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COPRIMENESS AMONG IRREDUCIBLE CHARACTER DEGREES OF FINITE SOLVABLE GROUPS

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ABSTRACT. Given a finite solvable group G, we say that G has property P_k if every set of k distinct irreducible character degrees of G is (setwise) relatively prime. Let $k(G)$ be the smallest positive integer such that G satisfies property P_k. We derive a bound, which is quadratic in $k(G)$, for the total number of irreducible character degrees of G. Three exceptional cases occur; examples are constructed which verify the sharpness of the bound in each of these special cases.

1. INTRODUCTION

Suppose G is a finite solvable group and let $\operatorname{cd}(G)$ denote the set $\{\chi(1) \mid \chi \in \operatorname{Irr}(G)\}$. We say that G has property P_k if every set of k distinct elements of $\operatorname{cd}(G)$ is (setwise) relatively prime. Every finite group G satisfies P_k at least for $k \geq |\operatorname{cd}(G)|$, since $1 \in \operatorname{cd}(G)$. The main result of this paper is the following:

Theorem A. *Let G be a nonabelian finite solvable group and let k be the smallest positive integer such that G satisfies property P_k. Then*

$$|\operatorname{cd}(G)| \leq \begin{cases} 3 & \text{if } k = 2; \\ 6 & \text{if } k = 3; \\ 9 & \text{if } k = 4; \\ k^2 - 3k + 4 & \text{if } k \geq 5. \end{cases}$$

Following the proof of Theorem A, a collection of examples is presented. In each of the exceptional cases $k = 2, 3, 4$ the bound is attained. For $k = 2$, an example is provided by the group $SL(2,3)$. This group satisfies P_2 and has 3 irreducible character degrees: $\operatorname{cd}(SL(2,3)) = \{1, 2, 3\}$. For $k = 3$, we construct a group Γ with $\operatorname{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$, where q, r, s are any three primes satisfying $q \equiv 3 \pmod 4$ and $q \equiv 1 \pmod{rs}$. The group Γ attains the bound in this case. Next, for infinitely many values of k, we construct a group which satisfies property P_k and has $3(k-1)$ irreducible character degrees. Observe that, for $k = 4$, such a group satisfies P_4 and has 9 irreducible character degrees, verifying the sharpness of the bound in this case. It also follows from this infinite set of examples that the best possible bound for $|\operatorname{cd}(G)|$ in terms of k cannot be better than the linear bound $3(k-1)$.

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While this result belongs to a genre of problems and results concerning the irreducible character degrees of finite solvable groups (see §2 of [4]), it has a unique flavor. The investigation of property P_k was inspired by problem 12.3 of [1] which is, in fact, the $k = 2$ case of the result. At this point the author would like to express her appreciation to Professor Martin Isaacs for his direction and encouragement in this work, which is a portion of her thesis.

2. Preliminaries

The purpose of this section is to restate facts about the structure and character degrees of a factor group G/K of a finite nonabelian solvable group G with K chosen to be maximal such that G/K remains nonabelian. Notice that, in this situation every proper factor group of G/K is abelian and thus $(G/K)'$ is the unique minimal normal subgroup of G/K.

(2.1) Lemma. *Let G be a finite solvable group and assume that G' is the unique minimal normal subgroup of G. Then all the nonlinear irreducible characters of G have equal degree f and one of the following situations obtains:*

(a) *G is a p-group, $\mathbf{Z}(G)$ is cyclic and $G/\mathbf{Z}(G)$ is elementary abelian of order f^2.*
(b) *G is a Frobenius group with a cyclic Frobenius complement of order f. Also, G' is the Frobenius kernel and is an elementary abelian p-group.*

Proof. This is Lemma 12.3 of [1] with the observation that an abelian Frobenius complement is necessarily cyclic. □

(2.2) Theorem. *Let $K \lhd G$ be such that G/K is a Frobenius group with kernel N/K, an elementary abelian p-group. Let $\psi \in \mathrm{Irr}(N)$. Then one of the following holds:*

(a) $|G : N|\psi(1) \in \mathrm{cd}(G)$.
(b) *p divides $\psi(1)$.*

Proof. This is immediate from Theorem 12.4 of [1]. □

3. Proof of Theorem A

We begin by proving a key lemma.

