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HAUSDORFF MEASURES AND DIMENSION ON $\mathbb{R}^\infty$

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ABSTRACT. We consider the Hausdorff measures H^s, $0 \le s < \infty$, defined on $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}$ with the topology induced by the metric

$$\rho(x,y) = \sum_{i=1}^\infty |x_i - y_i|/2^i(1 + |x_i - y_i|),$$

for all $x = (x_i)_{i=1}^\infty, y = (y_i)_{i=1}^\infty \in \mathbb{R}^\infty$. We study its properties, their relation to the "Lebesgue measure" defined on $\mathbb{R}^\infty$ by R. Baker in 1991, and the associated Hausdorff dimension. Finally, we give some examples.

1. Introduction

In this paper we study the Hausdorff measures H^s, $0 \le s < \infty$, defined in the usual way on $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}$ with the metric

$$\rho(x,y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}\ , \quad x = (x_i)_{i=1}^\infty, y = (y_i)_{i=1}^\infty \in \mathbb{R}^\infty, \tag{1}$$

and the topology induced by it.

We define the Hausdorff dimension on $\mathbb{R}^\infty$ showing that the Lebesgue measure, defined on $\mathbb{R}^\infty$ by R. Baker in [1], operates as a measure H^∞.

Then we study the behaviour of the measures and dimension under some transformations defined on $\mathbb{R}^\infty$ or between this space and $\mathbb{R}^n$. Finally, we give some examples.

The plan of the paper is as follows. In §2 we present the preliminary results that we will need for our work. In §3 we study the Hausdorff measures and dimension on $\mathbb{R}^\infty$. In §4 we give some examples.

In this paper, we will use the word "measure" to mean "outer measure", and every infimum taken over an empty set of real numbers has the value $+\infty$.

2. Preliminaries

In $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R} = \{x = (x_i)_{i=1}^\infty = (x_1, x_2, ..., x_n, ...) : x_i \in \mathbb{R}, i \ge 1\}$ we consider the metric ρ defined in (1). Observe that ρ induces the product topology on $\mathbb{R}^\infty$.

For each $A \subset \mathbb{R}^\infty$ we will denote by $\rho(A)$ the diameter of A with respect to ρ. Since the metric is bounded by 1, then $0 \le \rho(A) \le 1$, for all $A \subset \mathbb{R}^\infty$.

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Let $\mathcal{R}$ be the class of all infinite-dimensional rectangles $R \subset \mathbb{R}^\infty$ of the form

$$R = \prod_{i=1}^{\infty}(a_i, b_i)\ , \qquad -\infty < a_i < b_i < +\infty,$$

such that $0 \leq \prod_{i=1}^{\infty}(b_i - a_i) < +\infty$, together with the empty set. Let τ be the set function on $\mathcal{R}$ defined by

$$\tau(R) = \begin{cases} 0, & \text{if } R = \emptyset, \\ \prod_{i=1}^{\infty}(b_i - a_i), & \text{if } R \neq \emptyset. \end{cases}$$

Then $(\tau, \mathcal{R})$ is a pre-measure. If we apply to it Method I of the generation of measures (see [3]) we obtain the measure

$$\nu(E) = \inf\left\{\sum_{i=1}^{\infty}\tau(R_i)\ :\ E \subset \bigcup_{i=1}^{\infty} R_i,\ \{R_i\}_{i=1}^{\infty} \subset \mathcal{R}\right\}\ ,$$

for all $E \subset \mathbb{R}^\infty$. R. Baker proves in [1] that this measure coincides with the measure obtained applying Method II (see [3]) to the same pre-measure. Then

$$\nu(E) = \lim_{\delta \downarrow 0}\nu_\delta(E) = \sup_{\delta > 0}\nu_\delta(E)\,,$$

for all $E \subset \mathbb{R}^\infty$, where

$$\nu_\delta(E) = \inf\left\{\sum_{i=1}^{\infty}\tau(R_i)\ :\ E \subset \bigcup_{i=1}^{\infty} R_i,\ \{R_i\}_{i=1}^{\infty} \subset \mathcal{R},\ \rho(R_i) \leq \delta\right\}\ ,$$

for each $\delta > 0$.

