

AFFINE SEMIGROUP RINGS THAT ARE COMPLETE INTERSECTIONS

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ABSTRACT. This paper presents a result concerning the structure of affine semigroup rings that are complete intersections. It generalizes to arbitrary dimensions earlier results for semigroups of dimension less than four. The proof depends on a decomposition theorem for mixed dominating matrices.

1. INTRODUCTION

Let $T = \{s_1, s_2, \dots, s_n\}$ be a set of nonzero vectors in \mathbf{Z}^k , and let S be the semigroup generated by T . If S contains no invertible elements, and if the group generated by T has rank d , then S is called an *affine* semigroup of dimension d . One may associate to S the space of relations W over the rationals \mathbf{Q} . If $\{u_1, u_2, \dots, u_r\}$ is a set of integral vectors that forms a basis for W over the rationals, we will call this set an integral basis for W . If $\{u_1, u_2, \dots, u_r\}$ is such a basis, then $\text{span}_{\mathbf{Z}}\{u_1, u_2, \dots, u_r\} = \mathbf{Z}^n \cap W$ if and only if the associated matrix M whose rows are the vectors u_i , $i = 1, \dots, r$, has content 1. By the *content* of M we mean the greatest common divisor of all $r \times r$ minors of M . Also note that M is an $r \times n$ matrix where $n - r$ equals the dimension of S . Since we will assume that S is an affine semigroup, every nonzero vector u in W must contain both a positive and a negative entry. We will call a matrix M *mixed* if every row of M contains a positive and a negative entry. We will call M *dominating* if it contains no square mixed submatrix (see [FS2], Proposition 2.6, where the name is motivated).

The semigroup ring $\mathbf{Z}[S]$ is the polynomial ring $\mathbf{Z}[X_1, X_2, \dots, X_n]$ modulo the ideal of relations $\{X^{u^+} - X^{u^-}\}$ where u is any integral vector in W and u^+ and u^- are respectively the positive and negative parts of u . This ideal is prime and has height $\dim_{\mathbf{Q}} W = r$ (see [FS1]). Hence, if this ideal is generated by r elements, then the semigroup S is called a complete intersection. We prove in this paper that an affine semigroup S that is not a free abelian semigroup is a complete intersection if and only if the minimal generating set T may be partitioned into two smaller sets T_1 and T_2 , so that the semigroups generated by these sets, denoted $\langle T_1 \rangle$ and $\langle T_2 \rangle$ respectively, are complete intersections and there exists $\alpha \in \langle T_1 \rangle \cap \langle T_2 \rangle$ such that $\alpha \neq 0$ and $G(\alpha) = G(T_1) \cap G(T_2)$, where $G(X)$ denotes the group generated by X .

This statement was proved for semigroups of dimension one by C. Delorme [D] and by [FS2], and recently by J.C. Rosales and P.A. García-Sánchez [RG-S] for semigroups with dimension less than 4. The technique used in [FS2] is to show

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that S is a complete intersection if and only if there exists an integral basis for the space of relations of S so that the associated matrix M is mixed dominating and has content 1. The above statement then follows from a decomposition theorem in [FS2] for $r \times (r+1)$ mixed dominating matrices. We show in section 2 that a mixed dominating matrix of arbitrary size decomposes (Theorem 2.2). The statement for arbitrary affine semigroups then follows in a straightforward manner (Theorem 3.1). We make explicit that the concept of a “gluing” of semigroups, defined in [RG-S], is the same as the decomposition of a mixed dominating matrix. Furthermore, a result (for dimension less than four) of [RG-S], concerning the number of extreme rays of the rational cone of a complete intersection S , generalizes to any dimension as a result of the decomposition theorem.

The properties of a mixed dominating matrix, whose applications to semigroups form the basis of this paper, will be explored more extensively in a forthcoming paper. Such matrices are a subclass of L -matrices as discussed by Brualdi [BCS] and properly generalize the S -matrices as discussed in Klee [K].

