, then the points on L which are fixed by τ consist precisely of a real line through the origin. We leave this easy exercise to the reader. □

Let $\varrho_0 : G \to \mathrm{SU}_2(\mathbb{C})$ be a non-abelian representation. Assume that $t(\varrho_0) \in X(G)$ is a regular point which is contained in a one-dimensional irreducible component X_0 of $X(G)$. Since $X(G)$ is defined over $\mathbb{Q}$ (see [4, 2.3]) it is invariant under complex conjugation. Therefore, Lemma 3 implies that there is a smooth arc

$\chi_s : \mathbb{R} \to X_0$, $s \in \mathbb{R}$, of characters such that $\chi_0 = t(\varrho_0)$ and $\chi_s(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Moreover, there exists $\varepsilon > 0$ such that if $|s| < \varepsilon$, the character χ_s corresponds to an irreducible representation.

Now, let $E \subset \mathcal{R}(G, \mathrm{SL}_2(\mathbb{C}))$ be the set of all irreducible representations. The restriction $t|_E : E \to t(E) \subset X(G)$ is a fiber bundle with fiber $\mathrm{PSL}_2(\mathbb{C})$. Hence any map $\mathbb{R} \to t(E) \subset X(G)$ can be lifted (since $\mathbb{R}$ is contractible).

This implies that there is a smooth arc $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, such that $t(\varrho_s) = \chi_s$ for all $s \in \mathbb{R}$. In order to get an arc $\varrho_s : G \to \mathrm{SU}_2(\mathbb{C})$ we need the following lemma.

Lemma 4. *Let $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$, $s \in \mathbb{R}$, be a smooth arc of representations such that $\varrho_0 : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) is irreducible and $\chi_{\varrho_s}(G) \subset \mathbb{R}$ for all $s \in \mathbb{R}$. Then there exists an $\varepsilon > 0$ and a smooth arc $A_s \in \mathrm{SL}_2(\mathbb{C})$ such that for each s, $|s| < \varepsilon$, $\varrho' : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) where $\varrho'(g) = A_s \varrho(g) A_s^{-1}$.*

Proof. As in the proof of Lemma 1 we choose $g, h \in G$ such that $\varrho_0(g) \neq \pm\mathbf{1}$, $\chi_{\varrho_0}(h) \neq \pm 2$ and ϱ_0 restricted to the subgroup generated by g and h is irreducible. Each of the conditions above is an open condition, i.e. we can assume that there is an $\varepsilon > 0$ such that the $\varrho_s(g) \neq \pm\mathbf{1}$, $\chi_{\varrho_s}(h) \neq \pm 2$ and ϱ_s restricted to the subgroup generated by g and h is irreducible.

For that reason, there is a smooth arc of matrices $A_s \in \mathrm{SL}_2(\mathbb{C})$ such that

$$A_s \varrho_s(h) A_s^{-1} = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix} \quad \text{and} \quad A_s \varrho_s(g) A_s^{-1} = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$$

where $\lambda_s \neq \pm 1$ and $b_s c_s \neq 0$ for every $|s| < \varepsilon$.

By following the proof of Lemma 1 we can find a smooth arc $\varrho'_s : G \to \mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$) such that $\varrho_s \sim \varrho'_s$ for all $|s| < \varepsilon$. □

Proposition 1. *Let G be a knot group and $\varrho_0 : G \to \mathrm{SU}_2(\mathbb{C})$ a non-abelian representation. If $\dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1$, then a neighborhood of $[\varrho_0]$ in $\Re(G)$ is a one-dimensional manifold, i.e. there is a neighborhood of $[\varrho_0] \in \Re(G)$ which is diffeomorphic to an open interval.*