To finish this section we will give a result on the behaviour of the Hausdorff measures under some transformations between metric spaces.

Let (Ω, d) be a metric space and $0 \leq s < \infty$. We define the s-dimensional Hausdorff measure by

$$H^s(E) = \lim_{\delta \downarrow 0} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E)\,,$$

for all $E \subset \Omega$, where

$$H^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} d(U_i)^s\ :\ E \subset \bigcup_{i=1}^{\infty} U_i,\ d(U_i) \leq \delta\right\}\ ,$$

for all $\delta > 0$. The sets $\{U_i\}_{i=1}^{\infty}$ in this definition are arbitrary subsets of Ω, and $d(A)$ denotes the diameter of A with respect to the metric d. We will denote by H^s the generic s-dimensional Hausdorff measure defined on an arbitrary metric space.

The proof of the next theorem can be found in [2, 3].

Theorem 2.1. *If (Ω, d) and (Ω', d') are metric spaces, $E \subset \Omega$ and $f : E \longrightarrow \Omega'$ satisfies*

$$a \cdot d(x, y) \leq d'(f(x), f(y)) \leq b \cdot d(x, y)\ , \quad \forall x, y \in E$$

where a and b are positive and finite constants, then

$$a^s H^s(E) \leq H^s(f(E)) \leq b^s H^s(E)$$

for each $s \geq 0$.

3. Hausdorff measures and dimension on $\mathbb{R}^\infty$

In this section we consider the s-dimensional Hausdorff measures H^s, $0 \le s < \infty$, defined on $(\mathbb{R}^\infty, \rho)$ as in section §2, where the metric ρ is defined in (1). In this definition we have considered covers by arbitrary sets, but if we consider covers by convex, open or closed sets we obtain the same measures (see [2, 3]).

By the standard theory of Hausdorff measures (see [3]), the measure H^s is a regular, $\mathcal{G}_\delta$-regular, metric measure, the Borel sets are H^s-measurable and each H^s-measurable set of finite H^s-measure contains an $\mathcal{F}_\sigma$-set of the same measure. The Lebesgue measure ν satisfies these properties too.

We now study the behaviour of the Hausdorff measures with respect to the *embedding* $i_n : \mathbb{R}^n \longrightarrow \mathbb{R}^\infty$ defined by

$$i_n(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n, 0, 0, \dots)\,,$$

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and with respect to the *projection* $p_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ defined by

$$p_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n)\,,$$

for all $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$. We consider the euclidean metric d on $\mathbb{R}^n$.

Lemma 3.1. *For every bounded set $E \subset \mathbb{R}^n$ we have*

$$\left(\frac{1}{2^n(1+d(E))}\right)^s H^s(E) \le H^s(i_n(E)) \le \left(\frac{n}{2}\right)^s H^s(E)\,,$$

and for every $E \subset \mathbb{R}^\infty$ such that $p_n(E)$ is a bounded set in $\mathbb{R}^n$, we have

$$H^s(p_n(E)) \le [2^n\,(1+d(p_n(E)))]^s\, H^s(E)\,.$$

Proof. For all $x, y \in E \subset \mathbb{R}^n$, we have

$$\rho\,(i_n(x), i_n(y)) = \sum_{i=1}^n \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)} \le \frac{1}{2}\sum_{i=1}^n |x_i - y_i| \le \frac{n}{2} d(x,y)$$

and

$$\begin{aligned}\rho\,(i_n(x), i_n(y)) &= \sum_{i=1}^n \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)} \ge \frac{1}{2^n(1+d(E))}\sum_{i=1}^n |x_i - y_i| \\ &\ge \frac{1}{2^n(1+d(E))} d(x,y)\,.\end{aligned}$$

Now, for all $x, y \in E \subset \mathbb{R}^\infty$, we have

$$\begin{aligned}d\,(p_n(x), p_n(y)) &= \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2} \le \sum_{i=1}^n |x_i - y_i| \\ &\le 2^n\,(1 + d(p_n(E)))\,\rho(x,y)\,.\end{aligned}$$

Then, applying Theorem 2.1, we obtain the results of this lemma. □

It is easy to show (see [2]) that, for all $E \subset \mathbb{R}^\infty$ and $0 \le s < t < \infty$,

$$\begin{aligned}H^t(E) > 0 &\implies H^s(E) = \infty,\\ H^s(E) < \infty &\implies H^t(E) = 0.\end{aligned}$$

Moreover, we prove in the next theorem that the Lebesgue measure ν operates as a measure H^∞.

