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A RENEWAL THEOREM IN THE FINITE-MEAN CASE

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Abstract. Let $F(.)$ be a c.d.f. on $(0,\infty)$ such that $\overline{F}(.) \equiv 1-F(.)$ is regularly varying with exponent $-\alpha$, $1<\alpha<2$. Then $U(t)-\frac{t}{\mu}-\frac{1}{\mu^2}\int_0^t\int_s^\infty \overline{F}(v)dvds = O(t^4\overline{F}(t)^2\overline{F}(t^2\overline{F}(t)))$ as $t\to\infty$, where $U(t)=EN(t)$ is the renewal function associated with $F(t)$. Moreover similar estimates are given for distributions in the domain of attraction of the normal distribution and for the variance of $N(t)$. The estimates improve earlier results of Teugels and Mohan.

1. Introduction and results

In this paper we assume that $X_1, X_2, ...$ is a sequence of i.i.d. real-valued positive random variables with d.f. F. Define the associated random walk by $S_0=0, S_n = X_1+...+X_n$ for $n\geq 1$, $N(t)=\max\{n\geq 0; S_n\leq t\}$ and the renewal function U by

$$U(t)\equiv EN(t)=\sum_{n=0}^{\infty}P(S_n\leq t).$$

If F is not arithmetic, Blackwell's theorem says that

$$U(t)-U(t-h)\to\frac{h}{\mu}\text{ as } t\to\infty$$

for every $h>0$ where $\mu=EX_1$. See Feller [5] for a proof and an extension to the arithmetic case. Extensions such as

$$U(t)-\frac{t}{\mu}\to\frac{\sigma^2+\mu^2}{2\mu^2}\text{ as } t\to\infty$$

in case σ^2 is finite are usually proved using so-called key renewal theorems which give the asymptotic behavior of the convolution

$$(U*Q)(t)\equiv\int_{0-}^{t}Q(t-u)U(du)$$

as $t\to\infty$ under suitable hypothesis on $Q(.)$ and $F(.)$.

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In this paper we consider the special case where F is not arithmetic and the distribution function tail $\overline{F} \equiv 1 - F$ is regularly varying, i.e. satisfies

$$\frac{\overline{F}(tx)}{\overline{F}(t)} \to x^{-\alpha} \text{ as } t \to \infty. \tag{1.1}$$

In this case we use the notation $\overline{F}(.) \in RV_{-\alpha}$. Following earlier work by Feller [4] and Smith [10, 11], Teugels [12] addressed the question of the asymptotic behavior of the convolution $U * Q(.)$ under the assumption $\overline{F} \in RV_{-\alpha}$ where $0 < \alpha < 2$. In case $\frac{1}{2} < \alpha < 1$ (which implies $\mu = \infty$) an improvement of Teugels' result is given by Anderson and Athreya [1] using a result of Erickson [3]. In the case of a regularly varying tail function with $1 < \alpha < 2$ (in which case $\mu < \infty$ and $\sigma^2 = \infty$) Teugels [12] proved that

$$\tau(t) \equiv U(t) - \frac{t}{\mu} \sim \frac{t^2 \overline{F}(t)}{\mu^2(\alpha-1)(2-\alpha)} \text{ as } t \to \infty \tag{1.2}$$

(whence $\tau(.) \in RV_{2-\alpha}$) under a supplementary condition. This condition was shown to be unnecessary in a paper by Mohan [8]. Besides this, Mohan proved that for $F \in D(\alpha)$, the domain of attraction of a stable law with exponent α where $1 < \alpha \leq 2$ (in case $\alpha = 2$ F is assumed to have infinite variance), the above function $\tau(.)$ satisfies the asymptotic relation

$$\tau(t) \sim \frac{1}{\mu^2} \int_0^t \int_x^\infty \overline{F}(v) dv dx \text{ as } t \to \infty. \tag{1.3}$$

It should be observed that the above asymptotic relation holds for non-arithmetic F with finite mean μ even without the assumption (1.1). See Frenk [6], Lemma 4.1.2. Note that in case $1 < \alpha < 2$ relation (1.3) is equivalent to (1.2). In this paper we show that the assumption $F \in D(\alpha)$ with $1 < \alpha \leq 2$ permits a stronger conclusion than (1.3). In particular we have the following results.

