COPRIMENESS AMONG IRREDUCIBLE CHARACTER DEGREES OF FINITE SOLVABLE GROUPS

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Abstract. Given a finite solvable group $G$, we say that $G$ has property $P_k$ if every set of $k$ distinct irreducible character degrees of $G$ is (setwise) relatively prime. Let $k(G)$ be the smallest positive integer such that $G$ satisfies property $P_k$. We derive a bound, which is quadratic in $k(G)$, for the total number of irreducible character degrees of $G$. Three exceptional cases occur; examples are constructed which verify the sharpness of the bound in each of these special cases.

1. Introduction

Suppose $G$ is a finite solvable group and let $\text{cd}(G)$ denote the set $\{\chi(1) \mid \chi \in \text{Irr}(G)\}$. We say that $G$ has property $P_k$ if every set of $k$ distinct elements of $\text{cd}(G)$ is (setwise) relatively prime. Every finite group $G$ satisfies $P_k$ at least for $k \geq |\text{cd}(G)|$, since $1 \in \text{cd}(G)$. The main result of this paper is the following:

Theorem A. Let $G$ be a nonabelian finite solvable group and let $k$ be the smallest positive integer such that $G$ satisfies property $P_k$. Then

$$|\text{cd}(G)| \leq \begin{cases} 3 & \text{if } k = 2; \\ 6 & \text{if } k = 3; \\ 9 & \text{if } k = 4; \\ k^2 - 3k + 4 & \text{if } k \geq 5. \end{cases}$$

Following the proof of Theorem A, a collection of examples is presented. In each of the exceptional cases $k = 2, 3, 4$ the bound is attained. For $k = 2$, an example is provided by the group $SL(2,3)$. This group satisfies $P_2$ and has 3 irreducible character degrees: $\text{cd}(SL(2,3)) = \{1, 2, 3\}$. For $k = 3$, we construct a group $\Gamma$ with $\text{cd}(\Gamma) = \{1, r, s, rs, q^4, q^5\}$, where $q, r, s$ are any three primes satisfying $q \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{rs}$. The group $\Gamma$ attains the bound in this case. Next, for infinitely many values of $k$, we construct a group which satisfies property $P_k$ and has $3(k - 1)$ irreducible character degrees. Observe that, for $k = 4$, such a group satisfies $P_4$ and has 9 irreducible character degrees, verifying the sharpness of the bound in this case. It also follows from this infinite set of examples that the best possible bound for $|\text{cd}(G)|$ in terms of $k$ cannot be better than the linear bound $3(k - 1)$.

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While this result belongs to a genre of problems and results concerning the irreducible character degrees of finite solvable groups (see §2 of [4]), it has a unique flavor. The investigation of property $P_k$ was inspired by problem 12.3 of [1] which is, in fact, the $k = 2$ case of the result. At this point the author would like to express her appreciation to Professor Martin Isaacs for his direction and encouragement in this work, which is a portion of her thesis.

2. Preliminaries

The purpose of this section is to restate facts about the structure and character degrees of a factor group $G/K$ of a finite nonabelian solvable group $G$ with $K$ chosen to be maximal such that $G/K$ remains nonabelian. Notice that, in this situation every proper factor group of $G/K$ is abelian and thus $(G/K)'$ is the unique minimal normal subgroup of $G/K$.

(2.1) Lemma. Let $G$ be a finite solvable group and assume that $G'$ is the unique minimal normal subgroup of $G$. Then all the nonlinear irreducible characters of $G$ have equal degree $f$ and one of the following situations obtains:

(a) $G$ is a $p$-group, $Z(G)$ is cyclic and $G/Z(G)$ is elementary abelian of order $f^2$.
(b) $G$ is a Frobenius group with a cyclic Frobenius complement of order $f$. Also, $G'$ is the Frobenius kernel and is an elementary abelian $p$-group.

Proof. This is Lemma 12.3 of [1] with the observation that an abelian Frobenius complement is necessarily cyclic.