Theorem 3.2. *Let $E \subset \mathbb{R}^\infty$ and $0 \leq s < \infty$ satisfy the condition $H^s(E) = 0$. Then*

$$\nu(E) = 0.$$

Proof. Let $n \geq s$ be an integer with $H^n(E) = 0$, and denote by $E_n = p_n(E)$ the projection of E on $\mathbb{R}^n$. For each $N \geq 1$ we consider

$$E^N = \{x \in E : |x_i| \leq N,\, i \geq 1\} = E \cap [-N, N]^\infty .$$

Since $H^n(E^N) = 0$, and using Lemma 3.1, we have that $H^n(p_n(E^N)) = 0$. In consequence

$$H^n(E_n) = H^n\left(\bigcup_{N=1}^{\infty} p_n(E^N)\right) \leq \sum_{N=1}^{\infty} H^n\left(p_n(E^N)\right) = 0$$

and hence $L^n(E_n) = 0$, where L^n is the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

If we call

$$E_n^* = \{y = (y_i)_{i=1}^\infty : \exists x \in \mathbb{R}^n \text{ with } (x_1, \dots, x_n, y_1, y_2, \dots) \in E\},$$

then $E \subset E_n \times E_n^* = E^*$. To prove the theorem it is enough to prove that $\nu(E^*) = 0$.

Let $\varepsilon > 0$ be given. Let $\{R_j^*\}_{j=1}^\infty \subset \mathcal{R}$ be a cover of E_n^*. As $L^n(E_n) = 0$, for all $j \geq 1$ we can cover E_n by a family $\{R_i^j\}_{i=1}^\infty$ of n-dimensional open rectangles such that

$$\sum_{i=1}^{\infty} V(R_i^j) < \begin{cases} \frac{\varepsilon}{2^j \tau(R_j^*)}, & \text{if } \tau(R_j^*) > 0, \\ \frac{\varepsilon}{2^j}, & \text{if } \tau(R_j^*) = 0, \end{cases}$$

where, for each open rectangle $R = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$, $V(R) = \prod_{i=1}^n (b_i - a_i)$ is its volume. Then the set of infinite-dimensional open rectangles $\{R_i^j \times R_j^*\}_{i,j=1}^\infty \subset \mathcal{R}$ is a cover of E^* that satisfies

$$\sum_{i,j=1}^{\infty} \tau(R_i^j \times R_j^*) = \sum_{j=1}^{\infty} \tau(R_j^*) \sum_{i=1}^{\infty} V(R_i^j) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

and, since $\varepsilon > 0$ is arbitrary, we conclude that $\nu(E^*) = 0$. □

Thus, if we denote the measure ν by H^∞, we can affirm that for each $E \subset \mathbb{R}^\infty$ there exists a unique number $s \in [0, \infty]$ such that

$$H^t(E) = \begin{cases} \infty, & \text{if } 0 \leq t < s, \\ 0, & \text{if } s < t \leq \infty. \end{cases}$$

This number is called *Hausdorff dimension* of E and it will be denoted by $s = \dim E$.

To finish this section we will give some properties of these Hausdorff measures and dimension.

Properties 3.3. 1. The Hausdorff measures and dimension are invariant by translations. For each $E \subset \mathbb{R}^\infty$ and $a \in \mathbb{R}^\infty$ we have that $H^s(a + E) = H^s(E)$, for all $s \geq 0$, and $\dim(a + E) = \dim E$, where $a + E = \{a + x : x \in E\}$.

2. It is easy to show that for all s, $0 \leq s < \infty$, $E \subset \mathbb{R}^\infty$ and $\lambda > 0$

$$\begin{array}{rlll} \lambda^s H^s(E) & \leq H^s(\lambda E) \leq & H^s(E), & \text{if } 0 < \lambda < 1, \\ H^s(E) & \leq H^s(\lambda E) \leq & \lambda^s H^s(E), & \text{if } \lambda > 1, \end{array}$$

where $\lambda E = \{\lambda x : x \in E\}$. In consequence $\dim(\lambda E) = \dim E$.

