THE SLICE GENUS AND THE THURSTON-BENNEQUIN
INARIANT OF A KNOT

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Abstract. For any knot $K \subset S^3$, $g_s(K) \geq (TB(K) + 1)/2$.

Let $K \subset S^3 = \partial D^4 \subset C^2 \subset \mathbb{CP}^2$ be a (smooth) knot. The slice genus $g_s(K)$ is the smallest genus of a smooth, oriented surface in $D^4$ with boundary $K$. If $K$ is Legendrian, that is, everywhere tangent to the field of complex lines tangent to $S^3$, then the Thurston-Bennequin invariant $tb(K)$ is the integer associated in the usual way (using linking numbers in $S^3$, with the orientation induced by $C^2$) to the normal line field to $K$ which assigns to each point of $K$ the tangent line to $K$ multiplied by $\sqrt{-1}$; cf. [1]. For arbitrary $K$, the maximal Thurston-Bennequin invariant $TB(K)$ is $\sup\{tb(K') : K' \subset S^3$ is a Legendrian knot ambient isotopic to $K\}$.

Appealing to a theorem of Donaldson, I showed [4] that if $g_s(K) = 0$, then $TB(K) \leq -1$. The following inequality is more general; the proof requires more than Donaldson’s theorem, namely, the truth of the local Thom Conjecture [2].

Theorem. $g_s(K) \geq (TB(K) + 1)/2$.

Proof. Using the construction of entire totally-tangential C-links given in [4], and elementary rational approximation theory, it is easily seen that, for any integer $t \leq TB(K)$, there is a rational immersion $F : \mathbb{CP}^1 \to \mathbb{CP}^2$, in general position, such that $A = F^{-1}(D^4) \subset C \subset \mathbb{CP}^1$ is a neighborhood of $S^3$ diffeomorphic to an annulus, $F|A$ is an embedding, and $\partial F(A) = \partial A(K,t) \subset S^3$, where $\partial A(K,t) \subset S^3$ denotes an (embedded) annulus of type $K$ with $t$ twists (i.e., each component of $\partial A(K,t)$ is ambient isotopic to $K$, and the linking number of the components is $-t$).

Let $D_i \subset \mathbb{CP}^1$, $i = 1, 2$, be the two 2-disks complementary to Int $A$. Let the smoothley immersed 2-disk $F(D_i) \subset \mathbb{CP}^2 \setminus Int D^4$ have $d_i$ doublepoints; let $S_i \subset \mathbb{CP}^2$ be a smoothly immersed surface which is the union along a component of $\partial A(K,t)$ of $F(D_i)$ and a suitable smoothly embedded surface in $D^4$ of genus $g_s(K)$. Let $n_i$ be the degree of $S_i$. Then the degree of $F(\mathbb{CP}^1)$ is $n_1 + n_2$, and, by the local Thom Conjecture, $g_s(K) + d_i \geq (n_i - 1)(n_i - 2)/2$.

Let $p$ be the intersection number of the relative cycles $F(D_1)$ and $F(D_2)$ in $\mathbb{CP}^2 \setminus Int D^4$. Then $p - t = n_1 n_2$ is the intersection number of $S_1$ and $S_2$ in $\mathbb{CP}^2$, and $p + d_1 + d_2 = (n_1 + n_2 - 1)(n_1 + n_2 - 2)/2$. 

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By combining these equations and inequalities, we find that $2g_s(K) - 1 \geq t$. \qed

Remarks. (1) An advantage of the Theorem over the slice-Bennequin inequality is that a lower bound (however crude) for $TB(K)$, and thus for $g_s(K)$, can be read off from any (smooth or piecewise-linear) knot diagram $D(K) \subset \mathbb{R}^2$ for $K$, without a need for preliminary braiding. The global sign of a doublepoint $P$ of $D(K)$ is its sign (+ or −) in the usual sense (cf. Figure 1); the writhe $w(D(K))$ is the number of doublepoints of global sign +, less the number of doublepoints of global sign −. A linear form $\ell : \mathbb{R}^2 \to \mathbb{R}$ with $2m_\ell(D(K)) < \infty$ local extrema on $D(K)$, of which none is a doublepoint, assigns a local sign $s_\ell(P) \in \{+,-\}$ to each doublepoint $P$. If $s_\ell(P) = +$ for all $P$, then $D(K)$ is piecewise-smoothly isotopic to a front [4] and, as described in [4], $TB(K) \geq w(D(K)) - m_\ell$; in general (cf. Figure 1), $TB(K) \geq w(D(K)) - (m_\ell + \text{card } s_\ell^{-1}(-))$.

(2) Of course the genus $g(K)$ is greater than or equal to $g_s(K)$. Is there a direct, elementary, 3-dimensional proof that $g(K) \geq (\text{tb}(K) + |\text{r}(K)| + 1)/2$?

(3) Since this note was first circulated, Kronheimer and Mrowka [3] have improved the estimate for any Legendrian knot $K$ with $TB(K) > 0$; by using the adjunction inequality rather than just the local Thom Conjecture, they show that for any such $K$, $g_s(K) \geq (\text{tb}(K) + |\text{r}(K)| + 1)/2$, where $r(K)$ denotes the rotation number of $K$ (cf. [1]). (The restriction that $TB(K) > 0$ may be unnecessary.)

REFERENCES


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