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ON THE LENGTHS OF CLOSED GEODESICS ON A TWO-SPHERE

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ABSTRACT. Let c be an isolated closed geodesic of length L on a compact Riemannian manifold M which is homologically visible in the dimension of its index, and for which the index of the iterates has the maximal possible growth rate. We show that M has a sequence $\{c_n\}$, $n \in \mathbb{Z}^+$, of prime closed geodesics of length $m_nL - \varepsilon_n$ where $m_n \in \mathbb{Z}$ and $\varepsilon_n \downarrow 0$. The hypotheses hold in particular when M is a two-sphere and the "shortest" Lusternik-Schnirelmann closed geodesic c is isolated and "nonrotating".

Let M be a compact Riemannian manifold of dimension n. For $a \in \mathbb{R}$ let Λ^a be the space of H^1 closed curves on M of energy $< \frac{a^2}{2}$. In this paper we prove the

Theorem. *Let c be an isolated closed geodesic of length L on M. Let $\lambda \in \mathbb{Z}^+$. Assume*

(1) *c gives local homology in dimension λ, i.e.,*

$$H_\lambda(\Lambda^L \cup c, \Lambda^L) \neq 0.$$

(2) $\mathrm{Index}(c^m) \geq m\lambda + (m-1)(n-1)$ *for $m \geq 1$.*

Then there is an $m_0 \in \mathbb{Z}^+$ and a sequence $\sigma_m \downarrow 0$ so that if $m \geq m_0$, and if m is odd (or if n and λ have different parity), M has a closed geodesic γ_m with length $\ell \in (mL - \sigma_m, mL)$. It follows that M has infinitely many closed geodesics.

Corollary. *Suppose M is a two-sphere*; *suppose the "shortest" Lusternik-Schnirelmann closed geodesic c is isolated and "nonrotating". Then the above conclusion holds (with $\lambda = 1$).*

Background. See [2, 3, 9] for a more complete discussion. When M is a two-sphere, the theorem of Lusternik and Schnirelmann [6, 11] gives three simple (i.e. embedded) closed geodesics. If e.g. M has positive curvature, then associated to each simple closed geodesic c is the Birkhoff map B. This is an area-preserving self-map of the closed annulus which describes the geodesic flow on M [3]. Interior periodic points of B give closed geodesics on M. An elementary and well-known argument, which some say is due to Birkhoff, shows that B has infinitely many periodic orbits unless c is "*nonrotating*", i.e. unless there is a point on c whose second conjugate point along c occurs after exactly one circuit about c [3, 9, 13].

For each simple closed geodesic c on a two-sphere M we have three cases:

(i) B is defined, c not nonrotating.

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(ii) B is defined, c nonrotating.

(iii) B is not defined.

As mentioned above, M has infinitely many closed geodesics in case (i). Victor Bangert has proved [2] that there are infinitely many in case (iii) and John Franks [5] that there are infinitely many in case (ii). In [9] we gave an elementary variational proof that M has infinitely many closed geodesics if the longest of the three Lusternik-Schnirelmann geodesics is nonrotating. This followed from the

Complementary Theorem ([9]). *Let c be an isolated closed geodesic on a compact Riemannian manifold of dimension n. Let $k \in \mathbb{Z}^+$. Assume*

1) *c gives local homology in dimension k.*

2) $(\text{Index} + \text{nullity})(c^m) \leq mk - (m-1)(n-1)$ *for* $m \geq 1$.

Then there is an $m_0 \in \mathbb{Z}^+$ and a sequence $\sigma_m \downarrow 0$ so that if $m \geq m_0$, M has a closed geodesic γ_m with length $\ell \in (mL, mL + \sigma_m)$. It follows that M has infinitely many closed geodesics.

(The above statement is stronger than that in [9]; however, the above can be inferred from the proof.) In [9] we also showed, using the equivariant Morse complex [8], that in all cases the number $N(\ell)$ of closed geodesics of length $\leq \ell$ on a two-sphere M satisfies

$$\liminf N(\ell)\frac{\log \ell}{\ell} > 0.$$

The present theorem does nothing to improve this estimate.

One does *not* expect an analogous theorem about the "middle" Lusternik-Schnirelmann geodesic; it seems that this one could be nonrotating but nondegenerate and hyperbolic.