Theorem 1.1. *Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume σ^2 infinite) is not arithmetic. Suppose $Q(t) = \int_t^\infty q(s)ds < \infty,\ t \geq 0$ where $q(.)$ is nonnegative and nonincreasing. Suppose $\int_0^t Q(s)ds \in RV_{-\gamma+1}$ $(0 < \gamma \leq 1)$, where in case $\gamma = 1$ we assume $tQ(t) \to \infty$ as $t \to \infty$. Then as $t \to \infty$*

$$(U * Q)(t) = \begin{cases} \frac{1}{\mu}\int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) & \text{if} \quad 0 < \gamma < 1, \\ \frac{1}{\mu}\int_0^t Q(s)ds + o(\int_0^{\tau(t)} Q(s)ds) & \text{if} \quad \gamma = 1. \end{cases} \tag{1.4}$$

Theorem 1.2. *Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume σ^2 infinite) is not arithmetic. Then as $t \to \infty$*

$$U(t) - \frac{t}{\mu} - \frac{1}{\mu^2}\int_0^t \int_s^\infty \overline{F}(v)dvds = \begin{cases} O(\tau(\tau(t))) = O(t^4\overline{F}(t)^2\overline{F}(t^2\overline{F}(t))) & \text{if} \quad 1 < \alpha < 2, \\ o(\tau(\tau(t))), \text{where } \tau(.) \text{ satisfies (1.3)} & \text{if} \quad \alpha = 2. \end{cases} \tag{1.5}$$

We may use the same technique as Smith [11] to find the variance of $N(t)$. Using (1.5) and the method used in Smith's paper gives the following estimate which improves the estimates given in the papers of Teugels [12]) and Mohan [8]. We omit the details of the proof.

Theorem 1.3. *Under the assumptions of Theorem 1.2 we have*

$$\begin{aligned} &var\ N(t) - \frac{4}{\mu^3}\int_0^t\int_0^u\int_v^\infty \overline{F}(s)dsdvdu + \frac{2t}{\mu^3}\int_0^t\int_v^\infty \overline{F}(s)dsdv = \\ &\begin{cases} O(t^5\overline{F}(t)^2\overline{F}(t^2\overline{F}(t))) & \text{if} \quad 1<\alpha<2, \\ o(t\tau(\tau(t))), \text{where } \tau(.) \text{ satisfies (1.3)} & \text{if} \quad \alpha=2. \end{cases} \end{aligned} \tag{1.6}$$

2. Proofs

In the proofs below we write $\beta = 2-\alpha$. Before giving the proofs of the results we list some well-known properties of RV functions which are used in the sequel. For a proof the reader is referred to Bingham et al. [2], Geluk and de Haan [7] or Resnick [9].

Lemma 1. *Suppose $\phi \in RV_\delta$. It follows that*

(i) *Uniform convergence theorem for regularly varying functions. Convergence in $\phi(tx)/\phi(t) \to x^\delta (t\to\infty)$ is uniform on compact intervals of $(0,\infty)$.*

(ii) *Karamata's theorem. There exists $t_0>0$ such that $\phi(t)$ is positive and locally bounded for $t>t_0$. If $\delta \geq -1$, then*

$$\lim_{t\to\infty}\frac{t\phi(t)}{\int_{t_0}^t \phi(s)ds} = \delta+1.$$

A similar result holds for $\delta<-1$.

(iii) *Potter's inequality. If $\varepsilon_1,\varepsilon_2>0$ are arbitrary, there exists $t_0=t_0(\varepsilon_1,\varepsilon_2)$ such that for $t\geq t_0, tx\geq t_0$*

$$(1-\varepsilon_1)x^{\delta-\varepsilon_2} < \frac{\phi(tx)}{\phi(t)} < (1+\varepsilon_1)x^{\delta+\varepsilon_2}.$$

(iv) *Monotone density theorem. If $\delta\geq 0$ and $\phi(t)=\int_0^t f(s)ds$ for $t\geq 0$ with f monotone, then $\lim_{t\to\infty} tf(t)/\phi(t)=\delta$. Hence in case $\delta>0$ we have $f(.)\in RV_{\delta-1}$.*

Proof of Theorem 1.1. We write

$$\int_{0-}^t Q(t-y)U(dy) = I_1+I_2+I_3,$$

where I_1, I_2 and I_3 are the integrals over $(0,[L(t)]),([L(t)],[t])$ and $([t],t)$ respectively, where $[t]$ denotes the greatest integer not exceeding t. Take $L(t)\to\infty$ $(t\to\infty)$ a slowly varying (i.e. in RV_0) function. In case $\beta=0$ take in addition $L(t)=o(\tau(t))$ $(t\to\infty)$, which is possible since $\tau(t)\to\infty$ as $t\to\infty$. (Note that F has infinite variance.)

First we estimate $I_1=\int_0^{[L(t)]}Q(t-y)U(dy)$. By monotonicity of Q we have as $t\to\infty$

$$0\leq I_1\leq Q(t-[L(t)])U(L(t)).$$

Since $L(t)=o(t)$ and $\tau(.)\in RV_\beta$, $0\leq\beta<1$, we have $\tau(t)=o(t-L(t))$ as $t\to\infty$; hence $Q(t-[L(t)])\leq Q(\tau(t))$ for t sufficiently large. It follows that $I_1=O(L(t)Q(\tau(t)))=o(\tau(t)Q(\tau(t)))$.