(2.2) Theorem. Let $K \triangleleft G$ be such that $G/K$ is a Frobenius group with kernel $N/K$, an elementary abelian $p$-group. Let $\psi \in \text{Irr}(N)$. Then one of the following holds:

(a) $|G:N|\psi(1) \in \text{cd}(G)$.
(b) $p$ divides $\psi(1)$.

Proof. This is immediate from Theorem 12.4 of [1].

3. Proof of Theorem A

We begin by proving a key lemma.

(3.1) Lemma. Let $G$ be a finite nonabelian solvable group with $G' \leq O^p(G)$ for all primes $p$. Suppose that $K \triangleleft G$ and $K$ is maximal such that $G/K$ is nonabelian. Then $G/K$ is a Frobenius group with Frobenius kernel $N/K$, an elementary abelian $q$-group for some prime $q$, and a cyclic Frobenius complement. Let $f$ denote the order of the Frobenius complement and assume further that $K$ is chosen so that $f$ is minimal. Then for each linear character $\lambda$ of $N$, either $\lambda^G$ is irreducible or $\lambda$ extends to $G$. In particular, if $\chi \in \text{Irr}(G)$ lies over a linear character of $N$, then $\chi$ must have degree $1$ or $f$.

It will be handy in the proof to use the standard notation $b(G)$ to denote the largest irreducible character degree of $G$; that is, the maximum of the set $\text{cd}(G)$.

Proof. By hypothesis, if $M \triangleleft G$ with $K < M$, then the quotient $G/M$ is abelian. Since $G$ is solvable, it follows that $(G/K)'$ is the unique minimal normal subgroup of $G/K$. Now since $G$ has no nonabelian $p$-factor groups for any prime $p$, the Frobenius structure of $G/K$ follows from Lemma 2.1 (b).
Fix a linear character \( \lambda \in \text{Irr}(N) \) and let \( \chi \in \text{Irr}(G) \) lie over \( \lambda \). Set \( T = I_G(\lambda) \) and \( t = [G : T] \). Since \( T/N \) is cyclic, \( \lambda \) extends to a character \( \hat{\lambda} \in \text{Irr}(T) \) and further, by Corollary 6.17 of [1], every element of \( \text{Irr}(T) \lambda \) is an extension of \( \lambda \).

We may then assume without loss of generality that the extension \( \hat{\lambda} \) is the Clifford correspondent between \( \chi \) and \( \lambda \). Thus \( \hat{\lambda} = (\lambda)^G \) and \( \chi_N = \sum_{i=1}^{t} \lambda_i \), labeled so that \( \lambda_1 = \lambda \). Also note that \( \chi(1) = t \).

We are done if \( t = 1 \) or \( t = f \), so assume, for a contradiction, that neither happens. In this case \( 1 < t = \chi(1) < f \) and \( N < T < G \). Let \( M = \ker \lambda \). Since \( T \) fixes the linear character \( \lambda \), it follows that \( T \) centralizes \( N/M \) and so \([T, N] \leq M\). Also \([T, N] \triangleleft G \), since both \( T \) and \( N \) are normal. Let \( \equiv \) denote quotients mod \([T, N]\). Then \( \mathcal{N} \) is central in \( T \) and \( T/\mathcal{N} \) is cyclic since it is isomorphic to \( T/N \); thus \( T \) is abelian. We have \( T \) is normal and abelian in \( \mathcal{G} \). By Ito’s Theorem (6.15 of [1]) \( t = [\mathcal{G} : T] \geq b(\mathcal{G}) \). Also, since \( \ker \chi \geq \text{core}_G(\ker \lambda) \geq [T, N] \), we may view \( \chi \) as an element of \( \text{Irr}(\mathcal{G}) \). We have \( t = \chi(1) \in \text{cd}(\mathcal{G}) \) and thus \( \mathcal{G} \) is nonabelian.