Proof. By Lemma 2, $t(\varrho_0)$ is a regular point of a one-dimensional component X_0 of $X(G)$. Lemma 3 implies that there is a smooth arc of non-conjugate representations $\varrho_s : G \to \mathrm{SL}_2(\mathbb{C})$ through ϱ_0 such that $\chi_{\varrho_s}(G) \subset \mathbb{R}$, and by Lemma 4 we may assume that $\varrho_s : G \to \mathrm{SU}_2(\mathbb{C})$. Therefore, we get

$$1 = \dim_{\mathbb{R}} (X_0)_{\mathbb{R}} \leq \dim_{\mathbb{R}} T^{\mathrm{Zar}}_{[\varrho_0]}(\Re(G)) = \dim_{\mathbb{R}} H^1(G, \mathfrak{su}_{\varrho_0}) = 1.$$

Consequently there is neighborhood $U = U(t(\varrho_0)) \subset X_0$ such that $(X_0)_{\mathbb{R}} \cap U = \hat{t}(\Re(G)) \cap U$. So $\hat{t}$ is a diffeomorphism from a neighborhood of $[\varrho_0]$ in $\Re(G)$ to a smooth real arc in X_0. □

3. Deforming binary dihedral representations

During this section it is more convenient to work with unit quaternions. Therefore, we identify $\mathrm{SU}_2(\mathbb{C})$ with S^3 (see [8]). The Lie algebra of S^3 is the set $\mathbb{E}$ of pure quaternions and S^3 acts via Ad, i.e. $\mathrm{Ad}(q)X = qXq^{-1}$ for $q \in S^3$ and $X \in \mathbb{E}$.

The binary dihedral group $N \subset S^3$ is given by $N = S^1_A \cup S^1_B$ where $S^1_A := \{a + b\boldsymbol{i} | a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$ and $S^1_B := \{c\boldsymbol{j} + d\boldsymbol{k} | c, d \in \mathbb{R},\ c^2 + d^2 = 1\}$. The set $S^1_A \subset N$ is a subgroup of index 2 and hence normal (see [8]). Each element of S^1_B can be expressed as

$$e^{\boldsymbol{i}\theta}\boldsymbol{j} = \cos\theta\boldsymbol{j} + \sin\theta\boldsymbol{k}.$$

Let

$$G = \langle S_1, \dots, S_n | S_{j_l}^{\epsilon_l} S_l S_{j_l}^{-\epsilon_l} = S_{l+1},\ l = 1, \dots, n-1 \rangle, \quad \epsilon_l \in \{\pm 1\}, \tag{5}$$

be a Wirtinger presentation of G.

Every non-abelian representation $\varrho : G \to N \subset S^3$ is given by

$$\varrho : S_l \mapsto e^{\boldsymbol{i}\theta_l}\boldsymbol{j}, \quad 1 \leq l \leq n, \tag{6}$$

where the $\zeta_l := e^{\boldsymbol{i}\theta_l}$ satisfy $\zeta_{j_l}^2 = \zeta_l \zeta_{l+1}$.

Remark 3. It is implicit from [8] and [3] that all binary dihedral representations of a knot group have finite image. Futhermore if there exists a binary dihedral representation, it is easy to see there exists one of order $4p$ for some $p \in \mathbb{N}$ prime. By [3, Chapter 13] there exists a binary dihedral representation of order $4p$ if and only if $H_1(\hat{C}_2, \mathbb{Z}_p) \neq 0$ which is equivalent to the condition $p | \Delta(-1)$.

We would like to use Proposition 1 in order to deform binary dihedral representations of order $4p$ where p is a prime number. Let $\varrho : G \to N$ be such a binary dihedral representation. By Proposition 1 it is enough to prove that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$.

Remark 4. Following an argument given by Hodgson (for details see [7]) we have $\dim H^1(G, \mathfrak{su}_\varrho) \geq 1$. Namely, let C be the complement of k in S^3; we have $H^i(G, \mathfrak{su}_\varrho) \cong H^i(C, \mathfrak{su}_\varrho) =: H^i_\varrho(C)$ for $i = 0, 1$. The long exact sequence in cohomology gives:

$$H^1_\varrho(C) \xrightarrow{\alpha} H^1_\varrho(\partial C) \xrightarrow{\beta} H^2_\varrho(C, \partial C).$$