3. If $S : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ is a similitude of ratio $r > 0$, then $H^s(S(E)) = r^s H^s(E)$ for all $E \subset \mathbb{R}^\infty$ and $0 \leq s < \infty$. So the Hausdorff dimension is invariant with respect to similitudes.

4. For each $a = (a_1, a_2, \dots , a_n) \in \mathbb{R}^n$, we define the *shift operator* associated to a by

$$x = (x_1, x_2, \dots) \longrightarrow ax = (a_1, \dots , a_n, x_1, x_2, \dots)$$

for all $x \in \mathbb{R}^\infty$, and we denote $aE = \{ax : x \in E\}$. It is easy to see that

$$\rho(ax, ay) = \frac{1}{2^n}\rho(x, y)$$

for all $x, y \in \mathbb{R}^\infty$, and hence

$$H^s(aE) = \frac{1}{2^{ns}}H^s(E)$$

for all $E \subset \mathbb{R}^\infty$, $a \in \mathbb{R}^n$ and $0 \leq s < \infty$. Thus the Hausdorff dimension is invariant with respect to this operator.

5. If $a \in \mathbb{R}^m$, $b \in \mathbb{R}^\infty$ and $E \subset \mathbb{R}^n$ is a bounded set, we consider the function $f : E \longrightarrow \mathbb{R}^\infty$ defined by $f(x) = axb$. Using the preceding properties and Lemma 3.1 we have that

$$\left(\frac{1}{2^{n+m}(1 + d(E))}\right)^s H^s(E) \leq H^s\left(f(E)\right) \leq \left(\frac{n}{2^{m+1}}\right)^s H^s(E)$$

for each s, $0 \leq s < \infty$. Consequently the Hausdorff dimension is invariant with respect to this function.

4. Examples

In this section we give some examples of fractals in $\mathbb{R}^\infty$ and we compute their Hausdorff dimension.

Example 4.1. Let $I = \{0, 1\}$ and $0 < r < 1/2$. For each $i \geq 1$ we consider the set $E_i^r = (2r)^i I = \{0, (2r)^i\}$. Then the set $E^r = \prod_{i=1}^\infty E_i^r \subset \mathbb{R}^\infty$ has Hausdorff dimension $s = s(r) = -\log 2/ \log r$.

Proof. We consider the generalized Cantor set $C^r \subset [0, 1] \subset \mathbb{R}$, which is the self-similar set associated to the similitudes $S_1(x) = rx$ and $S_2(x) = rx + (1 - r)$ of Hausdorff dimension $s = s(r)$ in $\mathbb{R}$, and $H^s(C^r) = 1$ (see [2]).

Each point $x \in E^r$ has a unique representation $x = (x_i)$ where $x_i = (2r)^i a_i$ with $a_i = 0$ or 1. Now we consider the one-to-one function $f : E^r \longrightarrow C^r$ defined by

$$f(x) = \sum_{i=1}^\infty \frac{(1 - r)x_i}{2^i r} = \sum_{i=1}^\infty a_i r^{i-1}(1 - r)$$

that satisfies

$$|f(x) - f(y)| \leq \sum_{i=1}^\infty \frac{(1 - r)|x_i - y_i|}{2^i r} \leq \frac{2(1 - r)}{r}\rho(x, y).$$

Applying Theorem 2.1, we see that

$$0 < 1 = H^s(C^r) \leq \left(\frac{2(1 - r)}{r}\right)^s H^s(E^r).$$

To see that $H^s(E^r) < \infty$ we consider, for each $N \geq 1$, the set $P = \prod_{i=1}^{N} E_i^r \subset \mathbb{R}^N$ of 2^N points. Then

$$E^r = \bigcup_{x \in P} U_x\ , \quad \text{where } U_x = \left\{ xy\ :\ y \in \prod_{i=N+1}^{\infty} E_i^r \right\}$$

and

$$\rho(U_x) = \sup\left\{ \rho(xy, xz)\ :\ y, z \in \prod_{i=N+1}^{\infty} E_i^r \right\} \leq \frac{r^{N+1}}{1-r} = \delta(N)\,.$$