(3.1) Lemma. *Let G be a finite nonabelian solvable group with $G' \leq \mathbf{O}^p(G)$ for all primes p. Suppose that $K \lhd G$ and K is maximal such that G/K is nonabelian. Then G/K is a Frobenius group with Frobenius kernel N/K, an elementary abelian q-group for some prime q, and a cyclic Frobenius complement. Let f denote the order of the Frobenius complement and assume further that K is chosen so that f is minimal. Then for each linear character λ of N, either λ^G is irreducible or λ extends to G. In particular, if $\chi \in \mathrm{Irr}(G)$ lies over a linear character of N, then χ must have degree 1 or f.*

It will be handy in the proof to use the standard notation $b(G)$ to denote the largest irreducible character degree of G; that is, the maximum of the set $\mathrm{cd}(G)$.

Proof. By hypothesis, if $M \lhd G$ with $K < M$, then the quotient G/M is abelian. Since G is solvable, it follows that $(G/K)'$ is the unique minimal normal subgroup of G/K. Now since G has no nonabelian p-factor groups for any prime p, the Frobenius structure of G/K follows from Lemma 2.1 (b).

Fix a linear character $\lambda \in \mathrm{Irr}(N)$ and let $\chi \in \mathrm{Irr}(G)$ lie over λ. Set $T = I_G(\lambda)$ and $t = |G : T|$. Since T/N is cyclic, λ extends to a character $\hat{\lambda} \in \mathrm{Irr}(T)$ and further, by Corollary 6.17 of [1], every element of $\mathrm{Irr}(T|\lambda)$ is an extension of λ. We may then assume without loss of generality that the extension $\hat{\lambda}$ is the Clifford correspondent between χ and λ. Thus $\chi = (\hat{\lambda})^G$ and $\chi_N = \sum_{i=1}^{t} \lambda_i$, labeled so that $\lambda_1 = \lambda$. Also note that $\chi(1) = t$.

We are done if $t = 1$ or $t = f$, so assume, for a contradiction, that neither happens. In this case $1 < t = \chi(1) < f$ and $N < T < G$. Let $M = \mathrm{ker}\lambda$. Since T fixes the linear character λ, it follows that T centralizes N/M and so $[T, N] \leq M$. Also $[T, N] \lhd G$, since both T and N are normal. Let $^{-}$ denote quotients mod $[T, N]$. Then $\overline{N}$ is central in $\overline{T}$ and $\overline{T}/\overline{N}$ is cyclic since it is isomorphic to T/N; thus $\overline{T}$ is abelian. We have $\overline{T}$ is normal and abelian in $\overline{G}$. By Ito's Theorem (6.15 of [1]) $t = |\overline{G} : \overline{T}| \geq b(\overline{G})$. Also, since $\mathrm{ker}\chi \geq \mathrm{core}_G\big(\mathrm{ker}\lambda\big) \geq [T, N]$, we may view χ as an element of $\mathrm{Irr}(\overline{G})$. We have $t = \chi(1) \in \mathrm{cd}(\overline{G})$ and thus $\overline{G}$ is nonabelian.

Now, let $\overline{G}/\overline{L}$ be a minimal nonabelian factor of $\overline{G}$. Clearly the hypothesis on p-factors of G holds for p-factors of $\overline{G}$. It follows from Lemma 2.1 that $\overline{G}/\overline{L}$ is a Frobenius group with a Frobenius complement of order $\chi(1) = t < f$. Since factors of $\overline{G}$ are factors of G, this contradicts the minimality of f. □