2. A DECOMPOSITION OF MIXED DOMINATING MATRICES

Our goal in this section is to prove a decomposition theorem for mixed dominating matrices. It is important to note that the definition of a dominating $r \times n$ matrix $M = (m_{ij})$ is independent of the magnitude of the entries. Hence our concern will only be whether such an entry is positive (+) or negative (-). To a mixed matrix M we may associate a multigraph G with a vertex for every column of M and an edge of color s between vertices i and j if $m_{si}m_{sj} < 0$. It is clear that for each row s of M , the subgraph of G induced by the edges of color s is a complete bipartite graph. If M is not dominating, i.e., if M contains a square mixed submatrix, then it was shown in [FS2] that M (up to multiplication of rows by ± 1 and permutation of the rows and columns) necessarily contains a square $k \times k$ mixed submatrix of the form:

$$\begin{pmatrix} + & & & & - \\ - & + & & & \\ & & - & \ddots & \\ & & & - & + \\ & & & & - & + \end{pmatrix}$$

where all other entries are zero. Therefore, it is clear that M is mixed dominating if and only if the associated graph G has no circuits with edges of distinct colors. We say that such a circuit, and more generally a path of this sort, is multicolored.

If M is a mixed matrix, then the edge set induced by each color class is a complete bipartite graph; and if M contains no 2×2 mixed submatrices, then these color classes partition the edge set of G . In this regard, a theorem by Graham and Pollak [GP] (see also [BR] for a discussion and proof of this theorem) says that if G is complete, then $r \geq n - 1$. This result is actually stronger than what is needed. We have a combinatorial proof which will appear in a later paper of the fact that if M is mixed dominating and if G is complete, then $r \geq n - 1$. For the sake of brevity here, however, we will appeal to the Graham, Pollak result. This conclusion is used in the following “extension” theorem whose proof is deferred until after the proof of the decomposition theorem.

Theorem 2.1. *Suppose that M is an $r \times (r + d)$ mixed dominating matrix where $d > 1$. Then there exists a mixed $1 \times (r + d)$ row so that the $(r + 1) \times (r + d)$ matrix M^* obtained by appending this row to M is also mixed dominating.*

Using this it is not difficult to prove the decomposition theorem. In what follows we will define the “empty” $(0 \times d)$ matrix to be mixed dominating.

Theorem 2.2. *Let M be a mixed dominating $r \times (r + d)$ matrix with $r > 0$. Then there is a rearrangement of the rows and columns of M so that the resulting matrix has the form*

$$(1) \quad \left(\begin{array}{c|c} A & O \\ \hline O & B \\ \hline a & b \end{array} \right)$$

where A and B are mixed dominating matrices of sizes $t \times (t + d_1)$ and $s \times (s + d_2)$ respectively with $t \geq 0, s \geq 0$ and $s + t + 1 = r$ and $d_1 + d_2 - 1 = d$. Additionally, a and b are $1 \times (t + d_1)$ and $1 \times (s + d_2)$ nonzero, nonmixed matrices respectively of opposite sign.

Proof. Clearly it does no harm to assume that no column of M consists entirely of zeros. The proof of the theorem will be by induction on d , and we note that the case $d = 1$ has been shown in [FS2]. Assuming that $d > 1$, we may by Theorem 2.1 append a row to M so that the resulting $(r + 1) \times (r + d)$ matrix M^* is mixed dominating. The inductive assumption assures that M^* , upon permuting the rows and columns and multiplying rows by ± 1 , has the form (1). If the appended row is not the last row of this decomposition, then deleting it leaves M in the required form. If the appended row is the last row, then deletion of this row leaves M in the form of (1) but without the last row. Both A and B must have at least one row, for otherwise M would have a column of zeros. Since $d_1 + d_2 = d + 1$ and $d \geq 2$, then at least one of d_1 and d_2 , say d_2 , is less than d , so we again apply the inductive hypothesis and assume that B has the required form. Using the last row of B as the last row of M , it is clear that M is in the required form. \square