Now, let \( \mathcal{G}/\mathcal{L} \) be a minimal nonabelian factor of \( \mathcal{G} \). Clearly the hypothesis on \( p \)-factors of \( G \) holds for \( p \)-factors of \( \mathcal{G} \). It follows from Lemma 2.1 that \( G/\mathcal{L} \) is a Frobenius group with a Frobenius complement of order \( \chi(1) = t < f \). Since factors of \( \mathcal{G} \) are factors of \( G \), this contradicts the minimality of \( f \). \( \square \)

It will now be convenient to establish some notation for the proof of Theorem A. For a group \( G \), we define \( k(G) \) to be the smallest integer such that \( G \) satisfies property \( P_k \). Note that \( k(G) \leq |\text{cd}(G)| \) and if \( G \) is a \( q \)-group, for \( q \) prime, then equality holds. Given a positive integer \( q \), we define \( \text{cd}_q(G) = \{n \in \text{cd}(G) \mid (q, n) = 1\} \) and \( \text{cd}^q(G) = \{n \in \text{cd}(G) \mid q|n\} \). If \( q \) is prime, then \( \text{cd}(G) \) is the disjoint union of these two sets. Also, for \( N \triangleleft G \) with \( m \in \text{cd}(N) \) if there exists \( \psi \in \text{Irr}(N) \) and \( \chi \in \text{Irr}(G|\psi) \) with \( \chi(1) = n \) and \( \psi(1) = m \), then we will say that \( n \) lies over \( m \). Further, for each such \( m \), define a subset of \( \text{cd}(G) \) by \( s(m) = \{n \in \text{cd}(G) \mid n \text{ lies over } m\} \). Note that a given \( n \) may lie over many different \( m \) and each element of \( s(m) \) is divisible by \( m \).

To prove Theorem A, we bound each of \( |\text{cd}^q(G)| \) and \( |\text{cd}_q(G)| \) separately in terms of \( k(G) \) and add the results. Note that if \( k = k(G) \) for a group \( G \), then \( |\text{cd}^q(G)| \leq k - 1 \) for any positive integer \( q \). On the other hand, given \( k \), examples are available among \( q \)-groups, where \( q \) is prime, which satisfy \( |\text{cd}(G)| = k \). For instance, let \( Q \) be the direct product of \( k - 1 \) copies of a \( q \)-group \( A \) having \( \text{cd}(A) = \{1, q\} \). Then \( \text{cd}(Q) = \{1, q, q^2, \cdots, q^{k - 1}\} \); thus \( k = k(Q) \) and \( |\text{cd}^q(Q)| = k - 1 \). It follows that \( k - 1 \) is the best possible bound for \( |\text{cd}^q(G)| \). Our challenge in proving Theorem A will be to bound \( |\text{cd}_q(G)| \).

**Proof of Theorem A.** Let \( G \) be a nonabelian finite solvable group and let \( k = k(G) \).

Suppose first that \( G \) has a nonabelian \( p \)-factor group \( G/K \) for some prime \( p \). As we have observed, \( |\text{cd}^p(G)| \leq k - 1 \). Now we consider \( |\text{cd}_p(G)| \). Fix \( \psi \in \text{Irr}(G/K) \) with \( \psi(1) = p^2 > 1 \). For each character \( \chi \in \text{Irr}(G) \) with \( (p, \chi(1)) = 1 \) we have \( \chi_K \in \text{Irr}(K) \). By Corollary 6.17 of [1], we have \( \chi \psi \in \text{Irr}(G) \) with degree \( \chi \psi(1) \) divisible by \( p \), since \( \chi \psi(1) = \chi(1)p^2 \). This gives an injection from \( \text{cd}_p(G) \) into \( \text{cd}^p(G) \). Thus \( |\text{cd}_p(G)| \leq k - 1 \) and \( |\text{cd}(G)| \leq 2(k - 1) \). In this case the conclusion of the theorem holds. Henceforth we assume that \( G' \leq O_p(G) \) for all primes \( p \).

Now fix \( K \triangleleft G \) so that \( K \) is maximal with \( G/K \) nonabelian. By Lemma 2.1, \( G/K \) is a Frobenius group with kernel \( N/K \), an elementary abelian \( q \)-group, and with a
cyclic complement $H/K$ of order $f$. Also $\text{cd}(G/K) = \{1, f\}$. Assume further that $K$ is chosen so that $f$ is minimal. As before, we have $|\text{cd}^f(G)| \leq k - 1$.