It will now be convenient to establish some notation for the proof of Theorem A. For a group G, we define $k(G)$ to be the smallest integer such that G satisfies property P_k. Note that $k(G) \leq |\,\mathrm{cd}(G)|$ and if G is a q-group, for q prime, then equality holds. Given a positive integer q, we define $\mathrm{cd}_q(G) = \big\{n \in \mathrm{cd}(G) \;\big|\; (q, n) = 1\big\}$ and $\mathrm{cd}^q(G) = \big\{n \in \mathrm{cd}(G) \;\big|\; q|n\big\}$. If q is prime, then $\mathrm{cd}(G)$ is the disjoint union of these two sets. Also, for $N \lhd G$ with $m \in \mathrm{cd}(N)$ if there exists $\psi \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G|\psi)$ with $\chi(1) = n$ and $\psi(1) = m$, then we will say that n lies over m. Further, for each such m, define a subset of $\mathrm{cd}(G)$ by $s(m) = \{n \in \mathrm{cd}(G) \;\big|\; n \text{ lies over } m\}$. Note that a given n may lie over many different m and each element of $s(m)$ is divisible by m.

To prove Theorem A, we bound each of $|\mathrm{cd}^q(G)|$ and $|\mathrm{cd}_q(G)|$ separately in terms of $k(G)$ and add the results. Note that if $k = k(G)$ for a group G, then $|\mathrm{cd}^q(G)| \leq k - 1$ for any positive integer q. On the other hand, given k, examples are available among q-groups, where q is prime, which satisfy $|\,\mathrm{cd}(G)| = k$. For instance, let Q be the direct product of $k-1$ copies of a q-group A having $\mathrm{cd}(A) = \{1, q\}$. Then $\mathrm{cd}(Q) = \{1, q, q^2, \cdots, q^{k-1}\}$; thus $k = k(Q)$ and $|\mathrm{cd}^q(Q)| = k - 1$. It follows that $k-1$ is the best possible bound for $|\mathrm{cd}^q(G)|$. Our challenge in proving Theorem A will be to bound $|\mathrm{cd}_q(G)|$.

Proof of Theorem A. Let G be a nonabelian finite solvable group and let $k = k(G)$.

Suppose first that G has a nonabelian p-factor group G/K for some prime p. As we have observed, $|\mathrm{cd}^p(G)| \leq k - 1$. Now we consider $|\mathrm{cd}_p(G)|$. Fix $\psi \in \mathrm{Irr}(G/K)$ with $\psi(1) = p^a > 1$. For each character $\chi \in \mathrm{Irr}(G)$ with $\big(p, \chi(1)\big) = 1$ we have $\chi_K \in \mathrm{Irr}(K)$. By Corollary 6.17 of [1], we have $\chi\psi \in \mathrm{Irr}(G)$ with degree $\chi\psi(1)$ divisible by p, since $\chi\psi(1) = \chi(1)p^a$. This gives an injection from $\mathrm{cd}_p(G)$ into $\mathrm{cd}^p(G)$. Thus $|\mathrm{cd}_p(G)| \leq k - 1$ and $|\,\mathrm{cd}(G)| \leq 2(k - 1)$. In this case the conclusion of the theorem holds. Henceforth we assume that $G' \leq \mathbf{O}^p(G)$ for all primes p.

Now fix $K \lhd G$ so that K is maximal with G/K nonabelian. By Lemma 2.1, G/K is a Frobenius group with kernel N/K, an elementary abelian q-group, and with a cyclic complement H/K of order f. Also $\operatorname{cd}(G/K) = \{1, f\}$. Assume further that K is chosen so that f is minimal. As before, we have $|\operatorname{cd}^q(G)| \leq k-1$.