Lemma 2.3. *Suppose that a dominating matrix M has rows s and t and columns x, y , and z for which $m_{sx}m_{sy} < 0, m_{ty}m_{tz} < 0$, and $m_{sz} = m_{tx} = 0$. Then it is possible to change M , by making m_{sz} or m_{tx} nonzero, so that the resulting matrix is dominating.*

Proof. In order to make m_{sz} nonzero, we must do this so that $m_{sz}m_{sx} < 0$, for otherwise the condition $m_{sz}m_{sy} < 0$ would produce a 2×2 mixed submatrix in rows s and t and columns y and z . Similarly, in order to make m_{tx} nonzero, we must have $m_{tx}m_{tz} < 0$. If it were not possible to change m_{sz} so that $m_{sz}m_{sx} < 0$, this must be due to a pair (R, C) , where $R = (r_1, r_2, \dots, r_{k-1})$ is a sequence of distinct rows of M other than s , and $C = (c_1, c_2, \dots, c_k)$ is a sequence of distinct columns of M , such that $m_{r_i c_i} m_{r_i c_{i+1}} < 0$ for $i = 1, \dots, k - 1, m_{sx} m_{sc_1} > 0$, and $c_k = z$. To such a pair (R, C) we could append s to the end of R and get a multicolored circuit in the graph of the changed matrix. Clearly, any multicolored circuit in the graph of the changed matrix that was not in the graph of the original matrix can be obtained this way. To prevent the change of m_{tx} so that $m_{tz}m_{tx} < 0$, there must similarly be a pair (R', C') with $R' = (r'_1, r'_2, \dots, r'_{l-1})$ a sequence of distinct

rows of M other than t , and $C' = (c'_1, c'_2, \dots, c'_l)$ a sequence of distinct columns of M , so that $m_{r'_i c'_i} m_{r'_i c'_{i+1}} < 0$ for $i = 1, \dots, l - 1$, $m_{tz} m_{tc'_1} > 0$, and $c'_l = x$.

Suppose now that we choose R, C, R', C' as above so that k and l are minimal. If t does not appear in the sequence R , then the pair

$$((r_1, r_2, \dots, r_{k-1}, t, s), (c_1, c_2, \dots, c_k, y))$$

yields a multicolored circuit, contradicting the assumption that M is dominating. Suppose that $t = r_f$, where $f < k - 1$. If $m_{tc_f} m_{tz} > 0$, then $((r_{f+1}, \dots, r_{k-1}, t), (c_{f+1}, \dots, c_k))$ gives a multicolored circuit, again contradicting the assumption that M is dominating. If $m_{tc_f} m_{tz} < 0$, then the pair $((r_1, \dots, r_{f-1}, t), (c_1, \dots, c_f, z))$ contradicts the minimality of k . Thus $r_{k-1} = t$. A similar argument shows that $r'_{l-1} = s$.

If R and R' have no rows in common, then $((r_1, \dots, r_{k-2}, t, r'_1, \dots, r'_{l-2}, s), (c_1, \dots, c_{k-1}, c'_1, \dots, c'_{l-1}))$ produces a multicolored cycle, which clearly contains a multicolored circuit. Let $p = \min\{i : r_i \in R'\}$, and suppose that $r_p = r'_q$. There are two cases. If $m_{r_p c_p} m_{r'_q c'_q} > 0$, then the pair

$$((r_1, \dots, r_p, r'_{q+1}, \dots, r'_{l-1}), (c_1, \dots, c_p, c'_{q+1}, \dots, c'_{l-1}))$$

yields a multicolored cycle. If $m_{r_p c_p} m_{r'_q c'_q} < 0$, then the pair (R, C) with

$$R = (r_1, \dots, r_p, r'_{q-1}, r'_{q-2}, \dots, r'_1, t, s)$$

and

$$C = (c_1, \dots, c_p, c'_q, c'_{q-1}, \dots, c'_1, y)$$

gives a multicolored cycle. The minimality of k and l assures that the rows of the last cycle are distinct, because $t = r_{k-1} \notin (r_1, \dots, r_p)$ and $s = r'_{l-1} \notin (r'_{q-1}, \dots, r'_1)$. Thus the lemma is proved. \square