We have $\dim H^1_\varrho(\partial C) = \dim H^1(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{su}_\varrho) = 2$ and $\dim \mathrm{Im}(\beta) + \dim \ker(\beta) = \dim H^1_\varrho(\partial C)$. Moreover, α and β are dual maps via Poincaré duality. Hence, $\dim \mathrm{Im}(\alpha) = \dim \mathrm{Im}(\beta)$ and by $\mathrm{Im}(\alpha) = \ker(\beta)$ we obtain

$$2 \dim \mathrm{Im}(\alpha) = 2 \Rightarrow \dim H^1_\varrho(C) \geq 1.$$

Remark 5. The fact that $\dim H^1(G, \mathfrak{su}_\varrho) = 1$ if $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ is implicitly contained in [3, Proposition 14.10]. Nevertheless, to make this paper more complete we include the proof here.

Since ϱ is non-abelian it is irreducible and the orbit $\mathrm{SU}_2(\mathbb{C})_\varrho$ is 3-dimensional. By definition of the coboundary operator δ^0 (see [1]) we get $\dim B^1(G, \mathfrak{su}_\varrho) = 3$ because $\mathrm{Ad}(\varrho(G)) \subset \mathrm{Aut}\, \mathfrak{su}$ has no non-trivial fixed vector. Ergo,

$$\dim H^1(G, \mathfrak{su}_\varrho) = 1 \Leftrightarrow \dim Z^1(G, \mathfrak{su}_\varrho) = 4.$$

Remark 6. According to [8, Proposition 18] we are able to calculate $Z^1(G, \mathfrak{su}_\varrho)$ as follows. Specifically, each relator

$$S_{j_i}^{\epsilon_i} S_i S_{j_i}^{-\epsilon_i} = S_{i+1}$$

gives the equations:

$$\begin{aligned}
&(\mathbf{1} - \mathrm{Ad} \circ \varrho(S_{i+1}))X_{j_i} + \mathrm{Ad} \circ \varrho(S_{j_i})X_i - X_{i+1} &= 0 \quad \text{if } \epsilon_{j_i} = 1, \quad &(7)\\
&(\mathbf{1} - \mathrm{Ad} \circ \varrho(S_i))X_{j_i} + \mathrm{Ad} \circ \varrho(S_{j_i})X_{i+1} - X_i &= 0 \quad \text{if } \epsilon_{j_i} = -1. \quad &(8)
\end{aligned}$$

Altogether, we obtain a system of linear equations over $\mathbb{E}$ and

$$Z^1(G, \mathfrak{su}_\varrho) \cong \{(X_1, \dots X_n) \in (\mathbb{E})^n | (X_1, \dots, X_n) \text{ is a solution of the system}\}.$$

By choosing the basis $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ of $\mathbb{E}$ we obtain that

$$\mathrm{Ad}(\boldsymbol{j}) \quad \text{acts as} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathrm{Ad}(e^{\theta \boldsymbol{i}}) \quad \text{acts as} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

By setting $X_i = x_i\boldsymbol{i} + y_i\boldsymbol{j} + z_i\boldsymbol{k}$ and defining

$$u_i := \begin{pmatrix} \cos 2\theta_i & -\sin 2\theta_i \\ \sin 2\theta_i & \cos 2\theta_i \end{pmatrix} \quad \text{and} \quad t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we obtain from (7) and (8) the following system of equations:

$$2x_{j_i} - x_i - x_{i+1} = 0, \quad 1 \le i \le n-1, \tag{9}$$

and

$$(\mathbf{1} - u_{i+1}t)\begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i}t\begin{pmatrix} y_i \\ z_i \end{pmatrix} - \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = 0 \quad \text{if } \epsilon_i = 1, \tag{10}$$

$$(\mathbf{1} - u_i t)\begin{pmatrix} y_{j_i} \\ z_{j_i} \end{pmatrix} + u_{j_i}t\begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} - \begin{pmatrix} y_i \\ z_i \end{pmatrix} = 0 \quad \text{if } \epsilon_i = -1. \tag{11}$$

Let $J(t)$ be the $n \times (n-1)$ Jacobian obtained from the Wirtinger presentation (5) of G (see [3, Chapter 9]). A presentation matrix $A(t)$ for the Alexander module is obtained from any $(n-1)\times(n-1)$ minor of $J(t)$. Equation (9) is equivalent to

$$J(-1)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

This system of equations has only the solution $x_i = x \in \mathbb{R}$ for all $1 \le i \le n$ because $|\det A(-1)| = |\Delta(-1)|$ is an odd integer.