To assess $|\operatorname{cd}_q(G)|$ we will examine how many distinct elements of $\operatorname{cd}_q(G)$ lie over each element of $\operatorname{cd}(N)$. If we write $\operatorname{cd}(N) = \operatorname{cd}^q(N) \dot{\cup} \operatorname{cd}_q(N)$, then notice that elements of $\operatorname{cd}^q(G)$ can lie over only elements of $\operatorname{cd}^q(N)$, since $(q, f) = 1$, and elements of $\operatorname{cd}_q(G)$ lie over only elements of $\operatorname{cd}_q(N)$. Also, by Theorem 2.2 (a), for each element $z \in \operatorname{cd}_q(N)$ we must have $fz \in \operatorname{cd}(G)$. This gives an injection from $\operatorname{cd}_q(N)$ into $\operatorname{cd}^f(G)$. Again, by hypothesis, $|\operatorname{cd}^f(G)| \leq k-1$; thus $|\operatorname{cd}_q(N)| \leq k-1$. It follows that all the elements of $\operatorname{cd}_q(G)$ lie over the, at most $k-1$, elements of $\operatorname{cd}_q(N)$.

If $z \in \operatorname{cd}_q(N)$, how many elements of $\operatorname{cd}_q(G)$ can lie over z? By Lemma 3.1, if $z = 1$, then $s(z) = \{1, f\}$. If $z > 1$, then $|s(z)| \leq k-1$, since $s(z) \subseteq \operatorname{cd}^z(G) \leq k-1$, by hypothesis. It follows that $|\operatorname{cd}_q(G)| \leq 2 + (k-2)(k-1)$ and thus we have:

$$(*) \quad |\operatorname{cd}(G)| \leq |\operatorname{cd}^q(G)| + |\operatorname{cd}_q(G)| \leq (k-1) + 2 + (k-2)(k-1) = k^2 - 2k + 3.$$

Observe that, when $k = 2$ the bound $(*)$ yields $|\operatorname{cd}(G)| \leq 3$ and when $k = 3$ the bound $(*)$ yields $|\operatorname{cd}(G)| \leq 6$. Thus the first two special cases of Theorem A have been proved. Henceforth we assume that $k \geq 4$ and will improve $(*)$. We continue as before with the Frobenius factor group G/K.

If $|\operatorname{cd}_q(N)| < k-1$, then each of the, at most $k-3$, nonlinear character degrees of $\operatorname{cd}_q(N)$ has at most $k-1$ elements of $\operatorname{cd}_q(G)$ lying over it; thus $|\operatorname{cd}_q(G)| \leq 2 + (k-3)(k-1)$. This observation along with our bound on $\operatorname{cd}^q(G)$ yields $|\operatorname{cd}(G)| \leq (k-1) + 2 + (k-3)(k-1) = k^2 - 3k + 4$ and there is nothing further to prove in this case.

We may now assume that $|\operatorname{cd}_q(N)| = k-1$. In this case $\{fx | x \in \operatorname{cd}_q(N)\}$ is a subset of $\operatorname{cd}(G)$ of size $k-1$. We will show that $s(z) \subseteq \{z\} \cup \{fx | x \in \operatorname{cd}_q(N)\}$ for each $z \in \operatorname{cd}_q(N)$. Recall that an arbitrary member of $s(z)$ has the form rz, where $r \big| f$. If $rz \in s(z)$ with $r > 1$, then r divides every member of $\{rz\} \cup \{fx | x \in \operatorname{cd}_q(N)\}$. Since the latter set in this union has size $k-1$, it follows that $rz \in \{fx | x \in \operatorname{cd}_q(N)\}$ and thus we conclude that $s(z) \subseteq \{z\} \cup \{fx | x \in \operatorname{cd}_q(N)\}$ as claimed. It follows that all the members of $\operatorname{cd}_q(G)$ lie in $\operatorname{cd}_q(N) \cup \{fx | x \in \operatorname{cd}_q(N)\}$; hence $|\operatorname{cd}_q(G)| \leq 2(k-1)$. Since $|\operatorname{cd}^q(G)| \leq k-1$, we have $|\operatorname{cd}(G)| \leq 3k-3$, in this case.