By successive use of the above lemma, we may complete a mixed dominating matrix M to one in which every 2×3 mixed submatrix has at least five nonzero entries. We will call such a mixed dominating matrix *dense*.

Proof of Theorem 2.1. Suppose that M is an $r \times (r + d)$ mixed dominating matrix with $d > 1$. It is harmless to assume that M is dense, for if a dense completion of M can be extended, then so can M .

We say that columns k and l are connected by a multicolored path if the vertices in the graph of M corresponding to k and l are connected by such a path. Since M is dense, one can easily see that any two columns connected by a multicolored path are connected by a path of length one. Hence, if every pair of columns is connected by a multicolored path, then the associated graph of M is complete. The theorem of Graham and Pollak quoted earlier then says that $d = 1$, contradicting the hypothesis of the theorem.

Let k and l be a pair of columns that is not connected by a multicolored path. Consider any $1 \times (r + d)$ row s such that $s_k s_l < 0$ and all other entries of s are zero. It is now clear that the graph of the $(r + 1) \times (r + d)$ matrix M^* which is formed by appending s to M cannot contain a multicolored circuit and is, therefore, mixed dominating. \square

3. AFFINE SEMIGROUPS

Our goal in this section is to prove a characterization of affine semigroups that are complete intersections.

Theorem 3.1. *Let S be an affine semigroup that is not a free abelian semigroup. Then S is a complete intersection if and only if there exists a partition of the minimal generating set T of S into nonempty subsets T_1 and T_2 such that:*

1. *The semigroups $\langle T_1 \rangle$ and $\langle T_2 \rangle$ are complete intersections and*
2. *There exists $\alpha \in \langle T_1 \rangle \cap \langle T_2 \rangle$ such that $\alpha \neq 0$ and $G(\alpha) = G(T_1) \cap G(T_2)$.*

We note that the partition of the minimal generating set T into T_1 and T_2 in Theorem 3.1 is what is called a gluing of T_1 and T_2 in [RG-S, Theorem 0.1]. Before proving the theorem we first compute the content of a mixed dominating matrix M in terms of the content of submatrices A and B when M is in the form of Theorem 2.2. For any matrix M we write M_i to denote the matrix obtained from M by deleting the i th column. Denoting the content of the matrix M by $\text{cont}(M)$, it follows from the definition that $\text{cont}(M) = \text{gcd}(\text{cont}(M_i) : i = 1, 2, \dots, n)$. With A and B as in Theorem 2.2 we will use \overline{A} to denote the matrix obtained from A by adjoining a as the last row. Similarly \overline{B} is defined with b adjoined. We allow A to be a $0 \times t$ matrix, in which case we define $\text{cont}(A)$ to be 1, and similarly for B . If a matrix has more rows than columns, its content is defined to be zero.

Lemma 3.2. *Let M be an $r \times (r + d)$ matrix in the form of Theorem 2.2. Then*

$$\text{cont}(M) = \text{gcd}(\text{cont}(A)\text{cont}(\overline{B}), \text{cont}(\overline{A})\text{cont}(B)).$$

Proof. We induct on $n = r + d$, starting with $n = 2$. In this case the result is immediate.

We assume that A is $t \times k$.