In order to solve the equations (10) and (11) we introduce the following complex notation:

$$b_j := y_j + z_j\boldsymbol{i}; \quad t : \mathbb{C} \to \mathbb{C},\ t : z \mapsto \bar{z} \quad \text{and} \quad u_i : \mathbb{C} \to \mathbb{C},\ u_i : z \mapsto \zeta_i^2 z,$$

where $\zeta_i = e^{\boldsymbol{i}\theta_i}$. Then equation (10) is equivalent to

$$b_{j_i} - \zeta_{i+1}^2\bar{b}_{j_i} + \zeta_{j_i}^2\bar{b}_i - b_{i+1} = 0 \quad \text{if } \epsilon_i = 1 \tag{12}$$

and equation (11) transforms into

$$b_{j_i} - \zeta_i^2\bar{b}_{j_i} + \zeta_{j_i}^2\bar{b}_{i+1} - b_i = 0 \quad \text{if } \epsilon_i = -1. \tag{13}$$

Define new variables t_j and s_j in $\mathbb{R}$ by $b_j = \zeta_j(t_j + s_j\boldsymbol{i})$. Substituting in (12) (and (13)) yields

$$(\zeta_{j_i} + \zeta_{i+1}^2\bar{\zeta}_{j_i})s_{j_i}\boldsymbol{i} - \zeta_{j_i}^2\bar{\zeta}_i s_i\boldsymbol{i} - \zeta_{i+1}s_{i+1}\boldsymbol{i} + (\zeta_{j_i} - \zeta_{i+1}^2\bar{\zeta}_{j_i})t_{j_i} + \zeta_{j_i}^2\bar{\zeta}_i t_i - \zeta_{i+1}t_{i+1} = 0$$

if $\epsilon_{j_i} = 1$ and an analogous equation if $\epsilon_{j_i} = -1$. Now, multiplication of this equation with $-\bar{\zeta}_{i+1}\boldsymbol{i}$ if $\epsilon_{j_i} = 1$ (and multiplication of the analogous equation with $-\bar{\zeta}_i\boldsymbol{i}$ if $\epsilon_{j_i} = -1$) gives

$$(\bar{\zeta}_{j_i}\zeta_i + \bar{\zeta}_i\zeta_{j_i})s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i}t_{j_i}(\bar{\zeta}_{j_i}\zeta_i - \zeta_{j_i}\bar{\zeta}_i)\boldsymbol{i} - \epsilon_{j_i}(t_i - t_{i+1})\boldsymbol{i} = 0$$

(use the equation $\zeta_{j_i}^2 = \zeta_i\zeta_{i+1}$). Note that $\bar{\zeta}_{j_i}\zeta_i + \bar{\zeta}_i\zeta_{j_i} \in \mathbb{R}$ and $(\bar{\zeta}_{j_i}\zeta_i - \zeta_{j_i}\bar{\zeta}_i)\boldsymbol{i} \in \mathbb{R}$. The imaginary part yields $t_i =: t \in \mathbb{R}$ for $1 \le i \le n$ and from the real part we obtain the following system of linear equations over $\mathbb{R}$:

$$\alpha(j_i, i)s_{j_i} - s_i - s_{i+1} - \epsilon_{j_i}\beta(j_i, i)t = 0 \tag{14}$$

where $\alpha(j_i, i) := (\bar{\zeta}_{j_i}\zeta_i + \bar{\zeta}_i\zeta_{j_i})$ and $\beta(j_i, i) := (\bar{\zeta}_{j_i}\zeta_i - \bar{\zeta}_i\zeta_{j_i})\boldsymbol{i}$ are real numbers.