To assess $|\text{cd}_q(G)|$ we will examine how many distinct elements of $\text{cd}_q(G)$ lie over each element of $\text{cd}(N)$. If we write $\text{cd}(N) = \text{cd}^f(N) \cup \text{cd}_q(N)$, then notice that elements of $\text{cd}^f(N)$ can lie over only elements of $\text{cd}^f(N)$, since $(q, f) = 1$, and elements of $\text{cd}_q(G)$ lie over only elements of $\text{cd}_q(N)$. Also, by Theorem 2.2 (a), for each element $z \in \text{cd}_q(N)$ we must have $fz \in \text{cd}(G)$. This gives an injection from $\text{cd}_q(N)$ into $\text{cd}^f(G)$. Again, by hypothesis, $|\text{cd}^f(G)| \leq k - 1$; thus $|\text{cd}_q(N)| \leq k - 1$.

It follows that all the elements of $\text{cd}_q(G)$ lie over the, at most $k - 1$, elements of $\text{cd}_q(N)$.

If $z \in \text{cd}_q(N)$, how many elements of $\text{cd}_q(G)$ can lie over $z$? By Lemma 3.1, if $z = 1$, then $s(z) = \{1, f\}$. If $z > 1$, then $|s(z)| \leq k - 1$, since $s(z) \subseteq \text{cd}^r(G) \leq k - 1$, by hypothesis. It follows that $|\text{cd}_q(G)| \leq 2 + (k - 2)(k - 1)$ and thus we have:

$$|\text{cd}(G)| \leq |\text{cd}^f(G)| + |\text{cd}_q(G)| \leq (k - 1) + 2 + (k - 2)(k - 1) = k^2 - 2k + 3.$$  

Observe that, when $k = 2$ the bound $(*)$ yields $|\text{cd}(G)| \leq 3$ and when $k = 3$ the bound $(*)$ yields $|\text{cd}(G)| \leq 6$. Thus the first two special cases of Theorem A have been proved. Henceforth we assume that $k \geq 4$ and will improve $(*)$. We continue as before with the Frobenius factor group $G/K$.

If $|\text{cd}_q(N)| < k - 1$, then each of the, at most $k - 3$, nonlinear character degrees of $\text{cd}_q(N)$ has at most $k - 1$ elements of $\text{cd}_q(G)$ lying over it; thus $|\text{cd}_q(G)| \leq 2 + (k - 3)(k - 1)$. This observation along with our bound on $|\text{cd}^f(G)|$ yields $|\text{cd}(G)| \leq (k - 1) + 2 + (k - 3)(k - 1) = k^2 - 3k + 4$ and there is nothing further to prove in this case.

We may now assume that $|\text{cd}_q(N)| = k - 1$. In this case $\{fx | x \in \text{cd}_q(N)\}$ is a subset of $\text{cd}(G)$ of size $k - 1$. We will show that $s(z) \subseteq \{z\} \cup \{fx | x \in \text{cd}_q(N)\}$ for each $z \in \text{cd}_q(N)$. Recall that an arbitrary member of $s(z)$ has the form $rz$, where $r \mid f$. If $rz \in s(z)$ with $r > 1$, then $r$ divides every member of $\{rz\} \cup \{fx | x \in \text{cd}_q(N)\}$. Since the latter set in this union has size $k - 1$, it follows that $rz \in \{fx | x \in \text{cd}_q(N)\}$ and thus we conclude that $s(z) \subseteq \{z\} \cup \{fx | x \in \text{cd}_q(N)\}$ as claimed. It follows that all the members of $\text{cd}_q(G)$ lie in $\text{cd}_q(N) \cup \{fx | x \in \text{cd}_q(N)\}$; hence $|\text{cd}_q(G)| \leq 2(k - 1)$. Since $|\text{cd}^f(G)| \leq k - 1$, we have $|\text{cd}(G)| \leq 3k - 3$, in this case.