Under the hypotheses of the theorem, the index of c^m has the fastest possible growth; a geodesic satisfying the hypothesis of the complement has the slowest possible growth. The idea of the theorem is the same as that of the complement but the Morse theory is "upside down". (The proof, however, is quite different.) In the complement the local negative cycle coming from c^m is "locally" homologous to a shorter cycle, while in the theorem the local homology class of the dual *positive* cycle is "locally" homologous to a *longer* cycle. The γ_m are "near" c^m in the sense that γ_m lies in the closure of the stable (resp. unstable) manifold of c^m. We think it is remarkable that in both cases we do not seem quite to "catch" the geodesics γ_m, but only see their shadows. (In the theorem this shadow is a point in a tubular neighborhood of c^m in the unstable manifold of γ_m.) In particular: The estimate for $N(\ell)$ mentioned above uses a "divisibility" result for surfaces [8]. In both the theorem and the complement it seems obvious, but does not quite follow from the proof, that the geodesic γ_m lies in a tubular neighborhood of c_m. If we could prove that this were the case, the multiplicity of γ_m would be a divisor of m (a stronger version of divisibility) and the growth estimate would follow immediately. That γ_m could move far enough away from c^m to develop a stability group which is not a subgroup of $\mathbb{Z}_m$ we find *almost* unbelievable. We invite the reader to try to do a better job of nabbing the γ_m.

In both the theorem and the complement the local homology of c^m becomes "unstable" as $m \to \infty$. It appears that if one were to stand back a little from M and squint one would "see" a closed geodesic with nondegenerate-type index growth instead of c.

Note that, given condition (2), (1) is equivalent to the weaker hypothesis that $H_\lambda(\Lambda^L \cup S^1 \cdot c, \Lambda^L) \neq 0$. Using the theorem, Birkhoff's result [3] and the elementary argument found in [13, 9] (which may or may not have been known to Birkhoff), one can prove that a metric of positive curvature on S^2 has infinitely many closed geodesics provided that Birkhoff's "minimax" geodesic has no self-intersections.

We first prove that the corollary follows from the theorem. Next we give a rough sketch of the proof of the theorem, and finally we prove the theorem.

Proof of Corollary (*from the Theorem*). The "shortest" Lusternik-Schnirelmann geodesic can be found by taking the minimax, using an appropriate curve shortening process [6], of the homology class in the space of unparameterized embedded curves on M represented by the image of the one-parameter family of circles on the standard sphere S^2 parallel to a fixed great circle under a diffeomorphism $S^2 \to M$. This produces a simple closed geodesic c with local homology in dimension 1 [1]. If c is nonrotating, then $\operatorname{index}(c^m) \geq 2m-1$ (counting conjugate points) and thus (1) and (2) hold with $\lambda = 1$. □

Sketch of proof of the theorem. We could not find a proof of the theorem as simple as that of the complement. To give the idea of the proof we first give a rough sketch in a (not so) special case. Condition (2) implies that the Poincaré map P of c satisfies $(P-1)^2 = 0$. In the case when the kernel of $(P-1)$ has dimension $n-1$ (the *least* degenerate case), we can choose a point $Q = c(0)$ which is *not* conjugate to $c(1)$ along c. In this case the proof is much simpler. Let W be a local transverse hypersurface to c at Q. If Ω_W is a neighborhood of c in the set of curves starting and ending on W, we have the fibration

$$f : \Omega_W \to W \times W.$$

The inverse image Λ_W of the diagonal Δ gives a neighborhood of c in Λ and the inverse image of o gives a neighborhood of c in the based loop space Ω_Q. Now c has index λ in Ω_Q and in Λ. In the "special case" each fiber of f contains a unique geodesic, also of index λ; thus "Morse theoretically"

$$\Omega_W \xrightarrow{\sim} W \times W \times V$$

where V is a disk of dimension λ and where on each fiber $w_0 \times w_1 \times V$, the length ℓ has a nondegenerate local maximum at the origin $x = 0$ in V. Moreover ℓ has a degenerate strict local minimum at the origin in $\Delta \times o$. Similarly, for a neighborhood Ω_m of c^m in Λ, we have