By the inductive hypothesis, for $i = 1, 2, \dots, k$,

$$\text{cont}(M_i) = \text{gcd}(\text{cont}(A_i)\text{cont}(\overline{B}), \text{cont}(\overline{A_i})\text{cont}(B)),$$

while for $i = k + 1, \dots, n$

$$\text{cont}(M_i) = \text{gcd}(\text{cont}(A)\text{cont}(\overline{B_{i-k}}), \text{cont}(\overline{A})\text{cont}(B_{i-k})).$$

Repeated use of the fact that for any $l \times m$ matrix X , $\text{cont}(X) = \text{gcd}(\text{cont}(X_i) : i = 1, \dots, m)$ applied to M , A , B , \overline{A} and \overline{B} leads to the desired equality. \square

Corollary 3.3. *Let M be a matrix that decomposes as in Theorem 2.2. Then M has content 1 if and only if A and B have content 1 and $\text{cont}(\overline{A})$ and $\text{cont}(\overline{B})$ are relatively prime.*

Proof. If $\text{cont}(A) = \text{cont}(B) = 1$ and $\text{cont}(\overline{A})$ and $\text{cont}(\overline{B})$ are relatively prime, then certainly the lemma proves that $\text{cont}(M) = 1$.

The converse follows from the observation that $\text{cont}(A)$ divides $\text{cont}(\overline{A})$. This can be seen by computing the determinant of each of the relevant submatrices of \overline{A} by expansion along the last row. \square

Proof of Theorem 3.1. First suppose that S is a complete intersection with minimal generating set $T = \{s_1, \dots, s_n\}$. By [FS2, Corollary 2.10] there exists an $r \times n$ mixed

dominating matrix M with content 1 whose rows are a basis of the relation space of S . We will assume that M has the form of Theorem 2.2 and that A is $m \times k$ and B is $t \times p$ where $n = k + p$. The matrix A is a mixed dominating matrix and by Corollary 3.3 has content 1. If $T_1 = \{s_1, \dots, s_k\}$, then the rows of A form a basis of the space of relations of the semigroup $\langle T_1 \rangle$, and so again by [FS2, Corollary 2.10], this semigroup is a complete intersection. A similar argument using B shows that $\langle T_2 \rangle$ is a complete intersection, where $T_2 = \{s_{k+1}, \dots, s_n\}$.

By examination of the last row of M it follows that

$$a_1 s_1 + \dots + a_k s_k = -b_1 s_{k+1} - \dots - b_p s_n.$$

Without loss of generality we may assume that $a_i \geq 0$ and $b_j \leq 0$. Hence, if we denote the above element by α , it follows that $\alpha \in \langle T_1 \rangle \cap \langle T_2 \rangle$. We will show that $G(T_1) \cap G(T_2) \subset G(\alpha)$, since the reverse containment is clear.

Let $0 \neq \beta \in G(T_1) \cap G(T_2)$. Then we can write

$$\beta = f_1 s_1 + \dots + f_k s_k = g_1 s_{k+1} + \dots + g_p s_n$$

where f_i, g_j are integers. Hence, $w = (f_1, \dots, f_k, -g_1, \dots, -g_p)$ is a relation on S . Therefore, we can write w as an integer combination of the rows of M , say

$$w = \sum_{i=1}^r n_i u_i.$$

Viewing S as a subsemigroup of \mathbf{Z}^d we let V be the $n \times d$ matrix where the i th row is s_i , for $i = 1, \dots, k$, while for $i = k+1, \dots, n$ the i th row is the zero vector. We claim that $\beta = n_r u_r V$. To see this, first observe that $\beta = wV$ and that $n_i u_i V$ is the zero vector for $i = 1, \dots, r-1$ since each of these u_i 's is a relation vector on $\{s_1, \dots, s_k\}$. Therefore, $\beta = n_r (a_1 s_1 + \dots + a_k s_k) \in G(\alpha)$.