For $k \geq 4$ (and $\text{O}^p(G) = 1$), it follows that $|\text{cd}(G)|$ is bounded by the maximum of the bounds derived in the two preceding paragraphs. That is,

$$|\text{cd}(G)| \leq \max \left\{ \begin{array}{l}
(k - 1) + 2 + \frac{(k - 3)(k - 1)}{2} = k^2 - 3k + 4,
(k - 1) + 2(k - 1) = 3k - 3.
\end{array} \right.$$

Observe that, for $k = 4$, the second formula yields a maximum of 9, giving $|\text{cd}(G)| \leq 9$. In the cases $k \geq 5$, the maximum is $k^2 - 3k + 4$. Thus Theorem A is proved. \(\Box\)

In the next section we give constructions which verify the sharpness of the bound in the exceptional cases $k = 3$ and $k = 4$.

4. CONSTRUCTIONS

For any three primes $q, r, s$ satisfying $q \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{rs}$, we construct a group $\Gamma$ as the semidirect product of a normal Sylow $q$-subgroup $Q$ and a cyclic group $H$ of order $rs$ such that $\text{cd}(\Gamma) = \{1, r, s, rs, q^3, q^5\}$. The group $\Gamma$ satisfies $P_3$ and has 6 irreducible character degrees; thus providing an example for
the sharpness of the bound in the case \( k = 3 \). Note that \( r = 2, s = 3, q = 7 \) satisfy the conditions.

First we construct \( Q \). Let \( q \) be prime with \( q \equiv 3 \pmod{4} \). Define the group \( Q \) of exponent \( q \) as follows, where all unspecified commutators are trivial:

\[
Q = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \mid \\
[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}], \\
[x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7] \rangle.
\]

A few observations about the group \( Q \) are helpful. First, for notational convenience, label \( z_1 = [x_1, x_2] = [x_3, x_4] = [x_5, x_6] = [x_7, x_8] = [x_9, x_{10}] \) and \( z_2 = [x_1, x_4] = [x_2, x_3] = [x_5, x_8] = [x_6, x_7] \). Notice that \( Z(Q) = Q' = \langle z_1, z_2 \rangle \). From this we see that \( |Z(Q)| = q^2 \) and \( |Q/Z(Q)| = q^{10} \). Thus the group \( Q \) is \( q \)-special of order \( q^{12} \) with exponent \( q \). Further, using additive notation, we may view \( Q/Z(Q) \) and \( Z(Q) \) as vector spaces over \( GF(q) \) with bases \( \{x_1, x_2, \ldots, x_{10} \} \) and \( \{z_1, z_2 \} \), respectively. (Here \( - \) denotes quotient mod \( Z(Q) \).)

What are the degrees of the irreducible characters of \( Q \)? Since \( Q \) is \( q \)-special with \( |Q : Z(Q)| = q^{10} \) it follows that \( Q \) has \( q^{10} \) linear characters. Notice that \( Q/(z_1) \) is isomorphic to the direct product of an extra-special group of order \( q^9 \) having exponent \( q \) with \( Z_9 \times Z_q \). Also \( Q/(z_2) \) is an extra-special group of order \( q^{11} \) having exponent \( q \). These quotients give some information about \( \text{Irr}(Q) \) and about \( \text{cd}(Q) \). In particular, we have \( \{1, q^4, q^5 \} \subseteq \text{cd}(Q) \). In fact, with the assumption \( q \equiv 3 \pmod{4} \), we can show that these are the only irreducible character degrees of \( Q \). The following fact is required:

Claim. For each nonlinear character \( \theta \in \text{Irr}(Q) \) we have

(i) \( Q/\ker(\theta) \) is an extra-special \( q \)-group with center \( Z(\theta)/\ker(\theta) \).

(ii) \( Z(\theta) \leq \langle z_1, z_2, x_9, x_{10} \rangle \).

In particular, we have \( \theta(1) \) is \( q^4 \) or \( q^5 \) and every automorphism that centralizes \( \langle z_1, z_2, x_9, x_{10} \rangle \) fixes \( \theta \).