For $k \geq 4$ (and $\mathbf{O}^p(G) = 1$), it follows that $|\operatorname{cd}(G)|$ is bounded by the maximum of the bounds derived in the two preceding paragraphs. That is,

$$|\operatorname{cd}(G)| \leq \max \begin{cases} (k-1) + 2 + (k-3)(k-1) = k^2 - 3k + 4, \\ (k-1) + 2(k-1) = 3k-3. \end{cases}$$

Observe that, for $k = 4$, the second formula yields a maximum of 9, giving $|\operatorname{cd}(G)| \leq 9$. In the cases $k \geq 5$, the maximum is $k^2 - 3k + 4$. Thus Theorem A is proved. $\square$

In the next section we give constructions which verify the sharpness of the bound in the exceptional cases $k = 3$ and $k = 4$.

4. Constructions

For any three primes q, r, s satisfying $q \equiv 3 \pmod 4$ and $q \equiv 1 \pmod{rs}$, we construct a group Γ as the semidirect product of a normal Sylow q-subgroup Q and a cyclic group H of order rs such that $\operatorname{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$. The group Γ satisfies P_3 and has 6 irreducible character degrees; thus providing an example for the sharpness of the bound in the case $k = 3$. Note that $r = 2$, $s = 3$, $q = 7$ satisfy the conditions.

First we construct Q. Let q be prime with $q \equiv 3 \pmod 4$. Define the group Q of exponent q as follows, where all unspecified commutators are trivial:

$$\begin{aligned} Q =&\langle x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9, x_{10} \mid \\ &[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}], \\ &[x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7]\rangle. \end{aligned}$$

A few observations about the group Q are helpful. First, for notational convenience, label $z_1 = [x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}]$ and $z_2 = [x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7]$. Notice that $\mathbf{Z}(Q) = Q' = \langle z_1, z_2\rangle$. From this we see that $|\mathbf{Z}(Q)| = q^2$ and $|Q/\mathbf{Z}(Q)| = q^{10}$. Thus the group Q is q-special of order q^{12} with exponent q. Further, using additive notation, we may view $Q/\mathbf{Z}(Q)$ and $\mathbf{Z}(Q)$ as vector spaces over $GF(q)$ with bases $\{\overline{x_1}, \overline{x_2}, \cdots, \overline{x_{10}}\}$ and $\{z_1, z_2\}$, respectively. (Here $^-$ denotes quotient mod $\mathbf{Z}(Q)$.)

What are the degrees of the irreducible characters of Q? Since Q is q-special with $|Q : \mathbf{Z}(Q)| = q^{10}$ it follows that Q has q^{10} linear characters. Notice that $Q/\langle z_1\rangle$ is isomorphic to the direct product of an extra-special group of order q^9 having exponent q with $Z_q \times Z_q$. Also $Q/\langle z_2\rangle$ is an extra-special group of order q^{11} having exponent q. These quotients give some information about $\mathrm{Irr}(Q)$ and about $\mathrm{cd}(Q)$. In particular, we have $\{1, q^4, q^5\} \subseteq \mathrm{cd}(Q)$. In fact, with the assumption $q \equiv 3 \bmod 4$, we can show that these are the only irreducible character degrees of Q. The following fact is required:

Claim. For each nonlinear character $\theta \in \mathrm{Irr}(Q)$ we have

(i) $Q/\ker(\theta)$ is an extra-special q-group with center $\mathbf{Z}(\theta)/\ker(\theta)$.

(ii) $\mathbf{Z}(\theta) \le \langle z_1, z_2, x_9, x_{10}\rangle$.

In particular, we have $\theta(1)$ is q^4 or q^5 and every automorphism that centralizes $\langle z_1, z_2, x_9, x_{10}\rangle$ fixes θ.