The second integral is estimated as follows.

$$\begin{aligned} I_2 &= \sum_{j=[L(t)]+1}^{[t]} \int_{j-1}^{j} Q(t-y)U(dy) \\ &\leq \sum_{j=[L(t)]+1}^{[t]} Q(t-j)(U(j)-U(j-1)) \\ &= \frac{1}{\mu} \sum_{j=[L(t)]+1}^{[t]} Q(t-j) + \sum_{j=[L(t)]+1}^{[t]} Q(t-j)(\tau(j)-\tau(j-1)) \qquad (2.1) \end{aligned}$$

and similarly

$$\begin{aligned} I_2 &\geq \sum_{j=[L(t)]+1}^{[t]} Q(t-j+1)(U(j)-U(j-1)) \\ &= \frac{1}{\mu} \sum_{j=[L(t)]}^{[t]-1} Q(t-j) + \sum_{j=[L(t)]+1}^{[t]} Q(t-j+1)(\tau(j)-\tau(j-1)). \qquad (2.2) \end{aligned}$$

Application of the Lemmas 2, 4 and 5 below shows that $I_2 = \frac{1}{\mu}\int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) + o(\int_0^{\tau(t)} Q(s)ds)$ as $t \to \infty$. From Lemma 1 (ii) and (iv) in case $0 < \gamma < 1$ it follows that $\int_0^t Q(s)ds \sim (1-\gamma)^{-1} tQ(t)$ and $tQ(t) \to \infty$ $(t \to \infty)$. In case $\gamma = 1$ we have $tQ(t) = o(\int_0^t Q(s)ds)$. The proof is complete since $I_3 = \int_{[t]}^t Q(t-y)U(dy) = O(U(t) - U([t])) = O(1)$ by Blackwell's theorem. □

In the Lemmas 2 to 5 below the assumptions of Theorem 1.1 are supposed to be satisfied.

Lemma 2.

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) = \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) \ (t \to \infty).$$

The same estimate holds for $\sum_{j=[L(t)]}^{[t]-1} Q(t-j)$.

Proof of Lemma 2. Since the second statement is equivalent to the first (note that $Q(0+) < \infty$) we only prove the first statement. Note that by the monotonicity of Q

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) \geq \int_{t-[t]}^{t-[L(t)]} Q(s)ds = \int_0^t Q(s)ds - \int_{t-[L(t)]}^t Q(s)ds + O(1).$$

The last integral can be estimated by $0 \leq \int_{t-[L(t)]}^t Q(s)ds \leq Q(t-[L(t)])L([t])$; hence the integral is $O(\tau(t)Q(\tau(t)))$, $t \to \infty$ using the same argument as in the proof above.

Similarly we have

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) \leq Q(0+) + \int_0^{t-[L(t)]-1} Q(s)ds = \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))).$$

□

Lemma 3. *As* $t \to \infty$

$$\phi(t) \equiv \int_{[L(t)]}^{[t]-[\tau(t)]} q(t-s)\tau(s)ds = O(\tau(t)Q(\tau(t)))$$

Proof of Lemma 3. We estimate

$$\phi(t) = t\int_{\frac{[L(t)]}{t}}^{\frac{[t]-[\tau(t)]}{t}} q(t(1-u))\tau(tu)du$$

as follows. Note that $L(t) \to \infty$ and $\tau(t) \to \infty$ $(t \to \infty)$; hence $\tau(.)$ is positive on the specified interval of integration for t sufficiently large. From (1.3) it follows that $\tau(.)$ is asymptotic to a non-decreasing function. It follows that for $\varepsilon > 0$ arbitrary there exists $t_0 = t_0(\varepsilon)$ such that $\tau(tu) \leq (1+\varepsilon)\tau([t]-[\tau(t)])$ uniformly for $u \in [\frac{[L(t)]}{t}, \frac{[t]-[\tau(t)]}{t}]$ as $t > t_0(\varepsilon)$. Hence we obtain

$$\phi(t) \leq (1+\varepsilon)t\tau([t]-[\tau(t)])\int_{\frac{[L(t)]}{t}}^{\frac{[t]-[\tau(t)]}{t}} q(t(1-u))du.$$

Regular variation of the function $\tau(.)$ with exponent $\beta \in [0,1)$ implies

$$\tau([t]-[\tau(t)]) \sim \tau(t) \quad \text{as } t \to \infty;$$

hence

$$\phi(t) \leq (1+\varepsilon)^2\tau(t)\int_{t-[t]+[\tau(t)]}^{t-[L(t)]} q(s)ds \leq (1+\varepsilon)^2\tau(t)Q([\tau(t)])$$

for t sufficiently large. □

Lemma 4.

$$S_1(t) \equiv \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} Q(t-j)(\tau(j)-\tau(j-1)) = O(\tau(t)Q(\tau(t)))$$

as $t \to \infty$. *The same estimate holds if* $Q(t-j)$ *is replaced with* $Q(t-j+1)$.