Fix a nonlinear character \( \theta \in \text{Irr}(Q) \). Since \( Q' = Z(Q) \) we have \( Z(Q) \nparallel \ker(\theta) \); also \( Z(Q) \cdot \ker(\theta) \leq Z(\theta) \). The nontrivial group \( Z(\theta)/\ker(\theta) \) is cyclic of exponent \( q \); thus it has order \( q \). Further \( Z(\theta) = \ker(\theta) \cdot Z(Q) \). It follows that \( Q/\ker(\theta) \) has a center of order \( q \) and that the factor group modulo the center is elementary \( q \)-abelian. Thus part (i) holds and from this we have \( \theta(1)^2 = |Q : Z(\theta)| \). Now since \( |\langle z_1, z_2, x_9, x_{10} \rangle| = q^4 \), the final statement of the claim will hold once part (ii) is established.

As before, let \( - \) denote quotient mod \( Z(Q) \). Using additive notation in each of the abelian groups \( Q \) and \( Z(Q) \), for an element \( y \in Z(\theta) \) we may write:

\[
y = c_1x_1 + c_2x_2 + \cdots + c_{10}x_{10} \quad \text{with} \quad c_i \in GF(q).
\]

For each element \( z \in Z(Q) \) and for each generator \( x_i \) we have \( [x_i, yz] = [x_i, y] \).

Using the defining relations for the group \( Q \), it follows that:

\[
[x_1, y] = c_2z_1 + c_4z_2 \quad \text{and} \quad [x_3, y] = c_4z_1 - c_2z_2.
\]

Observe that for each element \( y \in Z(\theta) \) and \( x \in Q \) we have \( [y, x] \in Q' \cap \ker(\theta) = Z(Q) \cap \ker(\theta) \); thus each of \( [x_1, y] \) and \( [x_3, y] \) lie in \( Z(Q) \cap K \). Viewing \( Z(Q) \cap K \) as a 1-dimensional subspace of \( Z(Q) \), the vectors \( c_2z_1 + c_4z_2 \) and \( c_4z_1 - c_2z_2 \) are dependent. If either \( c_2 \) or \( c_4 \) is nonzero, then both are nonzero and \( c_4 = \alpha c_2 \) and \( c_2 = -\alpha c_4 \) for some \( \alpha \in GF(q) \); in which case \( -\alpha^2 = 1 \). Invoking the hypothesis
$q \equiv 3 \mod 4$, there is no solution for $-\alpha^2 = 1$ in $GF(q)$. This forces $c_2 = c_4 = 0$. Under the same assumption, similar comparisons (e.g. $[x_2, y]$ and $[x_4, y]$) force $c_1 = c_3 = c_5 = c_6 = c_7 = c_8 = 0$. Thus we have $Z(\theta) \leq \langle x_9, x_{10}, z_1, z_2 \rangle$ and the claim is proved.

Now let $r$ and $s$ be distinct primes dividing $q - 1$. We will define the action of a cyclic group $H$ of order $rs$ on the group $Q$. First, we define actions on $Q$ by automorphisms $a$ and $b$ of orders $r$ and $s$ respectively. Let $\delta$ and $\epsilon$ be primitive $r$ and $s$ roots of unity in $GF(q)$ respectively. The actions of $a$ and $b$ are defined on the generators of $Q$, where all unspecified generators are fixed:

$$\text{for } a: x_1^a = x_1^\delta, \ x_2^a = x_2^{\delta^{-1}}, \ x_3^a = x_3^\epsilon, \ x_4^a = x_4^{\epsilon^{-1}};$$

$$\text{for } b: x_5^b = x_5^\epsilon, \ x_6^b = x_6^{\epsilon^{-1}}, \ x_7^b = x_7^\delta, \ x_8^b = x_8^{\delta^{-1}}.$$

We must verify that the proposed definitions interact well with the defining relations among the generators. That is, it must be shown that: if $[x_i, x_j] = [x_k, x_l]$, then $[x_i, x_j]^a = [x_k, x_l]^a$ and $[x_i, x_j]^b = [x_k, x_l]^b$. In fact, more is true. If we write $x_i^a = x_i^{\alpha_i}$ and $x_j^b = x_j^{\beta_j}$, then $[x_i, x_j]^a = [x_i^{\alpha_i}, x_j^{\alpha_j}] = [x_i^{\alpha_i}, x_j^{\alpha_j}]^{\alpha_i \alpha_j}$; and whenever $[x_i, x_j] \neq 1$ we have $\alpha_i \alpha_j = 1$. Thus every commutator $[x_i, x_j]$ is fixed by $a$. The same holds for $b$. It follows that $a$ and $b$ act as automorphisms on $Q$ and that the subgroup $\langle z_1, z_2, x_9, x_{10} \rangle$ is centralized by both $a$ and $b$. It is also clear that $ab = ba$. Since $a$ and $b$ are commuting automorphisms of $Q$ of relatively prime orders, the cyclic group $H = \langle a \rangle \times \langle b \rangle$ acts on $Q$. Note that $|H| = rs$ and $H$ centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$.