Fix a nonlinear character $\theta \in \mathrm{Irr}(Q)$. Since $Q' = \mathbf{Z}(Q)$ we have $\mathbf{Z}(Q) \not\leq \ker(\theta)$; also $\mathbf{Z}(Q) \cdot \ker(\theta) \le \mathbf{Z}(\theta)$. The nontrivial group $\mathbf{Z}(\theta)/\ker(\theta)$ is cyclic of exponent q; thus it has order q. Further $\mathbf{Z}(\theta) = \ker(\theta) \cdot \mathbf{Z}(Q)$. It follows that $Q/\ker(\theta)$ has a center of order q and that the factor group modulo the center is elementary q-abelian. Thus part (i) holds and from this we have $\theta(1)^2 = |Q : \mathbf{Z}(\theta)|$. Now since $|\langle z_1, z_2, x_9, x_{10}\rangle| = q^4$, the final statement of the claim will hold once part (ii) is established.

As before, let $^-$ denote quotient mod $\mathbf{Z}(Q)$. Using additive notation in each of the abelian groups $\overline{Q}$ and $\mathbf{Z}(Q)$, for an element $y \in \mathbf{Z}(\theta)$ we may write:

$$\overline{y} = c_1\overline{x_1} + c_2\overline{x_2} + \cdots + c_{10}\overline{x_{10}} \quad \text{with} \quad c_i \in GF(q).$$

For each element $z \in \mathbf{Z}(Q)$ and for each generator x_i we have $[x_i, yz] = [x_i, y]$. Using the defining relations for the group Q, it follows that:

$$[x_1, y] = c_2z_1 + c_4z_2 \quad \text{and} \quad [x_3, y] = c_4z_1 - c_2z_2.$$

Observe that for each element $y \in \mathbf{Z}(\theta)$ and $x \in Q$ we have $[y, x] \in Q' \cap \ker(\theta) = \mathbf{Z}(Q) \cap \ker(\theta)$; thus each of $[x_1, y]$ and $[x_3, y]$ lie in $\mathbf{Z}(Q) \cap K$. Viewing $\mathbf{Z}(Q) \cap K$ as a 1-dimensional subspace of $\mathbf{Z}(Q)$, the vectors $c_2z_1 + c_4z_2$ and $c_4z_1 - c_2z_2$ are dependent. If either c_2 or c_4 is nonzero, then both are nonzero and $c_4 = \alpha c_2$ and $c_2 = -\alpha c_4$ for some $\alpha \in GF(q)$; in which case $-\alpha^2 = 1$. Invoking the hypothesis

$q \equiv 3 \bmod 4$, there is no solution for $-\alpha^2 = 1$ in $GF(q)$. This forces $c_2 = c_4 = 0$. Under the same assumption, similar comparisons (e.g. $[x_2, y]$ and $[x_4, y]$) force $c_1 = c_3 = c_5 = c_6 = c_7 = c_8 = 0$. Thus we have $\mathbf{Z}(\theta) \leq \langle x_9, x_{10}, z_1, z_2 \rangle$ and the claim is proved.

Now let r and s be distinct primes dividing $q - 1$. We will define the action of a cyclic group H of order rs on the group Q. First, we define actions on Q by automorphisms a and b of orders r and s respectively. Let δ and ϵ be primitive r and s roots of unity in $GF(q)$ respectively. The actions of a and b are defined on the generators of Q, where all unspecified generators are fixed:

$$\text{for } a : x_1^a = x_1^{\delta},\ x_2^a = x_2^{\delta^{-1}},\ x_3^a = x_3^{\delta},\ x_4^a = x_4^{\delta^{-1}};$$

$$\text{for } b : x_5^b = x_5^{\epsilon},\ x_6^b = x_6^{\epsilon^{-1}},\ x_7^b = x_7^{\epsilon},\ x_8^b = x_8^{\epsilon^{-1}}.$$