$$\Omega_m \xrightarrow{\sim} W^m \times V^m$$

since a curve in Ω_m has m "pieces", each of which lies in Ω_W. Now $W^m \times V^m$ has dimension $m(n-1)+m\lambda$, and length increases in the direction of $\Delta \times o \subseteq W^m \times V^m$. Since $\Delta \times o$ has dimension $n-1$, and $\operatorname{index}(c^m) = (m-1)(n-1) + m\lambda$, a maximal negative disk N for o in $W^m \times V^m$ must be transverse to $\Delta \times o$ at 0. However, N cannot avoid the higher dimensional set $\Delta' \times o$ where

$$\Delta' = \{w_0 = \cdots = w_j; w_{j+1} = \cdots = w_m\}, \qquad j \approx m/2.$$

Curves in $\Delta' \times o$ have $m-2$ "long" pieces and only 2 "short" pieces. For large m, $(m-2)$ times a degenerate minimum will win out quickly over twice a nondegenerate maximum, and the boundary sphere S of N cannot be pushed much below the level mL.

Proof of the theorem. Conditions (1) and (2) imply that c has index λ. Let r be the injectivity radius of M. Let $k \in \mathbb{Z}^+$ with $L/k < r/8$. Let $W = W_0, W_1, \dots, W_k$, $W_{k+1} = W$ be closed $(n-1)$ disks transverse to c at the $c(i/k+1)$. Let

$$\Omega_W = W \times W_1 \times \cdots \times W_k \times W,$$
$$f : \Omega_W \to W \times W,$$
$$\Lambda_W = f^{-1}(\Delta), \qquad \Lambda_W \approx W \times W_1 \times \cdot \times W_k,$$

where $\Delta \subseteq W \times W$ is the diagonal, and

$$\Omega_Q = f^{-1}(0), \quad \Omega_Q \approx W_1 \times \cdots \times W_k,$$
$$\Omega_m = (W \times W_1 \times \cdots \times W_k)^m.$$

Each of the above is identified with a set of curves: Ω_W is identified with the set of piecewise geodesics starting and ending on W in the obvious way. $\Lambda_W \subset \Omega_W$ is the subset of closed curves, and $\Omega_Q \subset \Lambda_W$ the closed curves at $Q = c(0)$. Ω_m is in the obvious way identified with a set of *closed* curves near c^m; Ω_m is the subset of Ω_W^m given by incidence relations on the ends. The natural function on these spaces is length. Thus, e.g. $\Omega_m^a \subseteq \Omega_m$ is the subset of curves of length $< a$.

If η is the nullity of c, the length function has a critical point at the origin in Λ_W of index λ and nullity η. (This follows from arguments in [12] although our finite dimensional approximation is somewhat different, corresponding to a neighborhood of c in the space of unparameterized curves.) The space Λ_W carries the local topology of Λ: If U is a sufficiently small H^1 neighborhood of c in Λ, we get a continuous length decreasing map $U \to \Lambda_W$ as follows. Let N be such that $L/N < r/2 \le L/N + 1$. If $\gamma \in U$ has length ℓ, first divide γ into pieces of length ℓ/N. Next replace each such piece by a geodesic segment. If U is sufficiently small, the endpoints of these pieces will lie close to the points $c(i/N)$ and the curve will now intersect the W_i exactly once transversally; thus we have a retraction onto Λ_W. Thus in particular Λ_W contains a representative of the local homology class of c. After shrinking the disks W_i we can find coordinates $(x, y, z) \in D^\lambda \times D^{(k+1)(n-1)-\lambda-\eta} \times D^\eta$ on Λ_W so that O is the only critical point of ℓ in Λ_W and so that [7]

$$\ell = L - x^2 + y^2 + g(z).$$

Since the critical point has local homology in dimension λ, g has a strict local minimum at O. (The complement proof in [9] contains the proof of a similar fact.)

Let w be a coordinate on W and thus on Λ_W. Then

A) Given $u > 0$ $\exists \rho_u > 0$ so that

$$\ell > L + \rho_u \quad \text{on } \{|w| \ge u; x = 0\} \subseteq \Lambda_W.$$

B) $\exists \mu > 0$ so that $\{\ell \le L + \mu; x = 0\}$ lies in the interior of Λ_W.

We can assume ρ_u increases with u.