For the converse of the theorem, suppose that there exists a partition of the generating set into subsets $T_1 = \{s_1, \dots, s_k\}$ and $T_2 = \{s_{k+1}, \dots, s_n\}$ satisfying the hypothesis of the theorem. Let $\alpha \in \langle T_1 \rangle \cap \langle T_2 \rangle$ such that $\alpha \neq 0$ and $G(\alpha) = G(T_1) \cap G(T_2)$. Let A, B be mixed dominating matrices of relations on $\langle T_1 \rangle$ and $\langle T_2 \rangle$ respectively, such that both A and B have content 1. Write

$$\alpha = a_1 s_1 + \dots + a_k s_k = b_1 s_{k+1} + \dots + b_p s_n, \text{ where } a_i \geq 0, b_j \geq 0.$$

Let a be the k -tuple (a_1, \dots, a_k) and b the p -tuple $(-b_1, \dots, -b_p)$. If

$$M = \left(\begin{array}{c|c} A & O \\ \hline O & B \\ \hline a & b \end{array} \right),$$

then the rows of M form a basis for the space of relations on S . It is not difficult to see that M is mixed dominating. We will show that M has content 1 or, equivalently, we will show that every relation vector on S is an integer combination of the rows of M .

Let $c = (c_1, \dots, c_n)$ be a relation vector on S and let

$$\beta = c_1 s_1 + \dots + c_k s_k = -c_{k+1} s_{k+1} - \dots - c_n s_n.$$

If $\beta = 0$, then $(c_1, \dots, c_k, 0, \dots, 0)$ is an integer combination of the first m rows of M since the matrix A gives all the relations on the elements s_1, \dots, s_k . Similarly,

$(0, \dots, 0, c_{k+1}, \dots, c_n)$ is an integer combination of rows $m + 1, \dots, r - 1$ of M . Hence, if $\beta = 0$, c is a linear combination of the rows 1 through $r - 1$ of M .

Suppose $\beta \neq 0$. Then, since $\beta \in G(T_1) \cap G(T_2)$, we can write $\beta = q\alpha$ for some integer q . Hence, the n -tuple $c - qu_r$ is also a relation on S . Furthermore, since $\beta = q\alpha$, by the preceding paragraph it follows that $c - qu_r$ is an integer combination of the first $r - 1$ rows of M . Hence c is an integer combination of the rows of M . \square

For any subset X of \mathbf{Q}^k we define the *cone* of X as the set of all positive rational combinations of elements of X . A ray R in the cone of an affine semigroup S is said to be an *extreme ray* of S , if given any vector $u \in R$, positive integers n, c_1, \dots, c_t and vectors w_1, \dots, w_t in S such that

$$nu = c_1w_1 + \dots + c_tw_t,$$

then $w_j \in R$ for $j = 1, \dots, t$.

In [RG-S] it was shown that if S is a three-dimensional semigroup and a complete intersection, then S has no more than four extreme rays. As a corollary to the previous theorem we generalize this result. This then gives a simple necessary condition for S to be a complete intersection.

Corollary 3.4. *Let S be a d -dimensional affine semigroup that is a complete intersection and suppose that $d \geq 2$. Then the cone of S contains no more than $2d - 2$ extreme rays.*

Proof. We induct on r , the dimension of the relation space of S . If $r = 0$, then the d elements of the minimal generating set T are linearly independent. Hence there are exactly d extreme rays. Since we assume that $d \geq 2$, the stated bound must hold.

