THE SCHWARZ-PICK LEMMA FOR DERIVATIVES

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Abstract. The Schwarz-Pick Lemma states that any analytic function of the unit disc into itself is a contraction with respect to the hyperbolic metric. In this note a related result is proved for the derivative of an analytic function.

1. Introduction

In [2] Dieudonné showed that if \( f : \mathbb{D} \to \mathbb{D} \) is analytic, where \( \mathbb{D} \) is the open unit disc in the complex plane, and if \( f(0) = 0 \), then

\[
|f'(z)| \leq \begin{cases} 
1 & \text{if } |z| \leq \sqrt{2} - 1, \\
\frac{1 + |z|^2}{4|z|(1 - |z|^2)} & \text{if } |z| \geq \sqrt{2} - 1.
\end{cases}
\]

He then suggested (in the footnote on page 352) that this inequality, which is best possible for each value of \(|z|\), should be considered as a Schwarz Lemma for the derivative \( f' \). In this note we derive a more elegant version of this that is closer to the classical Schwarz-Pick Lemma.

As \( f \) maps \( \mathbb{D} \) into itself one can argue that any discussion of \( f \) should be in terms of the hyperbolic metric rather than the Euclidean metric. The disk \( \mathbb{D} \) is endowed with the hyperbolic metric \( \rho \) derived from \( 2|dz|/(1 - |z|^2) \), and the hyperbolic derivative \( f^*(z) \) of \( f \) at \( z \) is given by

\[
f^*(z) = \left( \frac{1 - |z|^2}{1 - |f(z)|^2} \right) f'(z).
\]

Pick’s version of Schwarz’s Lemma, namely that \( \rho(fz, fw) \leq \rho(z, w) \), guarantees that \( |f^*(z)| < 1 \) in \( \mathbb{D} \) (unless \( f \) is a Möbius map of \( \mathbb{D} \) onto itself) and this means that we can measure the hyperbolic distance between two hyperbolic derivatives. This leads to the following Schwarz-Pick Lemma for derivatives.

Theorem. If \( f : \mathbb{D} \to \mathbb{D} \) is analytic but not a conformal automorphism of \( \mathbb{D} \), and if \( f(0) = 0 \), then

\[
\rho(f^*(0), f^*(z)) \leq 2\rho(0, z).
\]

Further, equality holds in (2) for each \( z \) when \( f(z) = z^2 \).
If \( f(z) = z^2 \), then \( f^*(0) = 0 \) and \( f^*(z) = 2z/(1 + |z|^2) \), and a simple calculation using the fact that
\[
\rho(0, z) = \log \frac{1 + |z|}{1 - |z|}
\]
shows that equality holds in (2).

2. The proof of the Theorem

We begin with a preliminary result.

\textbf{Lemma 1.} Let \( z_0 \) and \( w_0 \) be points of \( D \) with \( |w_0| < |z_0| \). If \( f : D \to D \) is analytic, and if \( f(0) = 0 \) and \( f(z_0) = w_0 \), then both \( f^*(0) \) and \( f^*(z_0) \) lie in the closed hyperbolic disc
\[
D = \{ z : \rho(z, w_0/z_0) \leq \rho(0, z_0) \}.
\]

\textit{Proof.} Given \( z_0 \) and \( w_0 \), we define maps \( h : D \to D \) and \( g : D \to D \) by
\[
h(z) = \frac{f(z)}{z}, \quad \frac{f(z) - f(z_0)}{1 - f(z)f(z_0)} = g(z) \left( \frac{z - z_0}{1 - \overline{z}z_0} \right).
\]
It is immediate that
\[
h(0) = f'(0) = f^*(0), \quad h(z_0) = \frac{w_0}{z_0}, \quad g(0) = \frac{w_0}{z_0}, \quad g(z_0) = f^*(z_0),
\]
so that, by applying Pick’s Lemma to \( h \) and \( g \),
\[
\rho(f^*(0), w_0/z_0) \leq \rho(0, z_0), \quad \rho(f^*(z_0), w_0/z_0) \leq \rho(0, z_0).
\]
This shows that \( f^*(0) \) and \( f^*(z) \) lie in the closed disc \( D \).

The inequality (2) (with \( z \) replaced by \( z_0 \)) is an immediate consequence of (4) and the Triangle inequality, and this completes the proof of the Theorem. For a Euclidean version of Lemma 1, see [1], p. 19 and [3], p. 198.

We end with the remark that if equality holds in (2), then \( f^*(0) \) and \( f^*(z) \) must lie at diametrically opposite points on the boundary of \( D \), so that equality holds in both of the inequalities in (4). We deduce that \( h \) is a Möbius map of \( D \) onto itself and hence \( f(z) = zh(z) \), a Blaschke product of degree two.

\textbf{References}