We now move from c to c^m. Let Ω_m lie in the interior of a slightly larger version $\widetilde{\Omega}_m$ (extending the W_i). We define a map ψ_0 from an H^1 neighborhood U of c^m into $\widetilde{\Omega}_m$ as follows: Let N be such that $mL/N < r/2 \le mL/N + 1$. Given $\gamma \in U$, as before replace γ by N geodesic segments and then retract to $\widetilde{\Omega}_m$. We claim that, if the W_i and $\varepsilon > 0$ are sufficiently small (*independent of* m), ψ_0 extends to a continuous map

$$\psi : \Lambda^{mL+\varepsilon}/\Lambda^{mL-\varepsilon} \to \Omega_m/\partial\Omega_m \cup \Omega_m^{mL-\varepsilon}; \mathbb{Z}_m.$$

Here $\mathbb{Z}_m$ acts by permutations; we need to divide by the $\mathbb{Z}_m$-action since we allow all possible parameterizations in $\Lambda^{mL+\varepsilon}$. The map ψ does not increase length on points not mapped to the basepoint.

Proof. We expect $\varepsilon \ll r$. Assume U is a maximal open subset of Λ on which $\psi_0/\mathbb{Z}_m$ is defined. The map ψ_0 consists of two deformations. The first is always defined on $\Lambda^{mL+\varepsilon}$, and results in an intermediate curve which is short or which consists of N geodesic segments of length $\geq r/4$. If the W_i are sufficiently small, and if the intermediate curve meets each W_i transversally (in cyclic order) exactly m times, and is not significantly shortened after the *second* deformation, its pieces lie $\mathcal{C}^1$ close to c^m and thus the intermediate curve lies in the interior of the set where the second deformation is defined $\operatorname{mod} \mathbb{Z}_m$. Thus, if ε and the W_i are small,

$$\{\gamma | \gamma \in U \cap \Lambda^{mL+\varepsilon}; \psi_0/\mathbb{Z}_m(\gamma) \in \Omega_m/\mathbb{Z}_m; \ell\psi_0(\gamma) \geq mL - \varepsilon\}$$

is a closed set in the interior of U. Any point for which the second deformation is not defined maps to the basepoint. Since ψ_0 is continuous on U, ψ is continuous. From now on the W_i and ρ_u, μ, ε are fixed; we assume that the coordinates (x, y, z) are defined on Λ_W and that ψ is defined.

Next we construct a map

$$\begin{aligned} \varphi : \Omega_m &\to W^m \times V^m \\ \gamma &\mapsto (w_1, \dots, w_m, x_1, \dots, x_m) \end{aligned}$$

where $V \approx D^\lambda$ is the range of the coordinate x on Λ_W. The w_i are the m intersections of γ with W. The curve γ has m "pieces", each in Ω_W. Let

$$p : \Omega_W \to \Lambda_W \approx W \times W_1 \times \cdots \times W_k$$

by averaging the two coordinates on W; then $x \circ p$ on each piece of γ gives $x_1, \dots, x_m$. This step identifies the fiber of f. Suppose $\tau \in \Omega_W$ has coordinates $(w_0, w_1, x) \in W \times W \times V$ (by x we mean $x \circ p$). Then

$$\text{C)} \qquad \ell(\tau) + |w_1 - w_0| \geq \ell \circ p(\tau).$$

$$\text{D)} \qquad \ell \circ p(\tau) \geq L \quad \text{if } x(\tau) = 0.$$

C) follows from the triangle inequality if we choose a norm on W that exceeds the arclength on W.

Finally project $W^m \to \overline{W^m}$ where $\overline{W^m} \approx D^{(n-1)(m-1)}$ is the orthogonal complement $\sum w_i = 0$ of the diagonal Δ in W^m. This gives

$$\overline{\varphi} : \Omega_m \to \overline{W^m} \times V^m.$$

Let $j \leq m/2$ but $j \approx m/2$ and let

$$\Delta' = \{w_i = \cdots = w_j; w_{j+1} = \cdots = w_m\} \subseteq \overline{W^m}.$$