Since $2r^s = 1$, we have that

$$H^s_{\delta(N)}(E^r) \leq 2^N \left(\frac{r^{N+1}}{1-r} \right)^s = \left(\frac{r}{1-r} \right)^s .$$

Taking the limit when $N \uparrow \infty$, we obtain that

$$H^s(E^r) \leq \left(\frac{r}{1-r} \right)^s < \infty$$

and we conclude the proof of this example. $\square$

By similar methods we can prove the next example.

Example 4.2. If $I = \{0, 1\}$ is as in the previous example, the Hausdorff dimension of the set $I^\infty = \{x = (x_i)\ :\ x_i = 0 \text{ or } 1\} \subset \mathbb{R}^\infty$ is 1, and $H^1(I^\infty) = 1/2$.

Example 4.3. If $C \subset [0, 1] \subset \mathbb{R}$ is the generalized Cantor set of ratio $r = 1/4$ and $E = C \times I^\infty$, where I^∞ is as in the previous example, then

$$\dim E = \frac{3}{2}.$$

Proof. We consider the one-to-one function $f : E \longrightarrow C \times [0, 1]$ defined by

$$f(x) = \left(x_1, \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} \right)$$

for each $x = (x_i)_{i=1}^{\infty} \in E$. It is easy to see that

$$d\,(f(x), f(y)) \leq |x_1 - y_1| + \sum_{i=2}^{\infty} \frac{|x_i - y_i|}{2^{i-1}} \leq 4 \cdot \rho(x, y)$$

and applying Theorem 2.1, we obtain that

$$4^{3/2} H^{3/2}(E) \geq H^{3/2}(C \times [0, 1]) \geq b \cdot H^{1/2}(C) \cdot H^1([0, 1]) = b$$

where $b > 0$ is a positive constant (see [2]).

To see that $H^{3/2}(E) < \infty$ we consider, for all $N \geq 1$, the natural cover of C by the 2^N intervals $\{I_i^N\}_{i=1}^{2^N}$, each of length 4^{-N}, which appear in step N of its construction, and the set $P = \prod_{i=2}^{2N+1} I$ of 2^{2N} elements. Then

$$E \subset \bigcup_{i=1}^{2^N} \bigcup_{x \in P} U_{xi}$$

where $U_{xi} = \{yxz : y \in I_i^N,\, z \in I^\infty\}$ and

$$\begin{aligned}\rho(U_{xi}) &= \sup\left\{\rho(yxz, y'xz') : y, y' \in I_i^N,\, z, z' \in I^\infty\right\} \\ &\le \sup_{y,y' \in I_i^N} \frac{|y-y'|}{2(1+|y-y'|)} + \sum_{j=2N+2}^{\infty} \frac{1}{2^j \cdot 2} \\ &= \frac{4^{-N}}{2(1+4^{-N})} + \frac{1}{2^{2N+2}} < \frac{1}{2^{2N}} = \delta(N)\,.\end{aligned}$$

Then

$$H^{3/2}_{\delta(N)}(E) \le 2^N \cdot 2^{2N} \cdot \left(\frac{1}{2^{2N}}\right)^{3/2} = 1$$

and taking limits when $N \uparrow \infty$, we obtain that $H^{3/2}(E) \le 1 < \infty$. □

Applying property 3.3(5), it is easy to see that

Example 4.4. If $m \ge 0$ and $n \ge 1$ are two integer numbers, the set

$$Q_m^n = \{0\}^m \times [0,1]^n \times \{0\}^\infty$$

has Hausdorff dimension n and $0 < H^n(Q_m^n) < \infty$.

Example 4.5. If $G \subset \mathbb{R}^\infty$ is a non-empty open set, then $\dim G = \infty$.

Proof. Applying Lemma 3.1 and that $p_n(G) \subset \mathbb{R}^n$, for all $n \ge 1$, is a non-empty open set, we have that

$$\dim G \ge \dim p_n(G) = n$$

and hence $\dim G = \infty$. □

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