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THE SLICE GENUS AND THE THURSTON-BENNEQUIN INVARIANT OF A KNOT

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ABSTRACT. For any knot $K \subset S^3$, $g_s(K) \geq (\mathrm{TB}(K)+1)/2$.

Let $K \subset S^3 = \partial D^4 \subset \mathbf{C}^2 \subset \mathbf{CP}^2$ be a (smooth) knot. The *slice genus* $g_s(K)$ is the smallest genus of a smooth, oriented surface in D^4 with boundary K. If K is *Legendrian*, that is, everywhere tangent to the field of complex lines tangent to S^3, then the *Thurston-Bennequin invariant* $\mathrm{tb}(K)$ is the integer associated in the usual way (using linking numbers in S^3, with the orientation induced by $\mathbf{C}^2$) to the normal linefield to K which assigns to each point of K the tangent line to K multiplied by $\sqrt{-1}$; cf. [1]. For arbitrary K, the *maximal Thurston-Bennequin invariant* $\mathrm{TB}(K)$ is $\sup\{\mathrm{tb}(K') : K' \subset S^3$ is a Legendrian knot ambient isotopic to $K\}$.

Appealing to a theorem of Donaldson, I showed [4] that if $g_s(K) = 0$, then $\mathrm{TB}(K) \leq -1$. The following inequality is more general; the proof requires more than Donaldson's theorem, namely, the truth of the local Thom Conjecture [2].

Theorem. $g_s(K) \geq (\mathrm{TB}(K)+1)/2$.

Proof. Using the construction of *entire totally-tangential* $\mathbf{C}$*-links* given in [4], and elementary rational approximation theory, it is easily seen that, for any integer $t \leq \mathrm{TB}(K)$, there is a rational immersion $F : \mathbf{CP}^1 \to \mathbf{CP}^2$, in general position, such that $A = F^{-1}(D^4) \subset \mathbf{C} \subset \mathbf{CP}^1$ is a neighborhood of S^1 diffeomorphic to an annulus, $F|A$ is an embedding, and $\partial F(A) = \partial A(K,t) \subset S^3$, where $\partial A(K,t) \subset S^3$ denotes an (embedded) annulus of type K with t twists (i.e., each component of $\partial A(K,t)$ is ambient isotopic to K, and the linking number of the components is $-t$).

Let $D_i \subset \mathbf{CP}^1$, $i = 1, 2$, be the two 2-disks complementary to $\mathrm{Int}\, A$. Let the smoothly immersed 2-disk $F(D_i) \subset \mathbf{CP}^2 \setminus \mathrm{Int}\, D^4$ have d_i doublepoints; let $S_i \subset \mathbf{CP}^2$ be a smoothly immersed surface which is the union along a component of $\partial A(K,t)$ of $F(D_i)$ and a suitable smoothly embedded surface in D^4 of genus $g_s(K)$. Let n_i be the degree of S_i. Then the degree of $F(\mathbf{CP}^1)$ is $n_1 + n_2$, and, by the local Thom Conjecture, $g_s(K) + d_i \geq (n_i - 1)(n_i - 2)/2$.

Let p be the intersection number of the relative cycles $F(D_1)$ and $F(D_2)$ in $\mathbf{CP}^2 \setminus \mathrm{Int}\, D^4$. Then $p - t = n_1 n_2$ is the intersection number of S_1 and S_2 in $\mathbf{CP}^2$, and $p + d_1 + d_2 = (n_1 + n_2 - 1)(n_1 + n_2 - 2)/2$.

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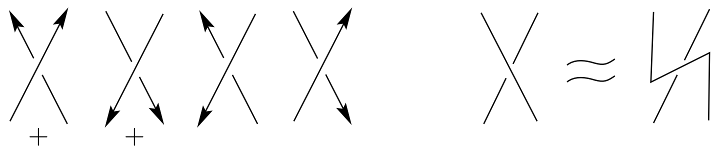

FIGURE 1. Left, global signs of the four types of crossings with local sign +. Right, changing a local sign from − to +.

By combining these equations and inequalities, we find that $2g_s(K) - 1 \geq t$. □

Remarks. (1) An advantage of the Theorem over the slice-Bennequin inequality is that a lower bound (however crude) for $\mathrm{TB}(K)$, and thus for $g_s(K)$, can be read off from any (smooth or piecewise-linear) knot diagram $\mathbf{D}(K) \subset \mathbf{R}^2$ for K, without a need for preliminary braiding. The *global sign* of a doublepoint $\mathbf{P}$ of $\mathbf{D}(K)$ is its sign (+ or −) in the usual sense (cf. Figure 1); the *writhe* $w(\mathbf{D}(K))$ is the number of doublepoints of global sign +, less the number of doublepoints of global sign −. A linear form $\ell : \mathbf{R}^2 \to \mathbf{R}$ with $2m_\ell(\mathbf{D}(K)) < \infty$ local extrema on $\mathbf{D}(K)$, of which none is a doublepoint, assigns a *local sign* $s_\ell(\mathbf{P}) \in \{+, -\}$ to each doublepoint $\mathbf{P}$. If $s_\ell(\mathbf{P}) = +$ for all $\mathbf{P}$, then $\mathbf{D}(K)$ is piecewise-smoothly isotopic to a *front* [4] and, as described in [4], $\mathrm{TB}(K) \geq w(\mathbf{D}(K)) - m_\ell$; in general (cf. Figure 1), $\mathrm{TB}(K) \geq w(\mathbf{D}(K)) - (m_\ell + \operatorname{card} s_\ell^{-1}(-))$.

(2) Of course the genus $g(K)$ is greater than or equal to $g_s(K)$. Is there a direct, elementary, 3-dimensional proof that $g(K) \geq (w(\mathbf{D}(K))-m_\ell-\operatorname{card} s_\ell^{-1}(-)+1)/2$?

(3) Since this note was first circulated, Kronheimer and Mrowka [3] have improved the estimate for any Legendrian knot K with $\mathrm{TB}(K) > 0$; by using the adjunction inequality rather than just the local Thom Conjecture, they show that for any such K, $g_s(K) \geq (\mathrm{tb}(K) + |r(K)| + 1)/2$, where $r(K)$ denotes the *rotation number* of K (cf. [1]). (The restriction that $\mathrm{TB}(K) > 0$ may be unnecessary.)

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