For simplicity assume $m = 2j$. (This is impossible if m is odd, but close to the truth if m is large.) The inverse image of a point $(w, \dots, w, -w, \dots, -w) \in \Delta'$ consists of points $(z + w, \dots, z + w, z - w, \dots, z - w) \in W^m$ for $z \in W$. Putting together A), C), and D), we have

$$\text{A)}^* \qquad \ell \geq mL + (j-1)\rho_{|w|} - 4|w| \quad \text{on } \overline{\varphi}^{-1}(\Delta' \times 0),$$

where w is the coordinate on Δ'. (Of the m pieces of a curve in $\overline{\varphi}^{-1}(\Delta' \times 0)$, two have length $\geq L - 2|w|$ by C), D); $j - 1$ have length $\geq L + \rho_{|z+w|}$ and $m - j - 1$ have length $\geq L + \rho_{|z-w|}$ by A). Either $\rho_{|z+w|}$ or $\rho_{|z-w|}$ is $\geq \rho_{|w|}$.)

Similarly, combining A), B), C), and D), if W contains a disk of radius η, we have

$$\text{B)}^* \qquad \ell \geq \min \begin{cases} mL + \mu - 4|w| \\ mL + (j-1)\rho_\eta - 4|w| \end{cases} \qquad \text{on } \overline{\varphi}^{-1}(\Delta' \times 0) \cap \partial\Omega_m.$$

(The second estimate holds if $z + w$ or $z - w \in \partial W$.)

Pick $\delta > 0$ but with $\delta < \min(\eta, \frac{\mu}{4}, \frac{\varepsilon}{4})$. Next find m_0 so that for $m \geq m_0$ we have

$$\text{E)} \qquad (j-1)\rho_\delta > 4\delta \qquad \left(j = \frac{m}{2} \text{ or } \frac{m-1}{2}\right).$$

Then also $(j-1)\rho_\eta > 4\delta$. Let Δ_δ be the subset of $\Delta' \times 0$ where $|w| \leq \delta$. By A)*

$$\ell > mL - \varepsilon \quad \text{on } \overline{\varphi}^{-1}(\Delta_\delta).$$

By B)*

$$\ell > mL \quad \text{on } \partial\Omega_m \cap \overline{\varphi}^{-1}(\Delta_\delta).$$

Thus, if

$$B = \text{closure}\, \overline{\varphi}(\Omega_m^{mL} \cap (\partial\Omega_m \cup \Omega_m^{mL-\varepsilon})),$$

B does not meet Δ_δ. Combining ψ and $\overline{\varphi}$, we get a continuous map for $m \geq m_0$

$$\Psi : \Lambda^{mL}/\Lambda^{mL-\varepsilon} \to \overline{W^m} \times V^m/B; \mathbb{Z}_m.$$

Now let N_m be a maximal negative definite subspace of the Hessian form on $T_{c^m}\Omega_m$. Consider

$$d\varphi : T_{c^m}\Omega_m \to T_0(W^m \times V^m).$$

The kernel of the map $T_{c^m}\Omega_m \to T_0 W^m$ is naturally $(T_c\Omega_Q)^m$. Since a negative definite vector in $T_c\Omega_Q$ cannot map to 0 in $T_0 V$, N_m injects into $T_0(W^m \times V^m)$. Now $\dim(W^m \times V^m) = \dim N_m + \dim(\Delta \times 0)$. By A)*, $\ell \geq mL$ on $\varphi^{-1}(\Delta \times 0)$ and thus

$$d\overline{\varphi} : N_m \xrightarrow{\sim} T_0(\overline{W^m} \times V^m)$$

is an isomorphism. Applying the exponential map to a sufficiently small sphere of codimension 1 in N_m gives a sphere S in $\Omega_m^{mL-\kappa}$ for some $\kappa > 0$ whose image in $\overline{W^m} \times V^m$ is a sphere encircling 0. We can assume that the image of S in $\overline{W^m}$ lies at a distance $< \delta$ from the origin, and that $\kappa < \varepsilon$. (In fact, the latter follows from the former.)