Now we consider the action of $H$ on $Z(Q)$ and on $Q/Z(Q)$. As observed, $H$ centralizes $Z(Q)$. To understand the action of $H$ on $Q/Z(Q)$, we return to a vector space point of view. As before, let $\overline{\cdot}$ denote quotients mod $Z(Q)$ and use additive notation. From this perspective, the set $\{x_1, x_2, \ldots, x_{10}\}$ is a basis for $Q/Z(Q)$ consisting of eigenvectors of $a$ and $b$ with corresponding eigenvalues—

$$\text{for } a: \delta, \delta^{-1}, \delta, \delta^{-1}, 1, 1, 1, 1, 1;$$

$$\text{for } b: 1, 1, 1, 1, 1, \epsilon, \epsilon^{-1}, \epsilon, \epsilon^{-1}, 1, 1.$$

It follows that $a$ and $b$ act diagonally on $\overline{Q}$ and that the orbits of the action of $H$ on $\overline{Q}$ have sizes 1, $r$, $s$, and $rs$.

To complete the construction, define the group: $\Gamma = Q \rtimes H$.

Now we consider $cd(\Gamma)$. The coprime action of $H$ on the abelian group $Q/Z(Q)$ is permutation isomorphic to the action of $H$ on Irr($Q/Z(Q)$). The orbit sizes of the former action are $\{1, r, s, rs\}$. Since $H$ is cyclic, it follows that $cd(\Gamma/Z(Q)) = \{1, r, s, rs\}$. On the other hand, we see from the claim that each nonlinear irreducible character of $Q$ is fixed by $H$, since $H$ centralizes $\langle z_1, z_2, x_9, x_{10} \rangle$. Again, since $H$ is cyclic and since the action of $H$ on $Q$ is coprime, the remaining elements of $cd(\Gamma)$ are exactly the nonlinear irreducible character degrees of $cd(Q)$. It follows that $cd(\Gamma) = \{1, r, s, rs, q^k, q^{k-1}\}$.

The next construction will show that for infinitely many values of $k$ there exist groups which satisfy property $P_k$ and have $3(k - 1)$ irreducible character degrees. It follows that the bound of Theorem A cannot be better than the linear bound $3(k - 1)$.

For any two distinct primes $p$ and $q$ and for an appropriate $n$, one can let a cyclic group of order $p$ act on an extra-special $q$-group of order $q^{2n+1}$ such that the
irreducible character degrees of the resulting semidirect product are \( \{1, p, q^n\} \). Let the group \( G \) be the direct product of \( m \) groups of this sort such that all of the primes involved are distinct. Then \( |\text{cd}(G)| = 3^m \); further, since a single prime divides no more than \( 3^{(m-1)} \) irreducible character degrees of \( G \), we have \( k = k(G) = 3^{(m-1)}+1 \). It follows that \( |\text{cd}(G)| = 3(k-1) \), showing that the bound of Theorem A can be no better than \( 3(k-1) \).

Finally, observe that if, as in the preceding construction, we let \( \Delta = A \times B \) for groups \( A \) and \( B \) with \( \text{cd}(A) = \{1, 2, 3\} \) and \( \text{cd}(B) = \{1, 5, 11\} \), then \( \Delta \) satisfies \( P_4 \) and has \( |\text{cd}(\Delta)| = 9 \). Thus \( \Delta \) verifies the sharpness of the bound in the case \( k = 4 \) and completes our collection of examples for each of the exceptional cases of Theorem A.

References


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