We must verify that the proposed definitions interact well with the defining relations among the generators. That is, it must be shown that: if $[x_i, x_j] = [x_k, x_l]$, then $[x_i, x_j]^a = [x_k, x_l]^a$ and $[x_i, x_j]^b = [x_k, x_l]^b$. In fact, more is true. If we write $x_i^a = x_i^{\alpha_i}$ and $x_j^a = x_j^{\alpha_j}$, then $[x_i, x_j]^a = [x_i^a, x_j^a] = [x_i^{\alpha_i}, x_j^{\alpha_j}] = [x_i, x_j]^{\alpha_i \alpha_j}$; and whenever $[x_i, x_j] \neq 1$ we have $\alpha_i \alpha_j = 1$. Thus every commutator $[x_i, x_j]$ is fixed by a. The same holds for b. It follows that a and of b act as automorphisms on Q and that the subgroup $\langle z_1, z_2, x_9, x_{10} \rangle$ is centralized by both a and b. It is also clear that $ab = ba$. Since a and b are commuting automorphisms of Q of relatively prime orders, the cyclic group $H = \langle a \rangle \times \langle b \rangle$ acts on Q. Note that $|H| = rs$ and H centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$.

Now we consider the action of H on $\mathbf{Z}(Q)$ and on $Q/\mathbf{Z}(Q)$. As observed, H centralizes $\mathbf{Z}(Q)$. To understand the action of H on $Q/\mathbf{Z}(Q)$, we return to a vector space point of view. As before, let $^{-}$ denote quotients mod $\mathbf{Z}(Q)$ and use additive notation. From this perspective, the set $\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{10}}\}$ is a basis for $Q/\mathbf{Z}(Q)$ consisting of eigenvectors of a and b with corresponding eigenvalues—

$$\text{for } a : \delta,\ \delta^{-1},\ \delta,\ \delta^{-1},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1;$$

$$\text{for } b : 1,\ 1,\ 1,\ 1,\ \epsilon,\ \epsilon^{-1},\ \epsilon,\ \epsilon^{-1},\ 1,\ 1.$$

It follows that a and b act diagonally on $\overline{Q}$ and that the orbits of the action of H on $\overline{Q}$ have sizes 1, r, s, and rs.

To complete the construction, define the group: $\Gamma = Q \rtimes H$.

Now we consider $\mathrm{cd}(\Gamma)$. The coprime action of H on the abelian group $Q/\mathbf{Z}(Q)$ is permutation isomorphic to the action of H on $\mathrm{Irr}(Q/\mathbf{Z}(Q))$. The orbit sizes of the former action are $\{1, r, s, rs\}$. Since H is cyclic, it follows that $\mathrm{cd}(\Gamma/\mathbf{Z}(Q)) = \{1, r, s, rs\}$. On the other hand, we see from the claim that each nonlinear irreducible character of Q is fixed by H, since H centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$. Again, since H is cyclic and since the action of H on Q is coprime, the remaining elements of $\mathrm{cd}(\Gamma)$ are exactly the nonlinear irreducible character degrees of $\mathrm{cd}(Q)$. It follows that $\mathrm{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$.

The next construction will show that for infinitely many values of k there exist groups which satisfy property P_k and have $3(k-1)$ irreducible character degrees. It follows that the bound of Theorem A cannot be better than the linear bound $3(k-1)$.

For any two distinct primes p and q and for an appropriate n, one can let a cyclic group of order p act on an extra-special q-group of order $q^{(2n+1)}$ such that the irreducible character degrees of the resulting semidirect product are $\{1,\, p,\, q^n\}$. Let the group G be the direct product of m groups of this sort such that all of the primes involved are distinct. Then $|\operatorname{cd}(G)| = 3^m$; further, since a single prime divides no more than $3^{(m-1)}$ irreducible character degrees of G, we have $k = k(G) = 3^{(m-1)}+1$. It follows that $|\operatorname{cd}(G)| = 3(k-1)$, showing that the bound of Theorem A can be no better than $3(k-1)$.

Finally, observe that if, as in the preceding construction, we let $\Delta = A \times B$ for groups A and B with $\operatorname{cd}(A) = \{1, 2, 3\}$ and $\operatorname{cd}(B) = \{1, 5, 11\}$, then Δ satisfies P_4 and has $|\operatorname{cd}(\Delta)| = 9$. Thus Δ verifies the sharpness of the bound in the case $k = 4$ and completes our collection of examples for each of the exceptional cases of Theorem A.

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