Now let $m \geq m_0$ and $R \in \partial\Delta_\delta$. By C), D) we can find a disk D about the origin in $\overline{W^m} \times V^m$ with $\ell > mL - \kappa$ in $\overline{\varphi}^{-1}(D)$. (Near the origin in $\overline{W^m}$ the term $|w_1 - w_0|$ in C) is small.) By shrinking D, we can assume $R \notin D$ and $D \cap B = \varnothing$, since B does not meet Δ_δ. Let $\overline{OR}$ denote the segment in Δ_δ from the origin to R. Let D' be a disk in $\overline{W^m} \times V^m$ with

$$\overline{OR} \subset \text{Interior}(D \cup D').$$

We can assume $D' \cap B = \varnothing$ and that $\mathbb{Z}_m \times D'$ consists of m disjoint sets. (The latter is possible since $\mathbb{Z}_m$ acts freely on Δ' except at 0.) Then $D \cup D' \to \overline{W^m} \times V^m$ induces a continuous map

$$\overline{W^m} \times V^m/B \to D \cup D'/\partial(D \cup D').$$

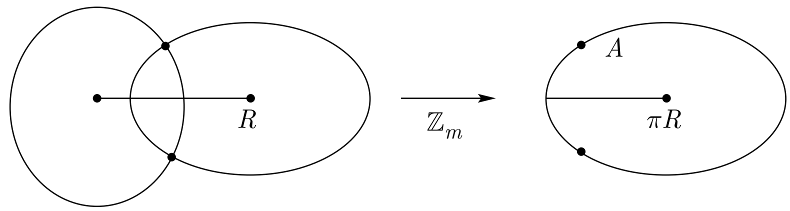

FIGURE 1

By A)* and E), $\ell > mL$ on $\overline{\varphi}^{-1}(R)$. Combining the above map with Φ gives a continuous map

$$\Psi : \Lambda^{mL-\kappa}/\Lambda^{mL-\varepsilon} \to A/B'$$

where

$$A = \mathrm{Image}(D' \setminus \{R\}) \subset \overline{W^m} \times V^m/\mathbb{Z}_m;$$
$$B' = \mathrm{Image}(\partial D' \cap \partial(D' \cap D)) \subset A.$$

Topologically A/B' is a sphere with two holes which are joined by the image of the segment $\overline{OR}$. (See Figure 1.)

Now $\mathbb{Z}_m$ will preserve orientation on $\overline{W^m} \times V^m$ if m is odd or if n and λ have different parity. Assume this is the case. The image of S in $\overline{W^m} \times V^m$ has intersection number m with $\mathbb{Z}_m \times \overline{OR}$, and thus the image of S in A/B' has intersection number m with the image $\pi\overline{OR}$ of $\overline{OR}$. If S and S' are homotopic in $\Lambda^{mL-\kappa}$, then $\Psi(S')$ must meet $\pi\overline{OR}$. Thus, by A)*, the minimax value for the homotopy class of S in $\Lambda^{mL-\kappa}$ is at least $mL - \sigma_m$, where σ_m is the maximum value of

$$4u - (j-1)\rho_u \quad \text{for } u \in [0, \delta].$$

Clearly $\sigma_m \to 0$ as $m \to \infty$. This minimax value will be achieved by a closed geodesic γ_m. Note that γ_m lies in the unstable manifold of c^m, and that some point in $\Psi^{-1}(\pi\overline{OR})$ lies in the unstable manifold of γ_m.

Finally, we show that the sequence γ_m must contain an infinite number of geometrically distinct closed geodesics. Let τ be a prime closed geodesic of length ℓ. If $L/\ell \in \mathbb{Q}$, then there is a constant $\varepsilon > 0$ so that

$$|mL - k\ell| > \varepsilon \quad \text{if } mL \neq k\ell.$$

Thus, only a finite number of the iterates τ^k could appear among the γ_m. Now suppose $L/\ell \notin \mathbb{Q}$. Given $N \in \mathbb{Z}$, let

$$\varepsilon_N = \min_{\substack{1 \le m \le N \\ k \in \mathbb{Z}}} |mL - k\ell| > 0.$$

Assume p is large enough so that $\sigma_m < \varepsilon_N$ for $m \geq p$; assume $q > p$ and that γ_p and γ_q are both iterates of τ. Then for some $k, j \in \mathbb{Z}$,

$$0 < pL - k\ell < \sigma_p \quad \text{and} \quad 0 < qL - j\ell < \sigma_q$$

so that

$$|(q-p)L - (j-k)\ell| < \varepsilon_N,$$

which implies $q - p > N$. Thus, the frequency with which the iterates of τ appear among the γ_m goes to 0.

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