Proof of Lemma 4. We only prove the first estimate, the second can be proved similarly. Using partial summation we have

$$\begin{aligned} |S_1(t)| &= |-\Sigma + Q(t-[t]+[\tau(t)])\tau([t]-[\tau(t)]) - Q(t-[L(t)])\tau([L(t)])| \\ &\leq \Sigma + Q(t-[t]+[\tau(t)])\tau([t]-[\tau(t)]) + Q(t-[L(t)])\tau([L(t)]), \end{aligned} \tag{2.3}$$

where

$$\Sigma = \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} (Q(t-j)-Q(t-j+1))\tau(j-1).$$

The middle term on the right-hand side in equation (2.3) is dominated by $(1+\varepsilon)\tau(t)Q(\tau(t)-1)$ and the last term is asymptotic to

$$\tau(L(t))Q(t-[L(t)]) = O(\tau(t)Q(\tau(t)))$$

by the same argument as in the proof of Theorem 1.1. Note that since $L(t) \to \infty$ and $\tau(t) \to \infty$ $(t \to \infty)$, monotonicity of Q implies that $\Sigma \geq 0$ for t sufficiently

large. We proceed with the upper estimate. For $\varepsilon > 0$ arbitrary and t sufficiently large we have

$$\begin{aligned}\Sigma &= \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} \tau(j-1) \int_{t-j}^{t-j+1} q(s)ds \\ &\le (1+\varepsilon) \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} \int_{t-j}^{t-j+1} \tau(t-s)q(s)ds \\ &= (1+\varepsilon) \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} \int_{j-1}^{j} \tau(s)q(t-s)ds \\ &= (1+\varepsilon) \int_{[L(t)]}^{[t]-[\tau(t)]} \tau(s)q(t-s)ds.\end{aligned}$$

Note that the above inequality follows from (1.3) as in the proof of Lemma 3 above.

Application of Lemma 3 completes the proof. □

Lemma 5.

$$S_2(t) \equiv \sum_{j=[t]-[\tau(t)]+1}^{[t]} Q(t-j+1)(\tau(j)-\tau(j-1)) = o(\int_0^{\tau(t)} Q(s)ds)$$

as $t \to \infty$. The same estimate holds if $Q(t-j+1)$ is replaced with $Q(t-j)$.

Proof of Lemma 5. Since $\tau(t) - \tau(t-1) = U(t) - U(t-1) - \frac{1}{\mu} \to 0$ as $t \to \infty$ by Blackwell's theorem, it follows that for $\varepsilon > 0$ and $t > t(\varepsilon)$

$$\begin{aligned}|S_2(t)| &\le \varepsilon \sum_{j=[t]-[\tau(t)]+1}^{[t]} Q(t-j+1) \\ &\le \varepsilon\{Q(0+) + \textstyle\int_0^{t-[t]+[\tau(t)]} Q(s)ds\}.\end{aligned}$$

Since $\int_0^t Q(s)ds$ is regularly varying, the integral on the right-hand side is asymptotic to $\int_0^{\tau(t)} Q(s)ds$. The statement of the lemma is proved since $\varepsilon > 0$ is arbitrary and $\int_0^{\tau(t)} Q(s)ds \ge \tau(t)Q(\tau(t)) \to \infty$ as $t \to \infty$. The proof of the second statement is similar. □

Proof of Theorem 1.2. Integrating the renewal equation $\int_{0-}^s \overline{F}(s-y)U(dy) = 1$ over the interval $[0,t]$ $(t > 0)$ gives $\int_{0-}^t \int_0^{t-y} \overline{F}(u)duU(dy) = t$. It follows that

$$U(t) - \frac{t}{\mu} = \int_{0-}^t Q(t-y)U(dy)$$

where $Q(t) = \frac{1}{\mu}\int_t^\infty \overline{F}(s)ds$.

Since the asymptotic behavior of τ is given by (1.3), application of Theorem 1.1 completes the proof. (Note that $t^2\overline{F}(t) \ge \int_{0-}^t x^2 F(dx) \to \infty$; hence $tQ(t) \to \infty$ as $t \to \infty$ in case $\alpha = 2$.) □

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