If $r \geq 1$, write T as the disjoint union of T_1 and T_2 as given in Theorem 3.1. By that theorem note that $d_1 + d_2 - 1 = d$, where $\langle T_1 \rangle, \langle T_2 \rangle$ have dimension d_1 and d_2 respectively and that the number of rows in a relation matrix of $\langle T_1 \rangle$ or $\langle T_2 \rangle$ is less than r . If R is an extreme ray for the cone of S , then we claim that R is an extreme ray for either the cone of T_1 or T_2 . To see this, observe that some element s of T must be in R . Hence, s is in either T_1 or T_2 and therefore R lies in either the cone of T_1 or the cone of T_2 . Clearly it must be an extreme ray in either cone. Therefore, the number of extreme rays of the cone of S is bounded by the sum of the number of extreme rays in the cones of T_1 and T_2 . Hence as long as $d_1, d_2 \geq 2$, the inductive assumption gives that the number of extreme rays of the cone of S is bounded by $2d_1 - 2 + 2d_2 - 2 = 2d - 2$. But if $d_1 = 1$ say, then since $\langle T_1 \rangle$ and $\langle T_2 \rangle$ intersect in a line, it follows that the cone of $\langle T_1 \rangle$ is contained in the cone of $\langle T_2 \rangle$ which has dimension $d_2 = d \geq 2$. Again the inductive assumption gives the result. \square

We present a family of examples to show that the bound in the above corollary is the best possible. If S is any affine semigroup of dimension 2, then the cone of S has exactly two extreme rays, which equals the bound of Corollary 3.4. For each integer $d > 2$, the following family of examples shows that there exists a semigroup S of dimension d that is a complete intersection and such that the cone of S has $2d - 2$ extreme rays. We observe that if M is any mixed dominating matrix with content 1, then it is not difficult to show that there is an affine semigroup S for which the rows of M form a basis for the space of relations of S .

Example 3.5. For $r = 1, 2, \dots$, let $M(r) = (m_{ij})$ denote the $r \times (2r + 2)$ matrix defined by

$$m_{ij} = \begin{cases} 0 & \text{if } j \leq 2i - 2 \text{ or } j \geq 2i + 3, \\ 1 & \text{if } j = 2i - 1, 2i, \\ -1 & \text{if } j = 2i + 1, 2i + 2. \end{cases}$$

Then $M(r)$ is a mixed dominating matrix with dimension $r + 2$. Furthermore, it can be easily checked that the matrix has content 1. Also note that each row of $M(r)$ has exactly two positive entries and two negative entries. Let S be an affine semigroup so that the rows of M are a basis of the space of relations of S . Then S is a complete intersection of dimension $r + 2$ generated by $2r + 2$ elements s_1, \dots, s_{2r+2} . We claim that each s_i , $i = 1, 2, \dots, 2r + 2$, determines a distinct extreme ray of the cone of S . Assume otherwise; then a positive multiple of some s_i is a nonnegative integer combination of the other s_j . In particular, there is a relation vector u of S that has exactly one positive entry. Since u must be a linear combination of the rows of $M(r)$, it follows by [FS2, Proposition 2.6] that some row of $M(r)$ has either exactly one positive entry or exactly one negative entry. But this contradicts the definition of $M(r)$, and hence every s_i is on a distinct extreme ray of the cone of S . Therefore, S is a complete intersection whose cone has the maximum number of extreme rays as allowed by Corollary 3.4.

An affine semigroup S , represented by a minimal generating set $T = \{s_1, \dots, s_n\}$, gives the decision problem: Is S a complete intersection? Our work implies that this problem is in the class NP. One can prove that S is a complete intersection (if indeed it is one) by displaying an $r \times n$ mixed dominating matrix of relations for S that has content 1. One can prove that M is mixed dominating by displaying an $(n - 1) \times n$ mixed dominating matrix N that contains M as a submatrix (such an N exists by Theorem 2.1), and proving that N is mixed dominating. To check that N is mixed dominating, one applies the $O(n^2)$ algorithm of Klee [K] for recognizing S -matrices, which are shown in [FS2] to be precisely the mixed dominating $(n - 1) \times n$ matrices. Furthermore, M can be shown to have content 1 by applying a polynomial-time algorithm to find its Hermite normal form ([S] is a standard reference). One can also show that if such an M exists, then there exists one of size comparable to the size of T . On the other hand, finding good criteria for establishing that S is not a complete intersection remains an interesting open problem - Corollary 3.4 is a step in this direction.

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