Let $X$ be a compact subset of the complex plane $\mathbb{C}$, and let $0 < \alpha \leq 1$. We show that the maximal ideal space of Banach algebras of Lipschitz functions, which are analytic on $\text{int}X$, coincides with $X$.

Let $X$ be a compact subset of the complex plane $\mathbb{C}$, and let $0 < \alpha \leq 1$. $\text{Lip}_{\text{Hol}}(X, \alpha)$ is the algebra of all complex-valued functions $f$ on $X$ which are analytic on $\text{int}X$ and such that $\|f\| = p_\alpha(f) + \|f\|_\infty < \infty$, and $\text{lip}_{\text{Hol}}(X, \alpha)$ is the subalgebra of $\text{Lip}_{\text{Hol}}(X, \alpha)$ consisting of all functions $f$ with $\|f\|_{\text{lip}}(X, \alpha) \rightarrow 0$ as $\text{dist}(x, y) \rightarrow 0$. $\text{Lip}_{\text{Hol}}(X, \alpha)$ and $\text{lip}_{\text{Hol}}(X, \alpha)$ are Banach algebras if equipped with the norm $\|f\| = p_\alpha(f) + \|f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm on $X$.

Mahyar [3] proved that for $0 < \alpha < 1$ the maximal ideal space of $\text{lip}_{\text{Hol}}(X, \alpha)$ is $X$, and asked to characterize the maximal ideal space of $\text{Lip}_{\text{Hol}}(X, \alpha)$. In this note we provide an elementary proof that the maximal ideal space of $\text{Lip}_{\text{Hol}}(X, \alpha)$ is also $X$.

We use the standard Banach algebra notation. For a commutative normed algebra $A$ we denote by $\mathfrak{M}(A)$ the maximal ideal space of $A$ and we identify maximal ideals with the corresponding linear and multiplicative functionals. For an algebra $A$ of functions on $X$ with pointwise multiplication we identify $X$ with a subset of $\mathfrak{M}(A)$.

**Theorem.** Let $X$ be a compact subset of $\mathbb{C}$, and let $0 < \alpha \leq 1$. Then the maximal ideal space of $\text{Lip}_{\text{Hol}}(X, \alpha)$ coincides with $X$.

**Proof.** Let $F \in \mathfrak{M}(\text{Lip}_{\text{Hol}}(X, \alpha))$, and let $z_0 = F(Z)$, where $Z(z) = z$ for $z \in X$. Notice that $z_0 \in X$. Otherwise $h = (Z - z_0)$ would be an invertible element in ker $F$.

Assume $f \in \text{Lip}_{\text{Hol}}(X, \alpha)$ is such that $f(z_0) = 0$ and let $n > \frac{3}{\alpha}$. Put $g(z) = \frac{(f(z))^n}{z - z_0}$ for $z \in X \setminus \{z_0\}$, and $g(z_0) = 0$.

We show that $g \in \text{Lip}_{\text{Hol}}(X, \alpha)$. To simplify the notation we can assume without loss of generality, by applying a suitable linear transformation to $X$, that $z_0 = 0$ and that $X$ is a subset of the unit disc. We can also assume that $p_\alpha(f) \leq 1$. 

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Since \( p_\alpha(f) \leq 1 \), we have \(|f(z)| = |f(z) - f(0)| \leq |z|^\alpha \) for \( z \in X \), so because of the choice of \( n \) we have

\[
|(f(z))^n| \leq |z|^3 \quad \text{for} \quad z \in X.
\]

Let \( z_1, z_2 \) be two distinct elements of \( X \). Assume that \(|z_1| \leq |z_2|\). By (1), and since \( p_\alpha(f) \leq 1 \), we have:

if \( z_1 = 0 \), then

\[
\frac{|g(z_2) - g(z_1)|}{|z_2 - z_1|} = \frac{|g(z_2)|}{|z_2 - z_1|} = \frac{|(f(z_2))^n|}{|z_2|^{\alpha+1}} \leq \frac{|z_2|^3}{|z_2|^{\alpha+1}} \leq 1 < \infty,
\]

and if \( z_1 \neq 0 \), we have

\[
\frac{|g(z_2) - g(z_1)|}{|z_2 - z_1|} = \frac{|z_1 (f(z_2))^n - z_2 (f(z_1))^n|}{|z_2 - z_1|} \leq \frac{|z_1| |(f(z_2))^n - (f(z_1))^n| + |z_2 - z_1| |(f(z_1))^n|}{|z_2 - z_1|} \leq \frac{|(f(z_2))^n - (f(z_1))^n|}{|z_2 - z_1|} + |z_2 - z_1| |(f(z_1))^n| \leq \frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|} \sum_{j=0}^{n-1} |f(z_2)|^j |f(z_1)|^{n-1-j} + |z_2 - z_1| |f(z_1)|^{n-1} \leq \frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|} n |z_2 - z_1|^{n-1} \leq n p_\alpha(f) + 2 < \infty.
\]

Hence \( g \in \text{Lip}_{H_\alpha}(X, \alpha) \).

We have \( F(f^n) = F([Z - z_0])g = F(Z - z_0)F(g) = 0 \), so \( F(f) = 0 \). Hence \( \{ f \in \text{Lip}_{H_\alpha}(X, \alpha) : f(z_0) = 0 \} \subseteq \ker F \), therefore \( F \) is the functional of evaluation at \( z_0 \), and \( X = \mathfrak{M}(\text{Lip}_{H_\alpha}(X, \alpha)) \).

A very similar proof, with only minor modifications, can be applied to several other algebras including \( \text{lip}_{H_\alpha}(X, \alpha) \) and some of the algebras studied by Dales and Davie [1].

One may be tempted to "prove" the Theorem with the following argument: The algebra \( \text{Lip}_{H_\alpha}(X, \alpha) \) is contained between \( \text{lip}_{H_\alpha}(X, \alpha) \) and the uniform algebra \( A(X) \) of all continuous functions on \( X \) that are analytic on \( \text{int} X \); the maximal ideal spaces of both algebras \( \text{lip}_{H_\alpha}(X, \alpha) \) and \( A(X) \) are equal to \( X \) so \( \mathfrak{M}(\text{Lip}_{H_\alpha}(X, \alpha)) = X \). However such a conclusion is, in general, incorrect - see an ingenuous example by Horany [2].

\begin{thebibliography}{10}


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