

$\text{Lip}_{Hol}(X, \alpha)$

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ABSTRACT. Let X be a compact subset of the complex plane \mathbb{C} , and let $0 < \alpha \leq 1$. We show that the maximal ideal space of Banach algebras of Lipschitz functions, which are analytic on $\text{int}X$, coincides with X .

Let X be a compact subset of the complex plane \mathbb{C} , and let $0 < \alpha \leq 1$. $\text{Lip}_{Hol}(X, \alpha)$ is the algebra of all complex-valued functions f on X which are analytic on $\text{int}X$ and such that

$$p_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{(\text{dist}(x, y))^\alpha} : x, y \in X, x \neq y \right\} < \infty,$$

and $\text{lip}_{Hol}(X, \alpha)$ is the subalgebra of $\text{Lip}_{Hol}(X, \alpha)$ consisting of all functions f with $\frac{|f(x) - f(y)|}{(\text{dist}(x, y))^\alpha} \rightarrow 0$ as $\text{dist}(x, y) \rightarrow 0$. $\text{Lip}_{Hol}(X, \alpha)$ and $\text{lip}_{Hol}(X, \alpha)$ are Banach algebras if equipped with the norm $\|f\| = p_\alpha(f) + \|f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm on X .

Mahyar [3] proved that for $0 < \alpha < 1$ the maximal ideal space of $\text{lip}_{Hol}(X, \alpha)$ is X , and asked to characterize the maximal ideal space of $\text{Lip}_{Hol}(X, \alpha)$. In this note we provide an elementary proof that the maximal ideal space of $\text{Lip}_{Hol}(X, \alpha)$ is also X .

We use the standard Banach algebra notation. For a commutative normed algebra A we denote by $\mathfrak{M}(A)$ the maximal ideal space of A and we identify maximal ideals with the corresponding linear and multiplicative functionals. For an algebra A of functions on X with pointwise multiplication we identify X with a subset of $\mathfrak{M}(A)$.

Theorem. *Let X be a compact subset of \mathbb{C} , and let $0 < \alpha \leq 1$. Then the maximal ideal space of $\text{Lip}_{Hol}(X, \alpha)$ coincides with X .*

Proof. Let $F \in \mathfrak{M}(\text{Lip}_{Hol}(X, \alpha))$, and let $z_0 = F(\mathbf{Z})$, where $\mathbf{Z}(z) = z$ for $z \in X$. Notice that $z_0 \in X$. Otherwise $h = (\mathbf{Z} - z_0)$ would be an invertible element in $\ker F$. Assume $f \in \text{Lip}_{Hol}(X, \alpha)$ is such that $f(z_0) = 0$ and let $n > \frac{3}{\alpha}$. Put $g(z) = \frac{(f(z))^n}{z - z_0}$ for $z \in X \setminus \{z_0\}$, and $g(z_0) = 0$.

We show that $g \in \text{Lip}_{Hol}(X, \alpha)$. To simplify the notation we can assume without loss of generality, by applying a suitable linear transformation to X , that $z_0 = 0$ and that X is a subset of the unit disc. We can also assume that $p_\alpha(f) \leq 1$.

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Since $p_\alpha(f) \leq 1$, we have $|f(z)| = |f(z) - f(0)| \leq |z|^\alpha$ for $z \in X$, so because of the choice of n we have

$$(1) \quad |(f(z))^n| \leq |z|^3 \quad \text{for } z \in X.$$

Let z_1, z_2 be two distinct elements of X . Assume that $|z_1| \leq |z_2|$. By (1), and since $p_\alpha(f) \leq 1$, we have:
if $z_1 = 0$, then

$$\frac{|g(z_2) - g(z_1)|}{|z_2 - z_1|^\alpha} = \frac{|g(z_2)|}{|z_2|^\alpha} = \frac{|(f(z_2))^n|}{|z_2|^{\alpha+1}} \leq \frac{|z_2|^3}{|z_2|^{\alpha+1}} \leq 1 < \infty,$$

and if $z_1 \neq 0$, we have

$$\begin{aligned} \frac{|g(z_2) - g(z_1)|}{|z_2 - z_1|^\alpha} &= \frac{|z_1 (f(z_2))^n - z_2 (f(z_1))^n|}{|z_2 - z_1|^\alpha |z_1| |z_2|} \\ &\leq \frac{|z_1| |(f(z_2))^n - (f(z_1))^n| + |z_2 - z_1| |(f(z_1))^n|}{|z_2 - z_1|^\alpha |z_1| |z_2|} \\ &= \frac{|(f(z_2))^n - (f(z_1))^n|}{|z_2 - z_1|^\alpha |z_2|} + \frac{|z_2 - z_1| |(f(z_1))^n|}{|z_2 - z_1|^\alpha |z_1| |z_2|} \\ &\leq \frac{|f(z_2) - f(z_1)| \sum_{j=0}^{n-1} |f(z_2)|^j |f(z_1)|^{n-1-j}}{|z_2 - z_1|^\alpha |z_2|} + \frac{|z_2 - z_1| |z_1|^2}{|z_2 - z_1|^\alpha |z_2|} \\ &\leq \frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|^\alpha} \frac{n |z_2|^{\alpha(n-1)}}{|z_2|} + |z_2 - z_1|^{1-\alpha} |z_1| \\ &\leq \frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|^\alpha} n + |z_2 - z_1|^{1-\alpha} \leq np_\alpha(f) + 2 < \infty. \end{aligned}$$

Hence $g \in \text{Lip}_{Hol}(X, \alpha)$.

We have $F(f^n) = F((\mathbf{Z} - z_0) \cdot g) = F(\mathbf{Z} - z_0)F(g) = 0$, so $F(f) = 0$. Hence $\{f \in \text{Lip}_{Hol}(X, \alpha) : f(z_0) = 0\} \subseteq \ker F$, therefore F is the functional of evaluation at z_0 , and $X = \mathfrak{M}(\text{Lip}_{Hol}(X, \alpha))$. \square

A very similar proof, with only minor modifications, can be applied to several other algebras including $\text{lip}_{Hol}(X, \alpha)$ and some of the algebras studied by Dales and Davie [1].

One may be tempted to “prove” the Theorem with the following argument: The algebra $\text{Lip}_{Hol}(X, \alpha)$ is contained between $\text{lip}_{Hol}(X, \alpha)$ and the uniform algebra $A(X)$ of all continuous functions on X that are analytic on $\text{int}X$; the maximal ideal spaces of both algebras $\text{lip}_{Hol}(X, \alpha)$ and $A(X)$ are equal to X so $\mathfrak{M}(\text{Lip}_{Hol}(X, \alpha)) = X$. However such a conclusion is, in general, incorrect - see an ingenious example by Honary [2].

REFERENCES

- [1] H. G. Dales and A. M. Davie. Quasianalytic Banach function algebras. *Journal of Functional Analysis*, 13:28–50, 1973. MR **49**:7782
- [2] T. G. Honary. Relations between Banach function algebras and their uniform closures. *Proc. Amer. Math. Soc.*, 109(2):337–342, June 1990. MR **91d**:46066
- [3] H. Mahyar. The maximal ideal space of $\text{lip}_A(X, \alpha)$. *Proc. Amer. Math. Soc.*, 122(1):175–181, September 1994. MR **95a**:46074

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