We summarize the discussion above in the following proposition:

Proposition 2. *Let $k \subset S^3$ be a knot and G its group. Assume there is a non-abelian representation $\varrho : G \to N$ given by* (6). (*By Remark 3, this assumption is equivalent to* $|\Delta(-1)| \neq 1$.)

The point $[\varrho] \in \Re(G)$ has a neighborhood in $\Re(G)$ which is diffeomorphic to an open interval if the system (14) *has a three-dimensional solution space* (*over* $\mathbb{R}$).

We are ready to give a proof of Theorem 1.

Proof of Theorem 1. (This part of the proof uses exactly the argument given in [3, Proposition 14.10].) We would like to give a lower bound for the rank of the linear system

$$\alpha(j_i, i)s_{j_i} - s_i - s_{i+1} = 0. \tag{15}$$

Since a lower bound for the rank of (15) is also a lower bound for the rank of (14) we will then be done.

The system (15) is a linear system of equations with coefficients in $\mathbb{Z}(\zeta)$ where ζ is a primitive p-th root of unity and $\mathbb{Z}(\zeta) \subset \mathbb{Q}(\zeta)$ denotes the ring of algebraic integers. There is a homomorphism

$$\psi_p : \mathbb{Z}(\zeta) \to \mathbb{Z}_p,$$

given by $\psi_p : \zeta \mapsto [1]_p$ and $\psi_p : n \mapsto [n]_p$ where $[n]_p$ denotes the coset of n modulo p. Now, (15) maps under ψ_p onto the system

$$2s_{j_i} - s_i - s_{i+1} = 0$$

over $\mathbb{Z}_p$ and the coefficient matrix of this system is the matrix $A(-1)$ where the coefficients are reduced mod p. Now, $A(-1)$ is an $(n-1) \times (n-1)$ presentation matrix for the first homology group of the double branched cover of (S^3, k) (see [3]). So its rank (mod$p$) is just $(n-1) - \dim_{\mathbb{Z}_p} H_1(\hat{C}_2, \mathbb{Z}_p)$.

But $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ by the assumption. Therefore, we have $\mathrm{rk}_{\mathbb{Z}_p} A(-1) = n-2$ and hence we obtain that the rank of (14) is at least $n-2$. □

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References

[1] Kenneth S. Brown, *Cohomology of groups*, Springer, 1982. MR **83k:**20002
[2] Gerhard Burde, *SU(2)–representation spaces for two–bridge knot groups*, Mathematische Annalen **288** (1990), 103–119. MR **92c:**57004
[3] Gerhard Burde and Heiner Zieschang, *Knots*, Walter de Gruyter, 1985. MR **87b:**57004
[4] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, *Plane curves associated to character varieties of 3–manifolds*, Invent. Math. **118** (1994), 47–84. MR **95g:**57029
[5] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3–manifolds*, Annals of Mathematics **117** (1983). MR **84k:**57005
[6] Charles D. Frohman and Eric P. Klassen, *Deforming representations of knot groups in* $su(2)$, Comment. Math. Helvetici **66** (1991). MR **93a:**57001
[7] Craig David Hodgson, *Degeneration and regeneration of geometric structures on three–manifolds*, Ph.D. thesis, Princeton University, 1986.
[8] Eric Paul Klassen, *Representations of knot groups in SU(2)*, Transaction of the AMS **326** (1991), no. 2. MR **91k:**57003
[9] R. C. Lyndon and J. L. Ullman, *Groups of elliptic linear fractorial transformations*, Proceedings AMS **18** (1967). MR **36:**5234
[10] Andy R. Magid and Alexander Lubotzky, *Varieties of representations of finitely generated groups*, Memoirs of the AMS **58** (1985). MR **87c:**20021
[11] André Weil, *Remarks on the cohomology of groups*, Annals of Mathematics **80** (1964), 149–157. MR **30:**199

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